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ON SOME PRODUCTS OF β-ELEMENTS IN THE STABLE HOMOTOPY OF L_2-LOCAL SPHERES

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§ 1. Introduction

The β-elements in the stable homotopy groups of spheres at the prime >3 are introduced by H. Toda ([22]) and generalized by L. Smith ([21]) and S. Oka ([4], [5], [6]). In [3], H. Miller, D. Ravenel and S. Wilson give the way to define the generalized Greek letter elements, including β-elements, in the E_2-term of the Adams-Novikov spectral sequence for computing the homotopy groups $\pi_*(S^0)$. S. Oka ([7], [8]) and H. Sadofsky ([12]) show that some of them are permanent cycles in the spectral sequence.

The second author has studied about the product of these β-elements ([9], [13], [14], [15], [16], [17]). The β-elements of the homotopy groups $\pi_*(M_p)$ of the mod p Moore spectrum M_p appear when we define those of $\pi_*(S^0)$. In fact, a β-element β'_t of $\pi_*(M_p)$ is sent to β_t in $\pi_*(S^0)$ by the projection map $\pi: M_p \to \Sigma^1 S^0$ to the top cell. It is also studied the non-triviality of products $\beta'_t \beta_E$ of β-elements β'_t in $\pi_*(M_p)$ and β_E in $\pi_*(S^0)$ for some subscript E (*cf.* [18], [1], [2]). In this paper, we study the projection map $\pi: M_p \to \Sigma^1 S^0$, and try to push out the non-trivial products of the homotopy groups of the Moore spectrum M_p to those of the sphere spectrum S^0. In other words, we study whether $\beta_t \beta_E$ is nontrivial in $\pi_*(S^0)$ when $\beta'_t \beta_E$ is non-trivial.

By the recent work [20], A. Yabe and the second author have determined the additive structure of the homotopy groups of L_2-local spheres, where L_2 stands for the Bousfield localization functor with respect to the Johnson-Wilson spectrum $E(2)$ whose coefficient ring is $Z/p[v_1, v_2, v_2^{-1}]$. Then we have the localization map $\pi_*(S^0) \to \pi_*(L_2 S^0)$. It would be fine if we obtain some information of $\pi_*(S^0)$ from the map, but we do not treat it here. Actually we study, in this paper, the localized map $L_2\pi: L_2 M_p \to L_2 \Sigma^1 S^0$ rather than π itself.

In particular, in [2] and [1], we have shown a relation

$$\beta'_t \beta_{sp^{n+r}/p^r a_{n-i,i+1}} \neq 0 \qquad \text{in} \quad \pi_*(L_2 M_p)$$

under the following condition on the integers appeared in the subscripts of β's.

$$p \nmid st \text{ for even } r \geqq 2, \text{ and}$$

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(1.1) $p|c$ and $p\nmid c+p$ for odd $r\geqq 1$.

Here a_i denotes the integer $p^i+p^{i-1}-1$ if $i>0$ and 1 if $i=0$, and c is an integer such that

$$t+sp^{n+r}-p^{n+r-i-1}+(p^r+1)/(p+1)=cp^l-(p^l-1)/(p-1) \text{ and } p\nmid c+1$$

for some $l\geqq 0$. Note that the definition of β-elements is slightly different from that of [3]. For our elements, see § 2. Further note that $\beta_{sp^{n+r}/p^r a_{n-i},i+1}$ is defined if $0<i+1\leqq r$ and $i\leqq n$. Our main result is that the above products of β's in $\pi_*(L_2M_p)$ all survive to $\pi_*(L_2S^0)$ under the map $L_2\pi_*$, and so we have

THEOREM. *Let t, s, n, r and i be non-negative integers such that $t, s, r>0$ and $i\leqq\min\{r-1, n\}$. In the homotopy groups $\pi_*(L_2S^0)$, the product $\beta_t\beta_{sp^{n+r}/p^r a_{n-i},i+1}$ is not null if the condition* (1.1) *is satisfied.*

As an example, taking $r=1$, we have

COROLLARY. *Let u, s and n be positive integers. Then,*

$$\beta_{up^2-1}\beta_{sp^{n+1}/p^{n+1}+p^n-p}\neq 0\in\pi_*(L_2S^0)$$

if $n>1$, and

$$\beta_{up^3-2}\beta_{sp^{n+1}/p^{n+1}+p^n-p}\neq 0\in\pi_*(L_2S^0)$$

if $n>2$.

§ 2. β-elements

Let (A, Γ) denote the Hopf algebroid associated to the Johnson-Wilson spectrum $E(2)$ with coefficient ring $E(2)_*=\boldsymbol{Z}_{(p)}[v_1, v_2, v_2^{-1}]$:

$$A=E(2)_* \qquad \Gamma=E(2)_*(E(2))=E(2)_*[t_1, t_2, \cdots]\otimes_{BP_*}E(2)_*,$$

in which BP_* acts on $E(2)_*$ by sending v_n to v_n if $n\leqq 2$, and to 0 if $n>2$. Then there is the Adams-Novikov spectral sequence converging to $\pi_*(L_2S_0)$ (resp. $\pi_*(L_2M_p)$) with E_2-term $E_2^*=\mathrm{Ext}_\Gamma^*(A, A)$ (resp. $E_2^*=\mathrm{Ext}_\Gamma^*(A, A/(p))$). Here in this paper, an element of the Ext-groups will be represented by an element of the cobar complex Ω_Γ^*A (resp. $\Omega_\Gamma^*A/(p)$). We shall abbreviate $\mathrm{Ext}_\Gamma^s(A, M)$ by

$$\mathrm{Ext}^s(M)$$

for a Γ-comodule M. We see that $E_2^s=0$ for $s>4$ by using Morava's theorem [10] (*cf.* [3, Th. 3.6], [11, Ch. 6]) and the chromatic spectral sequence [3, 3.A] (*cf.* [11, Ch. 5]). Therefore the spectral sequence collapses and arises no extension problem by its sparseness. Hence we identify the E_2-term with its abutment $\pi_*(L_2S^0)$ or $\pi_*(L_2M_p)$.

In order to define the β-elements, consider the connecting homomorphisms

$$\begin{aligned}&\delta_1: \mathrm{Ext}^1(A/(p^{i+1})) \longrightarrow \mathrm{Ext}^2(A), \quad \text{and}\\ &\delta_0: \mathrm{Ext}^0(A/(p^{i+1}, v_1^j)) \longrightarrow \mathrm{Ext}^1(A/(p^{i+1}))\end{aligned} \tag{2.1}$$

associated to the short exact sequences

$$0 \longrightarrow A \xrightarrow{p^{i+1}} A \longrightarrow A/(p^{i+1}) \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow A/(p^{i+1}) \xrightarrow{v_1^j} A/(p^{i+1}) \longrightarrow A/(p^{i+1}, v_1^j) \longrightarrow 0,$$

respectively. Here we assume that

$$p^i | j.$$

In [9], Miller, Ravenel and Wilson introduced the elements $x_n \in v_2^{-1}BP_*$ defined by

$$\begin{aligned}&x_0 = v_2,\\ &x_1 = v_2^p - v_1^p v_2^{-1} v_3\\ &x_2 = x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3\\ &x_n = x_{n-1}^p - 2v_1^{a_n-p} v_2^{p^n-p^{n-1}+1} \qquad \text{for} \quad v \geqq 3,\end{aligned} \tag{2.2}$$

where $a_n = p^n + p^{n-1} - 1$ for $n > 0$ and $a_0 = 1$, and showed that

$$d_0(x_n) = \varepsilon_n v_1^{a_n} v_2^{p^n - p^{n-1}} t_1 \qquad \text{in} \quad \Omega^1_{BP_*(BP)} v_2^{-1} BP_*/(p, v_1^{1+a_n}) \tag{2.3}$$

for $n > 0$ and $\varepsilon_n = \min\{n, 2\}$. Here

$$d_0 = \eta_R - \eta_L : v_2^{-1}BP_* \longrightarrow \Omega^1_{BP_*(BP)} v_2^{-1} BP_*/(p, v_1^{1+a_n})$$

for the right and the left units η_R and η_L of the Hopf algebroid $BP_*(BP)$. Note that x_n^s belongs to $BP_*/(p^{i+1}, v_1^j)$ if $p^i | j \leqq a_{n-i}$ (*cf.* [3]). In other words, if $p^i | j \leqq a_{n-i}$, $x_n^s \in v_2^{-1}BP_*/(p^{i+1}, v_1^j)$ is pulled back to $BP_*/(p^{i+1}, v_1^j)$ under the localization map $BP_* \hookrightarrow v_2^{-1}BP_*$. Thus we may consider that x_n^s is in $BP_*/(p^{i+1}, v_1^j)$ not in $v_2^{-1}BP_*$, and (2.3) shows

$$x_n^s \in \mathrm{Ext}^0_{BP_*(BP)}(BP_*, BP_*/(p^{i+1}, v_1^j)) \subset BP_*/(p^{i+1}, v_1^j)$$

under the condition, which yields the β-element $\beta_{sp^n/j, i+1}$ as the image under the composition of the connecting homomorphisms associated to the short exact sequences

$$0 \longrightarrow BP_* \xrightarrow{p^{i+1}} BP_* \longrightarrow BP_*/(p^{i+1}) \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow BP_*/(p^{i+1}) \xrightarrow{v_1^j} BP_*/(p^{i+1}) \longrightarrow BP_*/(p^{i+1}, v_1^j) \longrightarrow 0.$$

Considering this condition we have

(2.4) [3, Th. 2.6] *Let* $E_2^{s,t}=\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*)$ *denote the* E_2*-term of the Adams-Novikov spectral sequence for* $\pi_*(S^0)$. *Then* $E_2^{2,*}$ *consists of the* β*-elements* $\beta_{sp^n/j,i+1}$ *with*

$$p\nmid s, \qquad p^i|j\leqq a_{n-i} \text{ and } j\leqq p^{n-i} \text{ if } s=1.$$

In the following, we define the β-elements in the E_2-term $\mathrm{Ext}^*(A)$ of the Adams-Novikov spectral sequence computing $\pi_*(L_2S^0)$. As we have noted above, these β-elements are considered to be homotopy elements. Then β-elements in the $\pi_*(S^0)$ are obtained by pulling back those elements under the localization map $\eta: S^0\to L_2S^0$.

Consider the map $f: v_2^{-1}BP_*\to A$ given by sending v_n to v_n if $n\leqq 2$ and to 0 otherwise. We define the elements x_n in A by sending those in $v_2^{-1}BP_*$ to A under the map f. Actually they are obtained by setting $v_3=0$, and yield the same results (2.3). This with (2.3) implies that

$$x_n^s\in\mathrm{Ext}^0(A/(p^{i+1}, v_1^j)) \qquad \text{for} \quad p^i|j\leqq a_{n-i},$$

and further that

$$x_{n-i}^{sp^{r+i}}\in\mathrm{Ext}^0(A/(p^{i+1}, v_1^j)) \qquad \text{for} \quad p^i|j\leqq a_{n+r-i}.$$

Using these elements, we define the β-elements by

(2.5)
$$\beta'_{sp^{n+r}/j}=\delta_0(x_{n-i}^{sp^{r+i}})\in\mathrm{Ext}^2(A/(p)) \qquad \text{for} \quad 0<j\leqq a_n$$
$$\beta_{sp^{n+r}/j,i+1}=\delta_1\delta_0(x_{n-i}^{sp^{r+i}})\in\mathrm{Ext}^2(A)$$
$$\text{for} \quad p^i|j \text{ with } p^{r+1}a_{n-i-1}<j\leqq p^r a_{n-i}$$

in the E_2-terms of the Adams-Novikov spectral sequences computing $\pi_*(M_p)$ and $\pi_*(S^0)$. Here we notice that β-elements in [3] are defined by using x_n instead of $x_{n-i}^{p^i}$ as we have done here. The subscripts of β-elements are given as follows:

$$\beta_{a/b,c}=\delta_1\delta_0(v_2^a+v_1x)$$

for some $x\in BP_*$ such that

$$v_2^a+v_1x\in\mathrm{Ext}^0(A/(p^c, v_1^b)).$$

Thus our β's are good to be considered. We abbreviate $\beta_{sp^n/j,1}$ to $\beta_{sp^n/j}$, $\beta_{sp^n/1}$ to β_{sp^n} and $\beta'_{sp^n/1}$ to β'_{sp^n} as is our custom.

We end this section by stating the following.

LEMMA 2.6. ([1, Lemma 3.8]) *Let* s, n, r, j *and* i *be integers such that* $p\nmid s>0$, $r>0$, $n>i\geqq 0$, $p^i|j$, $1\leqq j\leqq p^r a_{n-i}$ *and* $r\geqq i$. *Then in* $\mathrm{Ext}^2(A)$, *we have*

$$\beta_{sp^{n+r}/j,i+1} \equiv \begin{cases} -\varepsilon_{n-i} s v_1^{p^r a_{n-i}-j} v_2^{e(s,n+r;i,r)} g_0 \mod (p, v_1^{p^r a_{n-i}-j+1}) \\ \qquad \textit{for even } r, \textit{ and} \\ -\varepsilon_{n-i} s v_1^{p^r a_{n-i}-j} v_2^{e(s,n+r;i,r)} g_1 \mod (p, v_1^{p^r a_{n-i}-j+1}) \\ \qquad \textit{for odd } r. \end{cases}$$

Here g_0 and g_1 are cocycles (cf. [18]) of the cobar complex $\Omega_\Gamma A/(p, v_1)$ as follows:

$$g_0 = v_2^{-p}(t_1 \otimes t_2^p + t_2 \otimes t_1^{p^2}) \quad \textit{and} \quad g_1 = v_2^{-1} g_0^p,$$

and the integers are defined by:

$$\varepsilon_n = \min\{2, n\}, \qquad a_n = p^n + p^{n-1} - 1 \quad \textit{and}$$

$$e(s, n; i, r) = sp^n - p^{n-i-1} + k(r),$$

for $k(r) = (p^n - (-1)^n)/(p+1)$.

§ 3. The map $L_2S^0 \to L_2M_p$

Consider the cofibering $S^0 \xrightarrow{p} S^0 \xrightarrow{i} M_p \xrightarrow{\pi} \Sigma^1 S^0$ defining the mod p Moore spectrum. Then by [11, Th. 2.3.4] the map π induces the map of E_2-terms

(3.1) $$\delta\colon \mathrm{Ext}^s(A/p) \longrightarrow \mathrm{Ext}^{s+1}(A).$$

By definition we have

(3.2) $$\delta(\beta_t') = \beta_t.$$

To study this, we consider Γ-comodules N_j^i and M_j^i introduced in [3]. These are characterized inductively by $N_0^0 = A$, $N_1^0 = A/(p)$, $M_j^i = v_{i+j}^{-1} N_j^i$ and the short exact sequences

(3.3) $$0 \longrightarrow N_j^i \longrightarrow M_j^i \longrightarrow N_j^{i+1} \longrightarrow 0.$$

Note that $M_j^i = N_j^i$ if $i+j=2$. Then by a result of [3], we see that the connecting homomorphisms yield isomorphisms

(3.4) $$\mathrm{Ext}^2(M_1^1) \overset{\delta_1}{\cong} \mathrm{Ext}^3(A/(p)) \quad \text{and} \quad \mathrm{Ext}^2(M_0^2) \overset{\delta_0'}{\cong} \mathrm{Ext}^3(M_0^1) \overset{\delta_0}{\cong} \mathrm{Ext}^4(A).$$

In fact, the first isomorphism follows from the fact $\mathrm{Ext}^s(M_1^0)=0$ for $s>1$ ([3, Th. 3.16]), the second follows from $\mathrm{Ext}^s(M_0^1)=0$ for $s>1$ ([3, Th. 4.2]), and the third from $\mathrm{Ext}^s(M_0^0)=0$ for $s>0$ ([3, Th. 3.16]). Furthermore, note that the isomorphism $\mathrm{Ext}^2(M_0^2) \cong \mathrm{Ext}^3(M_0^1)$ is valid at the internal degree $\neq 0$ by [3, Th. 4.2]. By definition, we have a canonical inclusions $\varphi\colon N_1^i \to N_0^{i+1}$ and $\varphi\colon M_1^i \to M_0^{i+1}$ given by $\varphi(x) = x/p$ in both cases. This gives rise to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_1^0 & \longrightarrow & M_1^0 & \longrightarrow & N_1^1 & \longrightarrow & 0 \\ & & \downarrow\varphi & & \downarrow\varphi & & \downarrow\varphi & & \\ 0 & \longrightarrow & N_0^1 & \longrightarrow & M_0^1 & \longrightarrow & N_0^2 & \longrightarrow & 0, \end{array}$$

in which two rows are the short exact sequences of (3.3). This diagram yields the commutative one

$$\begin{array}{ccc} \mathrm{Ext}^2(M_1^1) & \xrightarrow{\varphi_*} & \mathrm{Ext}^2(M_0^2) \\ \downarrow\delta_1 & & \downarrow\delta_0' \\ \mathrm{Ext}^3(A/p) & \xrightarrow{\varphi_*} & \mathrm{Ext}^3(N_0^1). \end{array}$$

Here note that $N_1^0=A/p$, $N_1^1=M_1^1$ and $N_0^2=M_0^2$. Therefore, the map δ of (3.1) is identified with

(3.5) $$\varphi_*:\ \mathrm{Ext}^2(M_1^1)\longrightarrow\mathrm{Ext}^2(M_0^2).$$

In fact, $\delta=\delta_0\varphi_*=\delta_0\delta_0'\varphi_*\delta_1^{-1}$, and $\delta_0\delta_0'$ and δ_1 are the isomorphisms in (3.4). We also have a short exact sequence

$$0\longrightarrow M_1^1\xrightarrow{\varphi} M_0^2\xrightarrow{p} M_0^2\longrightarrow 0,$$

which induces the exact sequence

(3.6) $$\mathrm{Ext}^1(M_0^2)\xrightarrow{\delta}\mathrm{Ext}^2(M_1^1)\xrightarrow{\varphi_*}\mathrm{Ext}^2(M_0^2).$$

Thus we have

LEMMA 3.7. *The kernel of δ in* (3.1) *is isomorphic to the image of δ in* (3.6).

§4. Proof of Theorem

As in Lemma 2.6, we have the cocycles g_0 and g_1 representing the generators of $\mathrm{Ext}^2(M_2^0)$ given by

$$g_0=v_2^{-p}(t_1\otimes t_2^p+t_2\otimes t_1^{p^2})\quad\text{and}\quad g_1=v_2^{-1}g_0^p.$$

Then in [13], it is shown that $\mathrm{Ext}^2(M_1^1)$ contains $F_p[v_1]$-module

$$G=F_p[v_1]\{v_2^{sp^n-(p^{n-1}-1)/(p-1)}g_1/v_1^{a_n}\mid n\geqq 1,\ p\nmid s+1\}$$
$$\oplus F_p\{v_2^s g_0/v_1\mid p\nmid s+1\}.$$

Here $a_0=1$ and $a_n=p^n+p^{n-1}-1\ (n>0)$. In [20, §9], the F_p-module $G_C=G/((\mathrm{Im}\,\delta)\cap G)$ is given by

$$G_C = F_p\{v_2^{sp^n-(p^{n-1}-1)/(p-1)}g_1/v_1^j \mid n \geqq 1,\ p \nmid s+1$$
$$1 \leqq j \leqq a_n,\ p^{i+1} \nmid j+A_{n-i+1}+1 \text{ for } s=up^i \text{ with } p \nmid u(u+1), \text{ or}$$
$$p^i \nmid j+A_{n-i}+1 \text{ for } s=up^i \text{ with } i>0 \text{ and } p^2 \mid u+1\}$$
$$\oplus F_p\{v_2^s g_0/v_1 \mid p \nmid s+1\}.$$

Here

$$A_n = (p+1)(p^n-1)/(p-1).$$

LEMMA 4.1. *Let a, b and t be positive integers.*

1) *Put* $\beta \equiv v_1^a v_2^b g_0 \mod (p, v_1^{a+1})$ *in the cobar complex* $\Omega_\Gamma A$. *Then,*

$$\beta_t \beta \neq 0$$

if $a=1$ *and* $p \nmid t+b+1$.

2) *Put* $\beta \equiv v_1^a v_2^b g_1 \mod (p, v_1^{a+1})$ *in the cobar complex* $\Omega_\Gamma A$. *Then,*

$$\beta_t \beta \neq 0$$

if $a=1$ *and* $p \mid c$ *and* $p^2 \nmid c+p$, *where* $t+b = cp^l-(p^l-1)/(p-1)$ *with* $p \nmid c+1$ *for some* $l \geqq 0$.

Proof. In the proof of [1, Lemma 4.4], we have seen that $v_2^t\beta/v_1$ is not zero in $\mathrm{Ext}^2(M_1^1)$ if the conditions of 1) or 2) is satisfied. By the assumption, $v_2^t\beta/v_1$ belongs to G and if it satisfies the conditions of 1) or 2), it belongs to G_C. By Lemma 3.7, G_C maps to $\mathrm{Ext}^2(M_0^2)$ monomorphically. Thus, noticing that $\beta_t\beta = \delta_0\delta_0'(v_2^t\beta/v_1)$, we have the non-trivial products. q.e.d.

Proof of Theorem. By Lemma 2.6,

$$\beta_{sp^n+r/p^r a_{n-r},\, i+1} = -\varepsilon_{n-i} s v_2^{sp^n-p^{n-i-1}+k(r)} g_{\varepsilon(r)}$$

for $\varepsilon(r) = (1-(-1)^r)/2$. Now apply Lemma 4.1, and we have Theorem. q.e.d.

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