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A REMARK ON THREE-SHEETED ALGEBROID SURFACES WHOSE PICARD CONSTANTS ARE FIVE

Dedicated to Professor Mitsuru Ozawa on his 70th birthday

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§ 1. Introduction

Let $\mathcal{M}(R)$ be the family of non-constant meromorphic functions on a Riemann surface R, and $P(f)$ be the number of values, which are not taken by $f\in\mathcal{M}(R)$. Then we put

$$P(R)=\sup_{f\in\mathcal{M}(R)} P(f),$$

which is called the Picard constant of R. In general $P(R)\geqq 2$ for every open Riemann surface R.

An n-sheeted algebroid surface is a proper existence domain of an n-valued algebroid function, which is defined by the following equation:

$$S_0(z)y^n-S_1(z)y^{n-1}+\cdots+(-1)^{n-1}S_{n-1}(z)y+(-1)^nS_n(z)=0,$$

where $S_i(z)$ $(i=0, 1, \cdots, n)$ are entire functions having no common zeros, all of which are not polynomials.

By Selberg's theory of algebroid functions [7], $P(R)\leqq 2n$ for every n-sheeted algebroid surface R.

If $S_i(z)/S_0(z)$ (for all i) are of finite order, then we call that the surface is of finite order.

In the following we shall consider 3-sheeted algebroid surfaces, that is, the case of $n=3$, with $S_0(z)\equiv 1$.

In [5] Ozawa and the first author listed up all of the 3-sheeted algebroid surfaces with $P(y)=5$ and showed that "if R is of finite order" their Picard constants are equal to 5 with three exceptional cases.

In this paper we shall prove that the assumption:

"R is of finite order"

can be taken off in the above result.

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Let R be a Riemann surface defined by

$$y^3-S_1(z)y^2+S_2(z)y-S_3(z)=0\,, \tag{1}$$

where $S_i(z)$ ($i=1, 2, 3$) are entire functions. If $P(y)=5$, then R must be one of the following three cases ([5]):

CASE (i)
$$\begin{cases} S_1(z)=y_1 \\ S_2(z)=y_0e^H+y_2 \\ S_3(z)=y_3 \end{cases}$$

where y_0 ($\neq 0$), y_1, y_2 and y_3 ($\neq 0$) are constants and H is a non-constant entire function with $H(0)=0$. Furthermore its discriminant D_R is

$$D_R=4y_0^3e^{3H}+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0\,, \tag{2}$$

where ζ_2, ζ_1, ζ_0 ($\neq 0$) are constants, which are suitable polynomials of y_1, y_2 and y_3.

CASE (ii)
$$\begin{cases} S_1(z)=y_0e^H+y_1 \\ S_2(z)=y_2 \\ S_3(z)=y_3 \end{cases}$$

where y_0 ($\neq 0$), y_1, y_2 and y_3 ($\neq 0$) are constants and H is a non-constant entire function with $H(0)=0$. Furthermore its discriminant D_R is

$$D_R=4y_0^3e^{3H}+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0\,,$$

where ζ_2, ζ_1, ζ_0 ($\neq 0$) are constants, which are suitable polynomials of y_1, y_2 and y_3.

CASE (iii)
$$\begin{cases} S_1(z)=2a_3+y_0e^H \\ S_2(z)=a_3^2+(a_1+a_2)y_0e^H \\ S_3(z)=a_1a_2y_0e^H \end{cases}$$

where y_0, a_1, a_2 and a_3 are non-zero constants and H is a non-constant entire function with $H(0)=0$. Furthermore its discriminant D_R is

$$D_R=y_0e^H(\zeta_3y_0^3e^{3H}+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0) \tag{3}$$

where ζ_3 ($\neq 0$), ζ_2, ζ_1 and ζ_0 ($\neq 0$) are constants, which are suitable polynomials of a_1, a_2 and a_3.

Now we prove the following

THEOREM. *Let R be the surface defined by* (1) *with $P(y)=5$, that is, one of the above three cases. If either $\zeta_2\neq 0$ or $\zeta_1\neq 0$ then $P(R)=5$.*

We assume that the reader is familiar with the Nevanlinna-Selberg theory of meromorphic functions and algebroid functions and notations, $T(r, f)$, $m(r, f)$, $N(r, 0, f)$, $N(r, \infty, f)$, $S(r, f)$ etc (cf. [1], [3], [7]).

§2. Transformation formula of discriminants

Let R be the surface defined by (1) with $P(y)=5$. We suppose that there exists an entire function f on R such that $P(f)=6$. Then we have the following

LEMMA 1 ([5]). *The above function f is representable as*

$$f=f_0+f_1y+f_2y^2, \tag{4}$$

where f_0, f_1 and f_2 are meromorphic functions on $\boldsymbol{C}$ having poles at most at zeros of H'.

Then f satisfies the following equation ([5]):

$$f^3-U_1(z)f^2+U_2(z)f-U_3(z)=0,$$

where $U_i(z)$ $(i=1, 2, 3)$ are entire functions, which can be expressed by f_0, f_1, f_2, S_1, S_2 and S_3. We denote the proper existence domain of f by $\boldsymbol{S}$. Then Ozawa and the first author proved that the discriminant D_S of $\boldsymbol{S}$ is ([5]):

$$D_S=-b_1^2x_0^4e^{4L}+\eta_3x_0^3e^{3L}+\eta_2x_0^2e^{2L}+\eta_1x_0e^L+\eta_0 \tag{5}$$

where b_1 $(\neq 0)$, x_0 $(\neq 0)$, η_3, η_2, η_1, η_0 $(\neq 0)$ are constants and L is a non-constant entire function with $L(0)=0$.

Furthermore the following equation holds (transformation formula of discriminants):

$$D_S=D_R\cdot G^2, \tag{6}$$

where

$$G=f_1^3+2S_1f_1^2f_2+(S_1^2+S_2)f_1f_2^2+(S_1S_2-S_3)f_2^3. \tag{7}$$

To prove the theorem, we need further investigation of the counting functions of poles and zeros of G.

§3. Poles and zeros of G

First of all, we consider the counting function of poles of G. We prove the following

LEMMA 2.

(8) $$N(r, \infty, G) \leqq 3N(r, 0, H') = O(\log T(r, e^H) + \log r)$$

as $r \to \infty$, $r \notin E$ (E: *of finite linear measure*)

Proof. Let z_0 be a pole of G of order p. Since D_R and D_S are entire functions on $\boldsymbol{C}$, z_0 is a zero of D_R by (6). On the other hand D_R has the following form:

$$D_R = 4y_0^3(e^H - \delta_1)(e^H - \delta_2)(e^H - \delta_3),$$

if R is the surface of the case of (i), (ii),

and

$$D_R = \zeta_3 y_0^4 e^H (e^H - \delta_1)(e^H - \delta_2)(e^H - \delta_3),$$

if R is the surface of the case of (iii).

Hence we may assume that z_0 is a zero of $e^H - \delta_1$. By the expression (7) of G, z_0 is a pole of either f_1 or f_2. Therefore z_0 is a zero of H' by Lemma 1. This suggests

$$\bar{N}(r, \infty, G) \leqq \bar{N}(r, 0, H').$$

And if z_0 is a zero of H' of order n, then

$$H(z) = H(z_0) + (z - z_0)^{n+1} h(z), \qquad h(z_0) \neq 0$$

and

$$e^H - \delta_1 = e^{H(z_0)}\{e^{(z-z_0)^{n+1}h(z)} - 1\}$$

$$= (z - z_0)^{n+1} e^{H(z_0)} h(z_0)(1 + o(1)).$$

Therefore D_R has the zero z_0 of order at most $3(n+1)$ by the form of D_R. Hence we have, by (6)

$$3(n+1) - 2p \geqq 0$$

and

$$p \leqq \frac{3}{2}(n+1) \leqq 3n.$$

Therefore

$$N(r, \infty, G) \leqq 3N(r, 0, H') = O(\log T(r, e^H) + \log r).$$

In the last estimation we need "the estimation of logarithmic derivatives".

Next we consider the counting function of zeros of G. We have

LEMMA 3. *The function y, defined by* (1) *with* $P(y) = 5$, *is an entire function on* $\boldsymbol{S}$ *and has the following form*:

(9) $$y = g_0 + g_1 f + g_2 f^2$$

where g_i ($i = 0, 1, 2$) *are meromorphic functions on* $\boldsymbol{C}$ *and every zero of* G *is a* *pole of at least one of* g_i $(i=0, 1, 2)$.

Proof. By (1) and (4)

$$\begin{aligned} f^2 &= (f_0+f_1y+f_2y^2)^2 \\ &= f_0^2+2f_1f_2S_3+f_2^2S_1S_3 \\ &\quad +[2f_0f_1-2f_1f_2S_2+(S_3-S_1S_2)f_2^2]y \\ &\quad +[f_1^2+2f_0f_2+2f_1f_2S_1+(S_1^2-S_2)f_2^2]y^2 \\ &= F_0+F_1y+F_2y^2 . \end{aligned}$$

Hence

$$\begin{bmatrix} 1 \\ f \\ f^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ f_0 & f_1 & f_2 \\ F_0 & F_1 & F_2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ y \\ y^2 \end{bmatrix} \equiv A \cdot \begin{bmatrix} 1 \\ y \\ y^2 \end{bmatrix} .$$

Then the determinant of A is

$$\begin{aligned} \det(A) &= f_1F_2-f_2F_1 \\ &= f_1^3+2f_1^2f_2S_1+(S_1^2+S_2)f_1f_2^2+(S_1S_2-S_3)f_2^3 . \end{aligned}$$

Therefore $\det(A)$ coincides the function G and $\det(A)$ $(=G)$ is not identically equal to zero, by (6). Then

$$y=g_0+g_1f+g_2f^2 ,$$

where

$$\text{(10)} \qquad \begin{cases} g_0=\dfrac{1}{G}(-f_0F_2+f_2F_0) \\ g_1=\dfrac{1}{G}F_2 \\ g_2=-\dfrac{1}{G}f_2 . \end{cases}$$

These expressions of g_i $(i=0, 1, 2)$ hold for all $z\in \boldsymbol{C}$, since every zero of G is at most a pole of g_i $(i=0, 1, 2)$.

Let us consider the expressions (10) more precisely. We assume that every zero of G is a regular point of g_i $(i=0, 1, 2)$.

Let z_0 be a zero of G of order n. By $g_2=-(1/G)f_2$, z_0 is a zero of f_2. Then let m_2 be the order of the zero z_0 of f_2. And by (7) z_0 is a zero of f_1, too. Therefore the order of the zero z_0 of f_1 is denoted by m_1. We may assume that $m_2 \geqq n \geqq 1$ and $m_1 \geqq 1$.

CASE 1). We assume that $m_1 \geqq m_2$. Then by

$$G=f_1^3+2f_1^2f_2S_1+(S_1^2+S_2)f_1f_2^2+(S_1S_2-S_3)f_2^3,$$

the order of the z_0 of G is $\geqq 3m_2>n$. This is a contradiction.

CASE 2). Next we assume that $m_1<m_2$. Then the order of the zero z_0 of G is equal to $3m_1$ $(=n\leqq m_2)$.

On the other hand, if f_0 is regular at z_0, then by

$$F_2=f_1^2+2f_0f_2+2f_1f_2S_1+(S_1^2-S_2)f_2^2,$$

the order of the zero z_0 of F_2 is equal to $2m_1$ $(<n)$. This contradicts the regularity of $g_1=F_2/G$ at z_0.

Therefore we assume that z_0 is a pole of f_0 of order p $(\geqq 1)$. Let us denote the order of the zero z_0 of F_2 by $\operatorname{ord}(z_0, F_2)$, then

$$\operatorname{ord}(z_0, F_2)=m_2-p<2m_1<n, \qquad \text{if } m_2-p<2m_1$$

and

$$\operatorname{ord}(z_0, F_2)=2m_1<n, \qquad \text{if } m_2-p>2m_1.$$

These contradict our assumption that g_1 is regular at z_0.

Therefore we consider the case that $m_2-p=2m_1$, that is, $p=m_2-2m_1$ $(m_2>2m_1)$. By the form of F_2,

$$f_1^2+2f_0f_2=A_k(z-z_0)^k+\cdots \ (k\geqq n),$$

since we have

$$\operatorname{ord}(z_0, f_1^2)=2m_1<n,$$

$$\operatorname{ord}(z_0, 2f_0f_2)=m_2-p=2m_1<n$$

and

$$\operatorname{ord}\{z_0, 2f_1f_2S_1+(S_1^2-S_2)f_2^2\}\geqq m_1+m_2>3m_1=n.$$

On the other hand by

$$\begin{aligned}-f_0F_2+f_2F_0=&-f_0\{-f_0f_2+(2f_0f_2+f_1^2)\}\\ &-\{2f_1f_2S_1+(S_1^2-S_2)f_2\}f_0\\ &+f_2^2(2f_1S_3+f_2S_1S_3),\end{aligned}$$

we have

$$\operatorname{ord}(z_0, -f_0F_2+f_2F_0)=m_2-p-p=m_2-2p<n.$$

This contradicts the regularity of $g_0=(-fF_2+f_2F_0)/G$ at z_0.

Above lemma suggests that R is conformally equivalent to $\boldsymbol{S}$. if $P(R)=6$. Next we need the following

LEMMA 4 ([5]). *Let g be an entire function on $\boldsymbol{S}$. Then g must have the following form:*

$$g=g_0+g_1f+g_2f^2,$$

where g_i $(i=0, 1, 2)$ *are meromorphic functions on* C *having poles at most at zeros of* L'. *(*L *is the function in the equation* (5).)

Therefore every zero of G must be a zero of L' by Lemma 3 and Lemma 4, that is,

$$\bar{N}(r, 0, G) \leqq \bar{N}(r, 0, L').$$

Now let z_0 be a zero of G of order q, then z_0 is a zero of D_S by (6).

On the other hand D_S has the form:

$$D_S=-b_1^2x_0^4(e^L-\xi_1)(e^L-\xi_2)(e^L-\xi_3)(e^L-\xi_4).$$

Therefore we may assume that z_0 is a zero of $e^L-\xi_1$. Further if z_0 is a zero of L' of order n, then

$$e^L-\xi_1=(z-z_0)^{n+1}e^{L(z_0)}l(z_0)(1+o(1)),$$

where $l(z)$ is an entire function with $l(z_0)\neq 0$. Hence the order of the zero z_0 of D_S is at most $4(n+1)$ and by (6) we have

$$4(n+1) \geqq 2q$$

and

$$q \leqq 2(n+1) \leqq 4n.$$

Therefore, using the estimation of logarithmic derivatives, we obtain the following

LEMMA 5.

(11) $$N(r, 0, G) \leqq 4N(r, 0, L')=O(\log T(r, e^L)+\log r)$$

as $r\to\infty$, $r\notin E$ *(*E *is a set of* r *of finite linear measure.)*

Then considering the counting functions of zeros of both sides of (6), we have

(12) $$4T(r, e^L)\sim N(r, 0, D_S)\sim N(r, 0, D_R)\sim 3T(r, e^H),$$

by (8) and (11).

§4. Representation of G in terms of its zeros and poles

We need the following

LEMMA 6 ([2]). *Let* $\{a_{\nu,\mu}\}$ *be* n *sequences* $(1\leqq\mu\leqq n)$ *of complex numbers satisfying* $1\leqq|a_{1,\mu}|\leqq|a_{2,\mu}|\leqq\cdots$, $\lim_{\nu\to\infty}|a_{\nu,\mu}|=+\infty$ *for each* μ.

Then we can construct Weierstrass products P_μ of the $\{a_{\nu,\mu}\}$ $(1\leqq\mu\leqq n)$ with the following property: There exists a set Ω in $[1, \infty)$ of infinite linear measure such that

$$\frac{\sum_{\mu=1}^{n} \log^+ m(r, P_\mu)}{\sum_{\mu=1}^{n} N(r, 0, P_\mu)} \longrightarrow 0 \qquad \text{as } r\to+\infty,\ r\in\Omega\,.$$

Therefore we can construct the Weierstrass products P, Q of zeros and poles of G, respectively, satisfying

$$\lim_{\substack{r\to\infty \\ (r\in\Omega)}} \frac{\log^+ m(r, P)+\log^+ m(r, Q)}{N(r, 0, P)+N(r, 0, Q)}=0\,, \tag{13}$$

where Ω is a set of r of infinite linear measure.

Here if G has a finite number of zeros and poles, then P and Q are polynomials. In this case the formulation of Theorem returns back to the case in [5], hence we may assume that G has an infinite number of zeros or poles.

Now G must have the following form:

$$G=\frac{P}{Q}e^M\,,$$

where M is an entire function with $M(0)=0$. And then the transformation formula of discriminants (6) becomes

$$D_S=D_R\cdot\frac{P^2}{Q^2}e^{2M}\,, \tag{6)'$$

where

$$D_S=-b_1^2x_0^4e^{4L}+\eta_3x_0^3e^{3L}+\eta_2x_0^2e^{2L}+\eta_1x_0e^L+\eta_0$$

and

$$D_R=4y_0^3e^{3H}+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0 \quad \text{in the case of (i), (ii)},$$

$$D_R=y_0e^H(\zeta_3y_0^3e^{3H}+\zeta_2y_0^2e^{2H}+\zeta_1y_0e^H+\zeta_0) \quad \text{in the case of (iii)}.$$

For simplicity's sake we put the above equation (6)′ by

$$\begin{aligned} &A_4e^{4L}+A_3e^{3L}+A_2e^{2L}+A_1e^L+A_0 \\ &=(B_3e^{3H}+B_2e^{2H}+B_1e^H+B_0)\frac{P^2}{Q^2}e^{2M+\delta H}\,, \end{aligned} \tag{14}$$

where $\delta=0$ in the case of (i), (ii) and $\delta=1$ in the case of (iii), and $A_4A_0B_3B_0\neq0$.

In the following we consider only the case of $\delta=0$. The same discussion holds for the case of $\delta=1$.

§ 5. Borel's unicity theorem

We need the following

LEMMA 7 ([8]). *Let* $a_0(z), a_1(z), \cdots, a_n(z)$ $(a_\nu(z)\not\equiv 0,\ \nu\geqq 1)$ *be meromorphic functions and* $g_1(z), g_2(z), \cdots, g_n(z)$ *be non-constant entire functions satisfying*

$$\lim_{\substack{r\to\infty \\ (r\in\Omega)}} \frac{\sum_{\nu=1}^{n}\log T(r, a_\nu)}{\operatorname{Max}_{1\leqq\nu\leqq n}\log m(r, e^{g_\nu})}\leqq 1,$$

where Ω *is a set of infinite linear measure. Suppose that the identity*

$$\sum_{\nu=1}^{n} a_\nu(z)e^{g_\nu(z)}\equiv a_0(z)$$

holds, then there exist n constants $(c_1, \cdots, c_n)\neq(0, \cdots, 0)$ *such that*

$$\sum_{\nu=1}^{n} c_\nu a_\nu(z)e^{g_\nu(z)}\equiv 0 .$$

First of all we assume that $3H+2M\not\equiv$constant, $2H+2M\not\equiv$constant, $H+2M\not\equiv$constant and $2M\not\equiv$constant. Then by Lemma 7 and (13), (14), there exist eight constants $(c_8, \cdots, c_1)\neq(0, \cdots, 0)$ such that

$$\begin{aligned}(15)\qquad & c_8A_4e^{4L}+c_7A_3e^{3L}+c_6A_2e^{2L}+c_5A_1e^{L}\\ &=(c_4B_3e^{3H}+c_3B_2e^{2H}+c_2B_1e^{H}+c_1B_0)\frac{P^2}{Q_2}e^{2M} .\end{aligned}$$

Here without loss of generality we may set the following condition (A):

A: $c_j=0$ $(j=7, 6, 5, 4, 3$ and $2)$,
if the corresponding coefficients A_i $(i=3, 2, 1)$,
B_k $(k=2, 1)$ are equal to 0.

By eliminating $(P^2/Q^2)e^{2M}$ from (14) and (15),

$$\begin{aligned}&(A_4e^{4L}+A_3e^{3L}+A_2e^{2L}+A_1e^{L}+A_0)(c_4B_3e^{3H}+c_3B_2e^{2H}+c_2B_1e^{H}+c_1B_0)\\ &=(c_8A_4e^{4L}+c_7A_3e^{3L}+c_6A_2e^{2L}+c_5A_1e^{L})(B_3e^{3H}+B_2e^{2H}+B_1e^{H}+B_0)\end{aligned}$$

and

$$\begin{aligned}(16)\qquad & (c_4-c_8)A_4B_3e^{4L+3H}+(c_4-c_7)A_3B_3e^{3L+3H}+(c_4-c_6)A_2B_3e^{2L+3H}\\ &+(c_4-c_5)A_1B_3e^{L+3H}+c_4A_0B_3e^{3H}+\cdots+(c_1-c_8)A_4B_0e^{4L}+(c_1-c_7)A_3B_0e^{3L}\\ &+(c_1-c_6)A_2B_0e^{2L}+(c_1-c_5)A_1B_0e^{L}+c_1A_0B_0=0 .\end{aligned}$$

Here we need the following

LEMMA 8 ([4]). *Let $\{\alpha_j\}$ be a set of non-zero constants and $\{g_j\}$ a set of functions satisfying*

$$\sum_{j=1}^{p} \alpha_j g_j = 1 .$$

Then

$$\sum_{j=1}^{p} \delta(0, g_j) \leqq p-1 ,$$

where $\delta(0, g_j)$ denotes the Nevanlinna-deficiency.

By the above lemma and (16), there exist only two possibilities:

CASE (1) all coefficients of left side of (16) vanish,

and

CASE (2) there exist two integers n, m ($n\cdot m\neq 0$) such that

$$n\cdot L+m\cdot H\equiv 0 . \qquad \text{(Since } L(0)=0 \text{ and } H(0)=0.)$$

If the case of (2) holds, we have $4L\equiv\pm 3H$ because of (12). Hence let us consider the case of (1). Then we have

$$c_1=c_8 \text{ by } A_4B_0\neq 0 , \qquad c_4=c_8 \text{ by } A_4B_3\neq 0$$

and

$$c_1=0 \text{ by } A_0B_0\neq 0 .$$

Therefore we have $c_1=c_4=c_8=0$. In this case, by (15), we have

$$e^L(c_7A_3e^{2L}+c_6A_2e^L+c_5A_1)=e^H(c_3B_2e^H+c_2B_1)\frac{P^2}{Q^2}e^{2M} . \tag{17}$$

If $c_3B_2\cdot c_2B_1\neq 0$ then

$$N\left(r, 0, e^H(c_3B_2e^H+c_2B_1)\frac{P^2}{Q^2}e^{2M}\right)\sim T(r, e^H) \qquad (r\in E) .$$

On the other hand

$$N(r, 0, e^L(c_7A_3e^{2L}+c_6A_2e^L+c_5A_1))\sim\begin{cases} 2T(r, e^L) \\ T(r, e^L) \\ \equiv 0 . \end{cases}$$

This contradicts (12), whence follows either $c_3B_2=0$ or $c_2B_1=0$. In this case, by (17), we have either

$$N(r, 0, e^L(c_7A_3e^{2L}+c_6A_2e^L+c_5A_1))\equiv 0 ,$$

$$\text{if } c_3B_2=0,\ c_2B_1\neq 0 \text{ or } c_3B_2\neq 0,\ c_2B_1=0 ,$$

or

$$e^L(c_7A_3e^{2L}+c_6A_2e^L+c_5A_1)\equiv 0\,, \qquad \text{if } c_3B_2=c_2B_1=0\,.$$

In the former case we have that just one of c_7A_3, c_6A_2, c_5A_1 is not equal to 0 and the others vanish. Then we have $P^2/Q^2\equiv$constant and, by Lemma 8 and (14), we have at least one of $3H+2M$, $2H+2M$, $H+2M$, $2M$ must be a constant. In the latter case we have $c_3B_2=c_2B_1=c_7A_3=c_6A_2=c_5A_1=0$ and $c_1=c_4=c_8=0$. This contradicts the condition (A).

Therefore we consider the cases:

(i) $3H+2M\equiv 0$, (ii) $2H+2M\equiv 0$, (iii) $H+2M\equiv 0$ and (iv) $2M\equiv 0$.

CASE (i). $2M=-3H$. Then (14) reduces to

$$\frac{P^2}{Q^2}(B_0e^{-3H}+B_1e^{-2H}+B_1e^{-H})-(A_4e^{4L}+A_3e^{3L}+A_2e^{2L}+A_1e^L)$$
$$=A_0-B_3\frac{P^2}{Q^2}\,.$$

By Lemma 7, there exist seven constants $(c_1, \cdots, c_7)\neq(0, \cdots, 0)$ such that

$$\frac{P^2}{Q^2}e^{-H}(c_1B_0e^{-2H}+c_2B_1e^{-H}+c_3B_2)=e^L(c_4A_4e^{3L}+c_5A_3e^{2L}+c_6A_2e^L+c_7A_1)\,.$$

Then

$$N\left(r,\,0,\,\frac{P^2}{Q^2}e^{-H}(c_1B_0e^{-2H}+c_2B_1e^{-H}+c_3B_2)\right)\sim\begin{cases}2T(r,\,e^H)\\ T(r,\,e^H)\\ S(r,\,e^H),\end{cases}$$

and

$$N(r,\,0,\,e^L(c_4A_4e^{3L}+c_5A_3e^{2L}+c_6A_2e^L+c_7A_1))\sim\begin{cases}3T(r,\,e^L)\\ 2T(r,\,e^L)\\ T(r,\,e^L)\\ S(r,\,e^L)\,.\end{cases}$$

By (12) there exists only one possibility:

$$N\left(r,\,0,\,\frac{P^2}{Q^2}e^{-H}(c_1B_0e^{-2H}+c_2B_1e^{-H}+c_3B_2)\right)=S(r,\,e^H)$$

and

$$N(r,\,0,\,e^L(c_4A_4e^{3L}+c_5A_3e^{2L}+c_6A_2e^L+c_7A_1))=S(r,\,e^L)\,.$$

Here we may assume that

$$(c_1B_0,\,c_2B_1,\,c_3B_2,\,c_4A_4,\,c_5A_3,\,c_6A_2,\,c_7A_1)\neq(0,\,0,\,\cdots,\,0)$$

because of the same reason as in the condition (A). Therefore we have only one possibility:

just two constants of c_1B_0, c_2B_1, c_3B_2 are equal to 0

and

just three constants of c_4A_4, c_5A_3, c_6A_2, c_7A_1 are equal to 0.

Hence, by (12), we have

$$\frac{P^2}{Q^2}\equiv\text{constant}=d\ (\neq 0),\ \text{say},$$

$$c_1B_0\neq 0,\quad c_2B_1=c_3B_2=0,\quad c_4A_4\neq 0,\quad c_5A_3=c_6A_2=c_7A_1=0$$

and

$$4L\equiv -3H\,.$$

Then (14) reduces to

$$d(B_0e^{-3H}+B_1e^{-2H}+B_2e^{-H})-(A_4e^{-3H}+A_3e^{-(9/4)H}+A_2e^{-(3/2)H}+A_1e^{-(3/4)H})$$
$$=A_0-B_3d$$

and

$$(dB_0-A_4)e^{-3H}+dB_1e^{-2H}+dB_2e^{-H}-A_3e^{-(9/4)H}-A_2e^{-(3/2)H}-A_1e^{-(3/4)H}$$
$$=A_0-B_3d\,.$$

By Lemma 8 and $H\not\equiv$constant, we have

$$dB_0-A_4=dB_1=dB_2=A_3=A_2=A_1=A_0-B_3d=0\,,$$

that is,

$$A_4/B_0=A_0/B_3=d\ (\neq 0)\quad\text{and}\quad A_1=A_2=A_3=B_1=B_2=0\,.$$

CASE (ii). $2M=-2H$. Then (14) becomes

$$\text{(18)}\qquad \frac{P^2}{Q^2}(B_0e^{-2H}+B_1e^{-H}+B_3e^{H})-(A_4e^{4L}+A_3e^{3L}+A_2e^{2L}+A_1e^{L})$$
$$=A_0-B_2\frac{P^2}{Q^2}\,.$$

By H, $L\not\equiv$constant and Lemma 7, there exist seven constants $(c_1,\cdots,c_7)\neq(0,\cdots,0)$ such that

$$\frac{P^2}{Q^2}e^{-2H}(c_3B_3e^{3H}+c_2B_1e^{H}+c_1B_0)=e^{L}(c_4A_4e^{3L}+c_5A_3e^{2L}+c_6A_2e^{L}+c_7A_1)\,.$$

Hence we have

$$N\Big(r,0,\frac{P^2}{Q^2}e^{-2H}(c_3B_3e^{3H}+c_2B_1e^{H}+c_1B_0)\Big)\sim\begin{cases}3T(r,e^H)\\2T(r,e^H)\\T(r,e^H)\\S(r,e^H)\end{cases}$$

and

$$N(r, 0, e^L(c_4A_4e^{3L}+c_5A_3e^{2L}+c_6A_2e^L+c_7A_1))\sim\begin{cases} 3T(r, e^L) \\ 2T(r, e^L) \\ T(r, e^L) \\ S(r, e^L). \end{cases}$$

By (12) we have

$$N\left(r, 0, \frac{P^2}{Q^2}e^{-2H}(c_3B_3e^{3H}+c_2B_1e^H+c_1B_0)\right)=S(r, e^H)$$

and

$$N(r, 0, e^L(c_4A_4e^{3L}+c_5A_3e^{2L}+c_6A_2e^L+c_7A_1))=S(r, e^L).$$

Therefore we have

just two constants of c_3B_3, c_2B_1, c_1B_0 are equal to 0

and

just three constants of c_4A_4, c_5A_3, c_6A_2, c_7A_1 are equal to 0.

In this case we have $P^2/Q^2=d$ $(\neq 0)$ and (18) gives

$$T(r, e^H)\sim m\cdot T(r, e^L)$$

with $m=4, 3, 2, 3/2, 1, 1/2$. This contradicts (12).

CASE (iii). $2M=-H$. Then we have, by Lemma 7,

$$\frac{P^2}{Q^2}e^{-H}(c_1B_0+c_2B_2e^{2H}+c_3B_3e^{3H})=e^L(c_4A_4e^{3L}+c_5A_3e^{2L}+c_6A_2e^L+c_7A_1).$$

This gives us a contradiction by the same method as in the case of (ii).

CASE (iv). $M\equiv 0$. Then we have, by Lemma 7,

$$\frac{P^2}{Q^2}e^H(c_3B_3e^{2H}+c_2B_2e^H+c_1B_1)=e^L(c_4A_4e^{3L}+c_5A_3e^{2L}+c_6A_2e^L+c_7A_1)$$

with $(c_1, c_2, \cdots, c_7)\neq(0, 0, \cdots, 0)$. Then, by the same method as in the case of (i), there is only one possibility:

$$c_2B_2=c_1B_1=0, \qquad c_5A_3=c_6A_2=c_7A_1=0$$

and

$$\frac{P^2}{Q^2}\equiv\text{constant}=d, \quad \text{say}, \quad (d\neq 0), \quad 3H=4L.$$

Therefore, by the same method as in the case of (i), we have

$$A_4/B_3=A_0/B_0=d \quad \text{and} \quad A_1=A_2=A_3=B_1=B_2=0.$$

§ 6. Proof of Theorem

Let R be the surface defined by (1) with $P(y)=5$. If $P(R)=6$, then there exists an entire function f on R of $P(f)=6$. Then f defines a 3-sheeted algebroid surface with $P(f)=6$, S, say, and further the following equation

$$D_S=D_R\cdot G^2$$

holds. By the result in § 5, we have $\zeta_2=0$ and $\zeta_1=0$ in the equation (2), (3). This result contradicts our assumption:

$$\text{either } \zeta_1\neq 0 \quad \text{or} \quad \zeta_2\neq 0 .$$

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