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PICARD CONSTANTS OF FOUR-SHEETED ALGEBROID SURFACES, I

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§ 1. Introduction

The notion of Picard constant of a Riemann surface R was introduced in [2]. Let $\mathcal{M}(R)$ be the class of non-constant meromorphic functions on R. Let $P(f)$ be the number of values which are not taken by f in $\mathcal{M}(R)$. Now we put

$$P(R)=\sup\{P(f)\,;\, f\in\mathcal{M}(R)\}.$$

This $P(R)$ is evidently a conformal invariant of R and is called the Picard constant of R. If R is open, then $P(R)\geqq 2$. If R is an n-sheeted algebroid surface, which is the proper existence domain of an n-valued algebroid function, then $P(R)\leqq 2n$ by Selberg's theory of algebroid functions [6]. In general it is very difficult to decide $P(R)$ of a given open Riemann surface R.

In our previous paper [4] we discussed the following problem: Is there any method to prove $P(R)=5$ for a three-sheeted algebroid surface R, which is defined by

$$y^3-S_1y^2+S_2y-S_3=0$$

with $P(y)=5$? Its discriminant is denoted by Δ. Then Δ has the following form: either

$$A_3y_0{}^3e^{3H}+\zeta_2y_0{}^2e^{2H}+\zeta_1y_0e^H+A_0$$

or

$$y_0e^H(A_3y_0{}^3e^{3H}+\zeta_2y_0{}^2e^{2H}+\zeta_1y_0e^H+A_0)$$

with non-zero constants A_0, A_3. Then we have the following result: If either $\zeta_2\neq 0$ or $\zeta_1\neq 0$, then $P(R)=5$ under an additional condition that H is a polynomial.

In this paper we consider a similar problem for a four-sheeted algebroid surface R, which is defined by

$$y^4-S_1y^3+S_2y^2-S_3y+S_4=0$$

with $P(y)=7$. Is there any method to prove $P(R)=7$ then? Again the discri-

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minant Δ of R plays a role firstly. We need quite hard computation in order to determine the form of Δ. In a subsequent paper II with the same title we shall consider a similar problem for four-sheeted algebroid surfaces R with $P(y)=6$.

§2. Surfaces with $P(R)=8$

Let us consider

$$F(z, y)\equiv y^4-S_1y^3+S_2y^2-S_3y+S_4=0\,,$$

which defines a four-sheeted algebroid surface R. Consider

$$\begin{pmatrix} F(z,0) \\ F(z,a_1) \\ F(z,a_2) \\ F(z,a_3) \\ F(z,a_4) \\ F(z,a_5) \\ F(z,a_6) \end{pmatrix} \overset{\text{(i)}}{=} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \\ \beta_4 e^{H_4} \end{pmatrix}, \quad \overset{\text{(ii)}}{=} \begin{pmatrix} \beta_1 e^{H_1} \\ c_1 \\ c_2 \\ c_3 \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \\ \beta_4 e^{H_4} \end{pmatrix},$$

where c_j, β_j are non-zero constants and H_j are non-constant entire functions satisfying $H_j(0)=0$.

CASE (i). Then $S_4=c_1$ and

$$a_1^4-a_1^3S_1+a_1^2S_2-a_1S_3+c_1=c_2\,,$$

$$a_2^4-a_2^3S_1+a_2^2S_2-a_2S_3+c_1=c_3\,,$$

$$a_3^4-a_3^3S_1+a_3^2S_2-a_3S_3+c_1=\beta_1e^{H_1}\,,$$

$$a_4^4-a_4^3S_1+a_4^2S_2-a_4S_3+c_1=\beta_2e^{H_2}\,,$$

$$a_5^4-a_5^3S_1+a_5^2S_2-a_5S_3+c_1=\beta_3e^{H_3}\,,$$

$$a_6^4-a_6^3S_1+a_6^2S_2-a_6S_3+c_1=\beta_4e^{H_4}\,.$$

From the first three equations we have

$$S_1=x_0e^{H_1}+x_1-x_2+x_3+a_1+a_2+a_3,$$

$$S_2=(a_1+a_2)x_0e^{H_1}+(a_1+a_2+a_3)x_1-(a_2+a_3)x_2$$

$$+(a_1+a_3)x_3+a_1a_2+a_1a_3+a_2a_3,$$

$$S_3=a_1a_2x_0e^{H_1}+(a_1a_2+a_1a_3+a_2a_3)x_1-a_2a_3x_2+a_1a_3x_3+a_1a_2a_3,$$

$$S_4=c_1=a_1a_2a_3x_1,$$

where

$$x_0a_3(a_1-a_3)(a_2-a_3)=-\beta_1, \qquad x_1a_1a_2a_3=c_1,$$

$$x_2a_1(a_1-a_2)(a_1-a_3)=c_2, \qquad x_3a_2(a_1-a_2)(a_2-a_3)=c_3.$$

Substituting these into the remaining three equations and making use of Borel's unicity theorem [1], [3] we have

$$H_1=H_2=H_3=H_4\ (\equiv H), \qquad \beta_2=-a_4(a_4-a_1)(a_4-a_2)x_0.$$

Hence we have finally

$$\frac{\beta_1}{a_3(a_1-a_3)(a_2-a_3)}=\frac{\beta_2}{a_4(a_1-a_4)(a_2-a_4)}=\frac{\beta_3}{a_5(a_1-a_5)(a_2-a_5)}$$
$$=\frac{\beta_4}{a_6(a_1-a_6)(a_2-a_6)}$$

and

$$\frac{x_1}{a_4}-\frac{x_2}{a_4-a_1}+\frac{x_3}{a_4-a_2}=1,$$

$$\frac{x_1}{a_5}-\frac{x_2}{a_5-a_1}+\frac{x_3}{a_5-a_2}=1,$$

$$\frac{x_1}{a_6}-\frac{x_2}{a_6-a_1}+\frac{x_3}{a_6-a_2}=1.$$

Then we have

$$x_1=\frac{a_4a_5a_6}{a_1a_2}, \qquad x_2=\frac{(a_4-a_1)(a_5-a_1)(a_6-a_1)}{a_1(a_2-a_1)},$$

$$x_3=\frac{(a_4-a_2)(a_5-a_2)(a_6-a_2)}{a_1(a_2-a_1)}.$$

Further $x_1-x_2+x_3=a_4+a_5+a_6-a_1-a_2$. Therefore

$$\begin{cases} S_1=x_0e^H+a_3+a_4+a_5+a_6 \\ S_2=(a_1+a_2)x_0e^H+a_3a_4+a_3a_5+a_3a_6+a_4a_5+a_4a_6+a_5a_6, \\ S_3=a_1a_2x_0e^H+a_3a_4a_5+a_3a_4a_6+a_3a_5a_6+a_4a_5a_6, \\ S_4=c_1=a_3a_4a_5a_6. \end{cases}$$

We denote this surface by X_1.

CASE (ii). Then $S_4=\beta_1e^{H_1}$ and

$$a_1^4 - a_1^3 S_1 + a_1^2 S_2 - a_1 S_3 + \beta_1 e^{H_1} = c_1,$$

$$a_2^4 - a_2^3 S_1 + a_2^2 S_2 - a_2 S_3 + \beta_1 e^{H_1} = c_2,$$

$$a_3^4 - a_3^3 S_1 + a_3^2 S_2 - a_3 S_3 + \beta_1 e^{H_1} = c_3,$$

$$a_4^4 - a_4^3 S_1 + a_4^2 S_2 - a_4 S_3 + \beta_1 e^{H_1} = \beta_2 e^{H_2},$$

$$a_5^4 - a_5^3 S_1 + a_5^2 S_2 - a_5 S_3 + \beta_1 e^{H_1} = \beta_3 e^{H_3},$$

$$a_6^4 - a_6^3 S_1 + a_6^2 S_2 - a_6 S_3 + \beta_1 e^{H_1} = \beta_4 e^{H_4}.$$

By the first three equations we have

$$S_1 = x_0 e^{H_1} - x_1 + x_2 - x_3 + a_1 + a_2 + a_3,$$

$$S_2 = (a_1 + a_2 + a_3) x_0 e^{H_1} - (a_2 + a_3) x_1 + (a_1 + a_3) x_2$$
$$- (a_1 + a_2) x_3 + a_1 a_2 + a_1 a_3 + a_2 a_3,$$

$$S_3 = (a_1 a_2 + a_1 a_3 + a_2 a_3) x_0 e^{H_1} - a_2 a_3 x_1 + a_1 a_3 x_2 - a_1 a_2 x_3 + a_1 a_2 a_3,$$

$$S_4 = \beta_1 e^{H_1} = a_1 a_2 a_3 x_0 e^{H_1}$$

with

$$x_0 = \frac{\beta_1}{a_1 a_2 a_3}, \quad x_1 = \frac{c_1}{a_1(a_1 - a_2)(a_1 - a_3)}, \quad x_2 = \frac{c_2}{a_2(a_1 - a_2)(a_2 - a_3)},$$

$$x_3 = \frac{c_3}{a_3(a_1 - a_3)(a_2 - a_3)}.$$

Substituting these into the remaining three equations and making use of Borel's unicity theorem we have

$$\beta_2 = -(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) x_0,$$

$$\beta_3 = -(a_5 - a_1)(a_5 - a_2)(a_5 - a_3) x_0,$$

$$\beta_4 = -(a_6 - a_1)(a_6 - a_2)(a_6 - a_3) x_0$$

and $H_1 = H_2 = H_3 = H_4$ $(\equiv H)$,

$$\frac{x_1}{a_j - a_1} - \frac{x_2}{a_j - a_2} + \frac{x_3}{a_j - a_3} + 1 = 0, \quad j = 4, 5, 6.$$

Then

$$x_1(a_2 - a_1)(a_3 - a_1) = -(a_4 - a_1)(a_5 - a_1)(a_6 - a_1),$$

$$x_2(a_2 - a_1)(a_3 - a_2) = -(a_4 - a_2)(a_5 - a_2)(a_6 - a_2),$$

$$x_3(a_3 - a_1)(a_3 - a_2) = -(a_4 - a_3)(a_5 - a_3)(a_6 - a_3).$$

Further

$$-x_1+x_2-x_3+a_1+a_2+a_3=a_4+a_5+a_6,$$

$$-(a_2+a_3)x_1+(a_1+a_3)x_2-(a_1+a_2)x_3$$
$$+a_1a_2+a_1a_3+a_2a_3=a_4a_5+a_4a_6+a_5a_6,$$

$$-a_2a_3x_1+a_1a_3x_2-a_1a_2x_3+a_1a_2a_3=a_4a_5a_6.$$

Hence we have

$$\begin{cases} S_1=x_0e^H+a_4+a_5+a_6, \\ S_2=(a_1+a_2+a_3)x_0e^H+a_4a_5+a_4a_6+a_5a_6, \\ S_3=(a_1a_2+a_1a_3+a_2a_3)x_0e^H+a_4a_5a_6, \\ S_4=a_1a_2a_3x_0e^H. \end{cases}$$

We denote this surface by X_2. If e^H is commonly used, then X_1 and X_2 are conformally equivalent by a suitable linear transformation $Y=\alpha y+\beta$. See the end of § 4.

§ 3. Discriminant of X_1

Let $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ define the surface X_1. Now we abbreviate S_j in the following manner: $S_1=X+x_1$, $S_2=(a_1+a_2)X+x_2$, $S_3=a_1a_2X+x_3$, $S_4=x_4$ with $X=x_0e^H$, $x_1=a_3+a_4+a_5+a_6$, $x_2=a_3a_4+a_3a_5+a_3a_6+a_4a_5+a_4a_6+a_5a_6$, $x_3=a_3a_4a_5+a_3a_4a_6+a_3a_5a_6+a_4a_5a_6$, $x_4=a_3a_4a_5a_6$. Let us put

$$L=-\frac{3}{8}S_1^2+S_2,$$

$$M=-\frac{1}{8}S_1^3+\frac{1}{2}S_1S_2-S_3,$$

$$N=-\frac{3}{256}S_1^4+\frac{1}{16}S_1^2S_2-\frac{1}{4}S_1S_3+S_4.$$

Then the discriminant D of X_1 is

$$-27M^4+144LM^2N-128L^2N^2+256N^3-4L^3M^2+16L^4N.$$

In this case we have

$$L=-\left(\frac{3}{8}X^2+\alpha_1X+\alpha_2\right),$$

$$M=-\left(\frac{1}{8}X^3+\beta_1X^2+\beta_2X+\beta_3\right),$$

$$N=-\left(\frac{3}{256}X^4+\gamma_1X^3+\gamma_2X^2+\gamma_3X+\gamma_4\right)$$

with

$$\alpha_1=\frac{3}{4}x_1-a_1-a_2, \qquad \alpha_2=\frac{3}{8}x_1^2-x_2,$$

$$\beta_1=\frac{3}{8}x_1-\frac{1}{2}(a_1+a_2). \qquad \beta_2=\frac{3}{8}x_1^2-\frac{1}{2}x_1(a_1+a_2)-\frac{1}{2}x_2+a_1a_2,$$

$$\beta_3=\frac{1}{8}x_1^3-\frac{1}{2}x_1x_2+x_3,$$

$$\gamma_1=\frac{3}{64}x_1-\frac{1}{16}(a_1+a_2), \qquad \gamma_2=\frac{9}{128}x_1^2-\frac{1}{8}x_1(a_1+a_2)-\frac{1}{16}x_2+\frac{1}{4}a_1a_2,$$

$$\gamma_3=\frac{3}{64}x_1^3-\frac{1}{8}x_1x_2-\frac{1}{16}x_1^2(a_1+a_2)+\frac{1}{4}a_1a_2x_1+\frac{1}{4}x_3,$$

$$\gamma_4=\frac{3}{256}x_1^3-\frac{1}{16}x_1^2x_2+\frac{1}{4}x_1x_3-x_4.$$

In this case we have $2\beta_1=16\gamma_1=\alpha_1$ and $\alpha_2=4\beta_2-16\gamma_2$. Then D looks like a polynomial of X of twelve degree at a glance but it reduces really to a polynomial of six degree. In order to prove this we need somewhat hard computation. It is comparatively easy to prove that coefficients of X^{12}, X^{11}, X^{10} are equal to zero. And the coefficient of X^9 is equal to the following expression:

$$-27\left[\frac{\beta_3}{64\cdot 2}+\frac{3}{16}\beta_1\beta_2+\frac{1}{2}\beta_1^3\right]$$

$$+144\left[\left(\frac{3}{8}\gamma_3+4\gamma_1\beta_2\right)\frac{1}{64}+\left(\frac{3}{32}\gamma_1\beta_2+\frac{3}{8}\gamma_1\gamma_2+32\gamma_1^3\right)\right.$$

$$\left.+\frac{9}{16}\gamma_1\left(\frac{1}{4}\beta_2+64\gamma_1^2\right)+\frac{9}{64\cdot 32}\left(\frac{1}{4}\beta_3+16\gamma_1\beta_2\right)\right]$$

$$-128\left[\frac{9}{64\cdot 16}\left(\frac{3}{8}\gamma_3+4\gamma_1\beta_2\right)+\frac{9}{8}\gamma_1\left(\frac{3}{64}\beta_2+\frac{3}{16}\gamma_2+16\gamma_1^2\right)\right]$$

$$-256\left[\frac{27}{64\cdot 64\cdot 16}\gamma_3+\frac{9}{64\cdot 2}\gamma_1\gamma_2+\gamma_1^3\right]$$

$$+4\left[\frac{\gamma_1}{4}(9\beta_2-36\gamma_2+256\gamma_1^2)+\left(\frac{27}{8}\gamma_1\beta_2-\frac{27}{2}\gamma_1\gamma_2+9\cdot 64\gamma_1^3\right)\right.$$

$$\left.+\frac{27}{4}\gamma_1\left(\frac{1}{4}\beta_2+64\gamma_1^2\right)+\frac{27}{64\cdot 8}\left(\frac{1}{4}\beta_3+16\gamma_1\beta_2\right)\right]$$

$$-16\left[\frac{81}{64}\gamma_1(\beta_2-4\gamma_2)+9\cdot 8\gamma_1^3+\frac{27}{32}(\beta_2-4\gamma_2)\gamma_1\right.$$

$$\left.+27\cdot 8\gamma_1^3+\frac{27}{8}\gamma_1\gamma_2+\frac{81}{64\cdot 64}\gamma_3\right].$$

All the coefficients of β_3, γ_3, $\gamma_1\beta_2$, $\gamma_1\gamma_2$ and γ_1^3 reduce to zero. Hence the coefficient of X^9 is equal to zero. Next the coefficient of X^8 has the following expression:

$$-27\Big[\frac{3}{16}\beta_1\beta_3+\frac{3}{32}\beta_2^2+\frac{3}{2}\beta_1^2\beta_2+\beta_1^4\Big]$$

$$+144\Big[\frac{1}{64}\Big(\frac{3}{8}\gamma_4+16\gamma_1\gamma_3+4\beta_2\gamma_2-16\gamma_2^2\Big)+\frac{3}{4}\gamma_1\gamma_3+8\gamma_1^2\beta_2$$

$$+\Big(\frac{3}{64}\beta_2+\frac{3}{16}\gamma_2+16\gamma_1^2\Big)\Big(\frac{1}{4}\beta_2+64\gamma_1^2\Big)+\frac{9}{16}\gamma_1\Big(\frac{1}{4}\beta_3+16\gamma_1\beta_2\Big)$$

$$+\frac{9}{64\cdot 32}(16\gamma_1\beta_3+\beta_2^2)\Big]$$

$$-128\Big[\frac{9}{64\cdot 16}\Big(\frac{3}{8}\gamma_4+16\gamma_1\gamma_3+4\beta_2\gamma_2-16\gamma_2^2\Big)+\frac{9}{8}\gamma_1\Big(\frac{3}{8}\gamma_3+4\gamma_1\beta_2\Big)$$

$$+\Big(\frac{3}{64}\beta_2+\frac{3}{16}\gamma_2+16\gamma_1^2\Big)^2\Big]$$

$$-256\Big[\frac{27}{64\cdot 64\cdot 16}\gamma_4+\frac{9}{64\cdot 4}(2\gamma_1\gamma_3+\gamma_2^2)+3\gamma_1^2\gamma_2\Big]$$

$$+4\Big[\frac{9}{32}(\beta_2-4\gamma_2)^2+3\cdot 16(\beta_2-4\gamma_2)\gamma_1^2+32\gamma_1^2(9\beta_2-36\gamma_2+256\gamma_1^2)$$

$$+\Big(\frac{27}{16}\beta_2-\frac{27}{4}\gamma_2+9\cdot 32\gamma_1^2\Big)\Big(\frac{1}{4}\beta_2+64\gamma_1^2\Big)$$

$$+\frac{27}{4}\gamma_1\Big(\frac{1}{4}\beta_3+16\gamma_1\beta_2\Big)+\frac{27}{64\cdot 8}(16\gamma_1\beta_3+\beta_2^2)\Big]$$

$$-16\Big[\frac{81}{512}(\beta_2-4\gamma_2)^2+27\cdot 2(\beta_2-4\gamma_2)\gamma_1^2+3\cdot 256\gamma_1^4+27\cdot 4(\beta_2-4\gamma_2)\gamma_1^2$$

$$+3\cdot 256\cdot 8\gamma_1^4+\frac{27}{32}(\beta_2-4\gamma_2)\gamma_2+27\cdot 8\gamma_1^2\gamma_2+\frac{27}{8}\gamma_1\gamma_3+\frac{81}{64\cdot 64}\gamma_4\Big].$$

Then all the coefficients of γ_4, $\gamma_1\beta_3$, $\gamma_1\gamma_3$, β_2^2, $\beta_2\gamma_2$, γ_2^2, $\gamma_1^2\beta_2$, $\gamma_1^2\gamma_2$ and γ_1^4 vanish. Hence the coefficient of X^8 reduces to zero. Next we consider the coefficient of X^7, which has the following expression:

$$-27\Big[\frac{3}{16}\beta_2\beta_3+\frac{3}{2}\beta_1^2\beta_3+\frac{3}{2}\beta_1\beta_2^2+4\beta_1^3\beta_2\Big]$$

$$+144\Big[\frac{1}{4}\gamma_1\gamma_4+\frac{1}{16}\beta_2\gamma_3-\frac{1}{4}\gamma_2\gamma_3+\frac{3}{4}\gamma_1\gamma_4+32\gamma_1^2\gamma_3+8\gamma_1\beta_2\gamma_2-32\gamma_1\gamma_2^2$$

$$+\left(\frac{3}{8}\gamma_3+4\gamma_1\beta_2\right)\left(\frac{1}{4}\beta_2+64\gamma_1^2\right)+\left(\frac{3}{64}\beta_2+\frac{3}{16}\gamma_2+16\gamma_1^2\right)\left(\frac{1}{4}\beta_3+16\gamma_1\beta_2\right)$$

$$+\frac{9}{16}\gamma_1(16\gamma_1\beta_3+\beta_2^2)+\frac{9}{64\cdot 16}\beta_2\beta_3\Big]$$

$$-128\Big[\frac{9}{64}\gamma_1\gamma_4+\frac{9}{256}\beta_2\gamma_3-\frac{9}{64}\gamma_2\gamma_3+\frac{9}{8}\gamma_1\left(\frac{3}{8}\gamma_4+16\gamma_1\gamma_3+4\beta_2\gamma_2-16\gamma_2^2\right)$$

$$+\left(\frac{3}{32}\beta_2+\frac{3}{8}\gamma_2+32\gamma_1^2\right)\left(\frac{3}{8}\gamma_3+4\gamma_1\beta_2\right)\Big]$$

$$-256\Big[\frac{9}{128}(\gamma_1\gamma_4+\gamma_2\gamma_3)+3\gamma_1^2\gamma_3+3\gamma_1\gamma_2^2\Big]$$

$$+4\Big[12\gamma_1(\beta_2-4\gamma_2)^2+36(\beta_2-4\gamma_2)^2\gamma_1+3\cdot 256\cdot 8(\beta_2-4\gamma_2)\gamma_1^3$$

$$+16\gamma_1(9\beta_2-36\gamma_2+256\gamma_1^2)\left(\frac{1}{4}\beta_2+64\gamma_1^2\right)+\frac{27}{4}\gamma_1(16\gamma_1\beta_3+\beta_2^2)$$

$$+\left(\frac{27}{16}\beta_2-\frac{27}{4}\gamma_2+9\cdot 32\gamma_1^2\right)\left(\frac{1}{4}\beta_3+16\gamma_1\beta_2\right)+\frac{27}{64\cdot 4}\beta_2\beta_3\Big]$$

$$-16\Big[\frac{27}{2}\gamma_1(\beta_2-4\gamma_2)^2+3\cdot 256\gamma_1^3(\beta_2-4\gamma_2)+\frac{27}{2}\gamma_1(\beta_2-4\gamma_2)^2+16^4\gamma_1^5$$

$$+9\cdot 512\gamma_1^3(\beta_2-4\gamma_2)+27\cdot 4\gamma_1\gamma_2(\beta_2-4\gamma_2)+3\cdot 256\cdot 8\gamma_1^3\gamma_2$$

$$+\frac{27}{32}(\beta_2-4\gamma_2)\gamma_3+27\cdot 8\gamma_1^2\gamma_3+\frac{27}{8}\gamma_1\gamma_4\Big].$$

Then all the coefficients of $\gamma_1\gamma_4$, $\beta_2\beta_3$, $\gamma_2\beta_3$, $\beta_2\gamma_3$, $\gamma_2\gamma_3$, $\gamma_1^2\beta_3$, $\gamma_1^2\gamma_3$, $\gamma_1\beta_2^2$, $\gamma_1\beta_2\gamma_2$, $\gamma_1\gamma_2^2$, $\gamma_1^3\beta_2$, $\gamma_1^3\gamma_2$ and γ_1^5 vanish and hence the coefficient of X^7 reduces to zero. We did not use any speciality of γ_4, β_3, γ_3, β_2, γ_2, α_2 and α_1, β_1, γ_1 excepting $2\beta_1=\alpha_1=16\gamma_1$, $\alpha_2=4\beta_2-16\gamma_2$ in order to prove that the degree of D is six. Anyway we have

$$D=A_6x_0^6e^{6H}+A_5x_0^5e^{5H}+A_4x_0^4e^{4H}+A_3x_0^3e^{3H}+A_2x_0^2e^{2H}+A_1x_0e^{H}+A_0$$

with non-zero coefficients A_0 and A_6. Why $A_0\neq 0$, $A_6\neq 0$?

Suppose $A_6=0$. Then firstly $4T(r, y)=(1+o(1))T(r, e^H)$ for X_1. Now by Ullrich-Selberg's ramification theorem or exactly speaking, by an analogue of the proof of Ullrich-Selberg's ramification theorem [6], [7] we have

$$4N(r, X_1)\leqq N(r, 0, D)\leqq 5(1+o(1))T(r, e^H).$$

Hence

$$N(r, X_1)\leqq 5(1+o(1))T(r, y),$$

$$\varepsilon=\varlimsup_{r\to\infty}\frac{N(r, X_1)}{T(r, y)}\leqq 5\,.$$

Therefore Selberg's deficiency relation [6] gives

$$\sum\delta(w_\nu)\leqq 2+\varepsilon\leqq 7\,,$$

where $\delta(w_\nu)$ is Nevanlinna-Selberg's deficiency at w_ν of y. We have just 8 lacunary values of y for X_1. Thus we have $\sum\delta(w_\nu)=8$. This is a contradiction. Similarly $A_0=0$ gives the same contradiction. By the way we give an explicit form of the coefficients of X^6 and X^5:

The coefficient of X^6 is just the following form:

$$\begin{aligned}
&-27\Big[\frac{3}{32}\beta_3^2+3\beta_1\beta_2\beta_3+4\beta_1^3\beta_3+\frac{1}{2}\beta_2^3+6\beta_1^2\beta_2^2\Big]\\
&+144\Big[\frac{9}{256\cdot 8}\beta_3^2+\Big(\frac{3}{256}\alpha_1+\frac{3}{8}\gamma_1\Big)2\beta_2\beta_3+\Big(\frac{3}{256}\alpha_2+\alpha_1\gamma_1+\frac{3}{8}\gamma_2\Big)(2\beta_1\beta_3+\beta_2^2)\\
&\quad+\Big(\alpha_2\gamma_1+\alpha_1\gamma_2+\frac{3}{8}\gamma_3\Big)\Big(\frac{1}{4}\beta_3+2\beta_1\beta_2\Big)+\Big(\alpha_2\gamma_2+\alpha_1\gamma_3+\frac{3}{8}\gamma_4\Big)\Big(\frac{1}{4}\beta_2+\beta_1^2\Big)\\
&\quad+(\alpha_2\gamma_3+\alpha_1\gamma_4)\frac{1}{4}\beta_1+\frac{1}{64}\alpha_2\gamma_4\Big]\\
&-128\Big[\frac{9}{256\cdot 4}\alpha_2\gamma_4+(\alpha_1\gamma_4+\alpha_2\gamma_3)\Big(\frac{3}{128}\alpha_1+\frac{3}{4}\gamma_1\Big)+\Big(\frac{3}{8}\gamma_3+\alpha_1\gamma_2+\alpha_2\gamma_1\Big)^2\\
&\quad+2\Big(\alpha_2\gamma_2+\alpha_1\gamma_3+\frac{3}{8}\gamma_4\Big)\Big(\frac{3}{256}\alpha_2+\alpha_1\gamma_1+\frac{3}{8}\gamma_2\Big)\Big]\\
&-256\Big[\Big(\frac{6}{256}\gamma_2+\gamma_1^2\Big)\gamma_4+\Big(\frac{6}{256}\gamma_3+2\gamma_1\gamma_2\Big)\gamma_3+\Big(\frac{6}{256}\gamma_4+2\gamma_1\gamma_3+\gamma_2^2\Big)\gamma_2\\
&\quad+(2\gamma_1\gamma_4+2\gamma_2\gamma_3)\gamma_1+\frac{6}{256}\gamma_2\gamma_4+\frac{3}{256}\gamma_3^2\Big]\\
&+4\Big[\frac{1}{64}\alpha_2^3+\frac{3}{4}\alpha_1\beta_1\alpha_2^2+\Big(\frac{9}{8}\alpha_2^2+3\alpha_1^2\alpha_2\Big)\Big(\frac{1}{4}\beta_2+\beta_1^2\Big)+\frac{27}{64}\cdot 2\alpha_1\beta_2\beta_3\\
&\quad+\Big(\frac{9}{4}\alpha_1\alpha_2+\alpha_1^3\Big)\Big(\frac{1}{4}\beta_3+2\beta_1\beta_2\Big)+\Big(\frac{27}{64}\alpha_2+\frac{9}{8}\alpha_1^2\Big)(2\beta_1\beta_3+\beta_2^2)+\frac{27}{64\cdot 8}\beta_3^2\Big]\\
&-16\Big[\Big(\frac{27}{64\cdot 2}\alpha_2+\frac{27}{32}\alpha_1^2\Big)\gamma_4+\Big(\frac{27}{16}\alpha_1\alpha_2+\frac{3}{2}\alpha_1^3\Big)\gamma_3+\Big(\frac{27}{32}\alpha_2^2+\frac{9}{2}\alpha_1^2\alpha_2+\alpha_1^4\Big)\gamma_2\\
&\quad+\Big(\frac{9}{2}\alpha_1\alpha_2^2+4\alpha_1^3\alpha_2\Big)\gamma_1+\Big(6\alpha_1^2\alpha_2^2+\frac{3}{2}\alpha_2^3\Big)\frac{3}{256}\Big],
\end{aligned}$$

which is equal to

$$-\frac{27}{16}(\beta_3-4\gamma_3)^2+\frac{9}{2}\alpha_1(\beta_2-8\gamma_2)(\beta_3-4\gamma_3)+\alpha_1^3(\beta_3-4\gamma_3)$$
$$+4(\beta_2-8\gamma_2)^3+\alpha_1^2(\beta_2-8\gamma_2)^2 .$$

The coefficients of γ_4 vanish in this case.

Next we consider the cofficient of X^5. The following form of the coefficient of X^5 is used in II (not in I).

$$-27\Big[\frac{3}{2}\beta_1\beta_3^2+\frac{3}{2}\beta_2^2\beta_3+12\beta_1^2\beta_2\beta_3+4\beta_1\beta_2^3\Big]$$
$$+144\Big[\frac{1}{4}\beta_1\alpha_2\gamma_4+\Big(\frac{\beta_2}{4}+\beta_1^2\Big)(\alpha_1\gamma_4+\alpha_2\gamma_3)+\Big(\frac{1}{4}\beta_3+2\beta_1\beta_2\Big)\Big(\alpha_2\gamma_2+\alpha_1\gamma_3+\frac{3}{8}\gamma_4\Big)$$
$$+(2\beta_1\beta_3+\beta_2^2)\Big(\alpha_2\gamma_1+\alpha_1\gamma_2+\frac{3}{8}\gamma_3\Big)+2\beta_2\beta_3\Big(\frac{3}{256}\alpha_2+\alpha_1\gamma_1+\frac{3}{8}\gamma_2\Big)$$
$$+\beta_3^2\Big(\frac{3}{256}\alpha_1+\frac{3}{8}\gamma_1\Big)\Big]$$
$$-128\Big[\Big(\frac{3}{128}\alpha_1+\frac{3}{4}\gamma_1\Big)\alpha_2\gamma_4+(\alpha_1\gamma_4+\alpha_2\gamma_3)\Big(\frac{3}{128}\alpha_2+2\alpha_1\gamma_1+\frac{3}{4}\gamma_2\Big)$$
$$+2\Big(\alpha_2\gamma_2+\alpha_1\gamma_3+\frac{3}{8}\gamma_4\Big)\Big(\alpha_2\gamma_1+\alpha_1\gamma_2+\frac{3}{8}\gamma_3\Big)\Big]$$
$$-256\Big[\frac{9}{128}\gamma_3\gamma_4+6\gamma_1\gamma_2\gamma_4+3\gamma_1\gamma_3^2+3\gamma_2^2\gamma_3\Big]$$
$$+4\Big[\frac{27}{64}\alpha_1\beta_3^2+\Big(\frac{27}{64}\alpha_2+\frac{9}{8}\alpha_1^2\Big)2\beta_2\beta_3+\Big(\frac{9}{4}\alpha_1\alpha_2+\alpha_1^3\Big)(2\beta_1\beta_3+\beta_2^2)$$
$$+\Big(\frac{9}{8}\alpha_2^2+3\alpha_1^2\alpha_2\Big)\Big(\frac{1}{4}\beta_3+2\beta_1\beta_2\Big)+3\alpha_1\alpha_2^2\Big(\frac{1}{4}\beta_2+\beta_1^2\Big)+\frac{1}{4}\alpha_2^3\beta_1\Big]$$
$$-16\Big[\Big(\frac{27}{16}\alpha_1\alpha_2+\frac{3}{2}\alpha_1^3\Big)\gamma_4+\Big(\frac{27}{32}\alpha_2^2+\frac{9}{2}\alpha_1^2\alpha_2+\alpha_1^4\Big)\gamma_3$$
$$+\Big(\frac{9}{2}\alpha_1\alpha_2^2+4\alpha_1^3\alpha_2\Big)\gamma_2+\Big(\frac{3}{2}\alpha_2^3+6\alpha_1^2\alpha_2^2\Big)\gamma_1+\frac{3}{64}\alpha_1\alpha_2^3\Big].$$

This is equal to the following expression:

$$\frac{27}{2}(\beta_3-4\gamma_3)\gamma_4-9\cdot 2\alpha_1(\beta_2-8\gamma_2)\gamma_4-4\alpha_1^3\gamma_4$$
$$-\frac{9}{2}\alpha_1(3\beta_3-8\gamma_3)(\beta_3-4\gamma_3)+\frac{9}{2}(\beta_2+8\gamma_2)(\beta_2-8\gamma_2)\beta_3$$
$$-6(11\beta_2-40\gamma_2)(\beta_2-8\gamma_2)\gamma_3$$

$$+30\alpha_1^2\beta_2\beta_3-24\cdot 8\alpha_1^2\gamma_2\beta_3-32\cdot 4\alpha_1^2\beta_2\gamma_3+26\cdot 32\alpha_1^2\gamma_2\gamma_3+4\alpha_1^4(\beta_3-4\gamma_3)$$
$$+2\alpha_1(\beta_2-8\gamma_2)^2(13\beta_2-88\gamma_2)+4\alpha_1^3(\beta_2-8\gamma_2)^2 .$$

These expressions shall play an important role later.

§4. Surfaces with $P(y)=7$

Let us consider

$$F(z,\ y)\equiv y^4-S_1y^3+S_2y^2-S_3y+S_4$$

and the following equations

$$\begin{pmatrix} F(z,0) \\ F(z,b_1) \\ F(z,b_2) \\ F(z,b_3) \\ F(z,b_4) \\ F(z,b_5) \end{pmatrix} \overset{\text{(i)}}{=} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \beta_1e^{H_1} \\ \beta_2e^{H_2} \\ \beta_3e^{H_3} \end{pmatrix},\ \overset{\text{(ii)}}{=} \begin{pmatrix} c_1 \\ c_2 \\ \beta_1e^{H_1} \\ \beta_2e^{H_2} \\ \beta_3e^{H_3} \\ \beta_4e^{H_4} \end{pmatrix},\ \overset{\text{(iii)}}{=} \begin{pmatrix} \beta_1e^{H_1} \\ c_1 \\ c_2 \\ c_3 \\ \beta_2e^{H_2} \\ \beta_3e^{H_3} \end{pmatrix},\ \overset{\text{(iv)}}{=} \begin{pmatrix} \beta_1e^{H_1} \\ c_1 \\ c_2 \\ \beta_2e^{H_2} \\ \beta_3e^{H_3} \\ \beta_4e^{H_4} \end{pmatrix},$$

where c_j and β_j are non-zero constants and H_j are non-constant entire functions satisfying $H_j(0)=0$.

CASE (i). We have $S_4=c_1$ and

$$\begin{cases} b_1^4-S_1b_1^3+S_2b_1^2-S_3b_1+c_1=c_2 , \\ b_2^4-S_1b_2^3+S_2b_2^2-S_3b_2+c_1=c_3 , \\ b_3^4-S_1b_3^3+S_2b_3^2-S_3b_3+c_1=\beta_1e^{H_1} , \\ b_4^4-S_1b_4^3+S_2b_4^2-S_3b_4+c_1=\beta_2e^{H_2} , \\ b_5^4-S_1b_5^3+S_2b_5^2-S_3b_5+c_1=\beta_3e^{H_3} . \end{cases}$$

Then by the first three equations

$$S_1=x+y+z+b_1+b_2+b_3+x_0e^{H_1} ,$$
$$S_2=(b_1+b_2+b_3)x+(b_2+b_3)y+(b_1+b_3)z+b_1b_2+b_2b_3+b_1b_3+(b_1+b_2)x_0e^{H_1},$$
$$S_3=(b_1b_2+b_2b_3+b_1b_3)x+b_2b_3y+b_1b_3z+b_1b_2b_3+b_1b_2x_0e^{H_1}$$

with

$$xb_1b_2b_3=c_1 , \qquad yb_1(b_1-b_2)(b_3-b_1)=c_2 ,$$

$$zb_2(b_1-b_2)(b_2-b_3)=c_3\,,\qquad x_0b_3(b_2-b_3)(b_3-b_1)=\beta_1\,.$$

Substituting these into two remaining equations we have by Borel's unicity theorem

$$H_2=H_3=H_1\ (\equiv H),\qquad \beta_2=-x_0b_4(b_4-b_1)(b_4-b_2)\,,$$

$$\beta_3=-x_0b_5(b_5-b_1)(b_5-b_2)\,,$$

$$\frac{x}{b_4}+\frac{y}{b_4-b_1}+\frac{z}{b_4-b_2}=1$$

and

$$\frac{x}{b_5}+\frac{y}{b_5-b_1}+\frac{z}{b_5-b_2}=1\,.$$

Now we impose the following condition: y does not have any other lacunary value, that is, excepting b_3, b_4, b_5 there is no lacunary value of the second kind. Hence

$$F(z,\,\alpha)=(\alpha-b_3)\{\alpha^3-(b_1+b_2+x+y+z)\alpha^2$$
$$+(b_1b_2+(b_1+b_2)x+b_2y+b_1z)\alpha-b_1b_2x\}$$
$$-\alpha(\alpha-b_1)(\alpha-b_2)x_0e^H$$

should be one of the following three forms:

$$(1)\quad (\alpha-b_3)^2(\alpha-b_4)(\alpha-b_5)-\alpha(\alpha-b_1)(\alpha-b_2)x_0e^H\,,$$

$$(2)\quad (\alpha-b_3)(\alpha-b_4)^2(\alpha-b_5)-\alpha(\alpha-b_1)(\alpha-b_2)x_0e^H\,,$$

$$(3)\quad (\alpha-b_3)(\alpha-b_4)(\alpha-b_5)^2-\alpha(\alpha-b_1)(\alpha-b_2)x_0e^H\,.$$

CASE (1). Then

$$\alpha^3-(b_1+b_2+x+y+z)\alpha^2+(b_1b_2+(b_1+b_2)x+b_2y+b_1z)\alpha-b_1b_2x$$
$$=\alpha^3-(b_3+b_4+b_5)\alpha^2+(b_3b_4+b_3b_5+b_4b_5)\alpha-b_3b_4b_5\,.$$

Hence

$$\begin{cases} b_1+b_2+x+y+z=b_3+b_4+b_5\,,\\ b_1b_2+(b_1+b_2)x+b_2y+b_1z=b_3b_4+b_3b_5+b_4b_5\,,\\ b_1b_2x=b_3b_4b_5\,.\end{cases}$$

Therefore

$$x=\frac{b_3b_4b_5}{b_1b_2},\quad y=\frac{(b_1-b_3)(b_1-b_4)(b_1-b_5)}{b_1(b_2-b_1)}\,,$$

$$z=\frac{-(b\ \ -b_3)(b_2-b_4)(b_2-b_5)}{b_2(b_2-b_1)}\,.$$

Then

$$c_1=xb_1b_2b_3=b_3{}^2b_4b_5\,,$$

$$c_2=yb_1(b_1-b_2)(b_3-b_1)=(b_3-b_1)^2(b_1-b_4)(b_1-b_5)\,,$$

$$c_3=zb_2(b_1-b_2)(b_2-b_3)=(b_2-b_3)^2(b_2-b_4)(b_2-b_5)\,.$$

Thus we have

$$\begin{cases} S_1=x_0e^H+2b_3+b_4+b_5\,, \\ S_2=(b_1+b_2)x_0e^H+b_3{}^2+2b_3b_4+2b_3b_5+b_4b_5\,, \\ S_3=b_1b_2x_0e^H+b_3{}^2b_4+b_3{}^2b_5+2b_3b_4b_5\,, \\ S_4=c_1=b_3{}^2b_4b_5\,. \end{cases}$$

We denote the surface $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ with the above S_1, S_2, S_3, S_4 by $R_1{}^*$.

CASE (2). Then

$$\alpha^3-(b_1+b_2+x+y+z)\alpha^2+(b_1b_2+(b_1+b_2)x+b_2y+b_1z)\alpha-b_1b_2x$$
$$=\alpha^3-(2b_4+b_5)\alpha^2+(b_4{}^2+2b_4b_5)\alpha-b_4{}^2b_5\,.$$

Hence

$$\begin{cases} b_1+b_2+x+y+z=2b_4+b_5\,, \\ (b_1+b_2)x+b_2y+b_1z+b_1b_2=b_4{}^2+2b_4b_5\,, \\ b_1b_2x=b_4{}^2b_5\,. \end{cases}$$

Then

$$x=\frac{b_4{}^2b_5}{b_1b_2}, \quad y=\frac{(b_1-b_4)^2(b_1-b_5)}{b_1(b_2-b_1)}, \quad z=\frac{(b_2-b_4)^2(b_2-b_5)}{b_2(b_1-b_2)}$$

and

$$c_1=b_3b_4{}^2b_5\,, \qquad c_2=(b_1-b_3)(b_1-b_4)^2(b_1-b_5)\,,$$

$$c_3=(b_2-b_3)(b_2-b_4)^2(b_2-b_5)\,.$$

Thus we have

$$\begin{cases} S_1=x_0e^H+b_3+2b_4+b_5\,, \\ S_2=(b_1+b_2)x_0e^H+b_4{}^2+2b_4b_5+2b_3b_4+b_3b_5\,, \\ S_3=b_1b_2x_0e^H+b_4{}^2b_5+b_3b^{\ 2}+2b_3b_4b_5\,, \\ S_4=c_1=b_3b_4{}^2b_5\,. \end{cases}$$

We denote the surface $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ with the above S_1, S_2, S_3, S_4 by $R_2{}^*$.

CASE (3). Then

$$\alpha^3-(b_1+b_2+x+y+z)\alpha^2+(b_1b_2+(b_1+b_2)x+b_2y+b_1z)\alpha-b_1b_2x$$
$$=\alpha^3-(b_4+2b_5)\alpha^2+(2b_4b_5+b_5^2)\alpha-b_4b_5^2\,.$$

Similarly we have

$$\begin{cases} S_1=b_3+b_4+2b_5+x_0e^H\,, \\ S_2=(b_1+b_2)x_0e^H+2b_3b_5+b_3b_4+2b_4b_5+b_5^2\,, \\ S_3=b_1b_2x_0e^H+b_3b_5^2+b_4b_5^2+2b_3b_4b_5\,, \\ S_4=c_1=b_3b_4b_5^2\,. \end{cases}$$

We denote the surface $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ with the above S_1, S_2, S_3, S_4 by R_3*.

CASE (ii). Then $S_4=c_1$ and

$$\begin{cases} b_1^4-S_1b_1^3+S_2b_1^2-S_3b_1+c_1=c_2\,, \\ b_2^4-S_1b_2^3+S_2b_2^2-S_3b_2+c_1=\beta_1e^{H_1}\,, \\ b_3^4-S_1b_3^3+S_2b_3^2-S_3b_3+c_1=\beta_2e^{H_2}\,, \\ b_4^4-S_1b_4^3+S_2b_4^2-S_3b_4+c_1=\beta_3e^{H_3}\,, \\ b_5^4-S_1b_5^3+S_2b_5^2-S_3b_5+c_1=\beta_4e^{H_4}\,. \end{cases}$$

From the second, third and fourth equations we have

$$\begin{cases} S_1=x_1e^{H_1}+x_2e^{H_2}+x_3e^{H_3}+x+b_2+b_3+b_4\,, \\ S_2=(b_3+b_4)x_1e^{H_1}+(b_2+b_4)x_2e^{H_2}+(b_2+b_3)x_3e^{H_3} \\ \qquad +(b_2+b_3+b_4)x+b_2b_3+b_3b_4+b_2b_4\,, \\ S_3=b_3b_4x_1e^{H_1}+b_2b_4x_2e^{H_2}+b_2b_3x_3e^{H_3}+(b_2b_3+b_3b_4+b_2b_4)x+b_2b_3b_4 \end{cases}$$

with $\beta_1=x_1b_2(b_2-b_3)(b_4-b_2)$, $\beta_2=x_2b_3(b_2-b_3)(b_3-b_4)$, $\beta_3=x_3b_4(b_3-b_4)(b_4-b_2)$ and $c_1=xb_2b_3b_4$. Substituting these into remaining two equations we have

$$H_2=H_3=H_4=H_1\ (\equiv H)\,,$$

$$\frac{x_1}{b_1-b_2}+\frac{x_2}{b_1-b_3}+\frac{x_3}{b_1-b_4}=0\,,$$

$$\frac{c_2}{(b_1-b_2)(b_1-b_3)(b_1-b_4)}+\frac{c_1}{b_2b_3b_4}=b_1\,,$$

$$\frac{x_1}{b_5-b_2}+\frac{x_2}{b_5-b_3}+\frac{x_3}{b_5-b_4}+\frac{\beta_4}{b_5(b_5-b_2)(b_5-b_3)(b_5-b_4)}=0$$

and

$$(x-b_5)(b_5-b_2)(b_5-b_3)(b_5-b_4)=0\ .$$

Hence $x=b_5$, which implies $c_1=b_2b_3b_4b_5$ and $c_2=(b_1-b_2)(b_1-b_3)(b_1-b_4)(b_1-b_5)$. Now we impose the following condition: y does not have any other lacunary value, that is, excepting 0, b_1, there is no lacunary value of the first kind. Hence

$$\begin{aligned} F(z,\ \alpha)=&(\alpha-b_2)(\alpha-b_3)(\alpha-b_4)(\alpha-b_5)\\ &-\alpha(\alpha-b_1)e^H\{\alpha(x_1+x_2+x_3)+(b_1-b_3-b_4)x_1\\ &+(b_1-b_2-b_4)x_2+(b_1-b_2-b_3)x_3\} \end{aligned}$$

satisfies one of the following conditions:

(a) $\{\ \}=k$ (const.)$\neq 0$,

(b) $\{\ \}=k\alpha\ (k\neq 0)$,

(c) $\{\ \}=k(\alpha-b_1)$.

CASE (a). Then $x_1+x_2+x_3=0$. Therefore

$$\left\{\begin{aligned} S_1&=b_2+b_3+b_4+b_5\ ,\\ S_2&=\frac{(b_4-b_2)(b_3-b_2)}{b_1-b_2}x_1e^H+b_2b_3+b_3b_4+b_2b_4+b_2b_5+b_3b_5+b_4b_5\\ &=\frac{-\beta_1}{b_2(b_1-b_2)}e^H+b_2b_3+b_3b_4+b_2b_4+b_2b_5+b_3b_5+b_4b_5\ ,\\ S_3&=-\frac{b_1\beta_1}{b_2(b_1-b_2)}e^H+b_2b_3b_4+b_2b_3b_5+b_2b_4b_5+b_3b_4b_5\ ,\\ S_4&=c_1=b_2b_3b_4b_5\ . \end{aligned}\right.$$

The surface defined by $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ with the above S_1, S_2, S_3 and S_4 is denoted by R_4*.

CASE (b). Then

$$(b_3+b_4-b_1)x_1+(b_2+b_4-b_1)x_2+(b_2+b_3-b_1)x_3=0\ ,$$

that is,

$$(b_3+b_4)x_1+(b_2+b_4)x_2+(b_2+b_3)x_3=b_1(x_1+x_2+x_3)\ .$$

By

$$\frac{x_1}{b_1-b_2}+\frac{x_2}{b_1-b_3}+\frac{x_3}{b_1-b_4}=0\ ,$$

$$\begin{aligned} b_1^2(x_1+x_2+x_3)-b_1((b_3+b_4)x_1+(b_2+b_4)x_2+(b_2+b_3)x_3)&\\ +b_3b_4x_1+b_2b_4x_2+b_2b_3x_3&=0 \end{aligned}$$

Hence

$$b_3b_4x_1+b_2b_4x_2+b_2b_3x_3=0 .$$

Eliminating x_3 we have

$$(b_3+b_4-b_1)x_1+(b_2+b_4-b_1)x_2$$
$$-\frac{b_2+b_3-b_1}{b_2b_3}(b_3b_4x_1+b_2b_4x_2)=0 .$$

Hence

$$x_2=-\frac{b_3(b_1-b_3)(b_2-b_4)}{b_2(b_1-b_2)(b_3-b_4)}x_1 .$$

Now we have

$$\begin{aligned} x_1+x_2+x_3&=x_1+x_2-\frac{b_4}{b_2}x_1-\frac{b_4}{b_3}x_2 \\ &=\frac{b_2-b_4}{b_2}x_1+\frac{b_3-b_4}{b_3}x_2 \\ &=\frac{b_2-b_4}{b_2}x_1+\frac{b_3-b_4}{b_3}\cdot\frac{(b_3-b_1)(b_2-b_4)}{b_2(b_1-b_2)(b_3-b_4)}x_1 \\ &=\frac{(b_2-b_4)(b_1-b_2+b_3-b_1)}{b_2(b_1-b_2)}x_1=\frac{\beta_1}{b_2^2(b_1-b_2)} . \end{aligned}$$

Further

$$(b_3+b_4)x_1+(b_2+b_4)x_2+(b_2+b_3)x_3=b_1(x_1+x_2+x_3)=\frac{b_1\beta_1}{b_2^2(b_1-b_2)} .$$

Therefore

$$\begin{cases} S_1=\dfrac{\beta_1}{b_2^2(b_1-b_2)}e^H+b_2+b_3+b_4+b_5 , \\ S_2=\dfrac{b_1\beta_1}{b_2^2(b_1-b_2)}e^H+b_2b_3+b_3b_4+b_2b_4+b_2b_5+b_3b_5+b_4b_5 , \\ S_3=b_2b_3b_4+b_2b_3b_5+b_2b_4b_5+b_3b_4b_5 , \\ S_4=c_1=b_2b_3b_4b_5 . \end{cases}$$

The surface defined by $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ with the above S_1, S_2, S_3 and S_4 is denoted by R_5^*.

CASE (c). Then $k=x_1+x_2+x_3$ and

$$2b_1(x_1+x_2+x_3)=(b_3+b_4)x_1+(b_2+b_4)x_2+(b_2+b_3)x_3 .$$

By

$$x_1(b_1-b_3)(b_1-b_4)+x_2(b_1-b_2)(b_1-b_4)+x_3(b_1-b_2)(b_1-b_3)=0$$

we have

$$b_1^2(x_1+x_2+x_3)-b_1((b_3+b_4)x_1+(b_2+b_4)x_2+(b_2+b_3)x_3)$$
$$+b_3b_4x_1+b_2b_4x_2+b_2b_3x_3=0\,.$$

Hence

$$b_1^2(x_1+x_2+x_3)=b_3b_4x_1+b_2b_4x_2+b_2b_3x_3\,.$$

Further

$$2b_1(x_1+x_2+x_3)=2b_1\left(\frac{b_4-b_2}{b_1-b_2}x_1+\frac{b_4-b_3}{b_1-b_3}x_2\right)$$

and

$$(b_3+b_4)x_1+(b_2+b_4)x_2+(b_2+b_3)x_3$$
$$=(b_3+b_4)x_1+(b_2+b_4)x_2+(b_2+b_3)\left(\frac{b_4-b_1}{b_1-b_2}x_1+\frac{b_4-b_1}{b_1-b_3}x_2\right).$$

Hence

$$\frac{(b_1-b_2)(b_4-b_3)}{b_1-b_3}x_2=-\frac{(b_1-b_3)(b_4-b_2)}{b_1-b_2}x_1\,.$$

Therefore

$$x_1+x_2+x_3=x_1+x_2-\frac{b_1-b_4}{b_1-b_2}x_1-\frac{b_1-b_4}{b_1-b_3}x_2$$
$$=\frac{b_4-b_2}{b_1-b_2}x_1+\frac{b_4-b_3}{b_1-b_3}x_2=\frac{(b_3-b_2)(b_4-b_2)}{(b_1-b_2)^2}x_1$$
$$=\frac{-\beta_1}{b_2(b_1-b_2)^2}\,.$$

Hence we have

$$\begin{cases} S_1=\dfrac{-\beta_1}{b_2(b_1-b_2)^2}e^H+b_2+b_3+b_4+b_5\,, \\ S_2=\dfrac{-2b_1\beta_1}{b_2(b_1-b_2)^2}e^H+b_2b_3+b_3b_4+b_2b_4+b_2b_5+b_3b_5+b_4b_5\,, \\ S_3=\dfrac{-b_1^2\beta_1}{b_2(b_1-b_2)^2}e^H+b_2b_3b_4+b_2b_3b_5+b_2b_4b_5+b_3b_4b_5\,, \\ S_4=c_1=b_2b_3b_4b_5\,. \end{cases}$$

The surface defined by $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ with the above S_1, S_2, S_3 and S_4 is denoted by R_6^*.

CASE (iii). In this case $S_4=\beta_1e^{H_1}$ and

$$\begin{cases} b_1^4 - S_1 b_1^3 + S_2 b_1^2 - S_3 b_1 + S_4 = c_1 \\ b_2^4 - S_1 b_2^3 + S_2 b_2^2 - S_3 b_2 + S_4 = c_2 \\ b_3^4 - S_1 b_3^3 + S_2 b_3^2 - S_3 b_3 + S_4 = c_3 \\ b_4^4 - S_1 b_4^3 + S_2 b_4^2 - S_3 b_4 + S_4 = \beta_2 e^{H_2} \\ b_5^4 - S_1 b_5^3 + S_2 b_5^2 - S_3 b_5 + S_4 = \beta_3 e^{H_3} . \end{cases}$$

We have $H_1 = H_2 = H_3$ $(\equiv H)$ and

$$\frac{\beta_1}{b_1 b_2 b_3} = \frac{\beta_2}{(b_1 - b_4)(b_2 - b_4)(b_3 - b_4)} = \frac{\beta_3}{(b_1 - b_5)(b_2 - b_5)(b_3 - b_5)},$$

$$\frac{x}{b_4 - b_1} - \frac{y}{b_4 - b_2} + \frac{z}{b_4 - b_3} = -1$$

and

$$\frac{x}{b_5 - b_1} - \frac{y}{b_5 - b_2} + \frac{z}{b_5 - b_3} = -1$$

with

$$x b_1 (b_1 - b_2)(b_1 - b_3) = c_1 , \qquad y b_2 (b_1 - b_2)(b_2 - b_3) = c_2 ,$$

$$z b_3 (b_1 - b_3)(b_2 - b_3) = c_3 .$$

We now impose a condition that $F(z, \alpha) = \alpha^4 - S_1 \alpha^3 + S_2 \alpha^2 - S_3 \alpha + S_4$ does not reduce to the form

$$\frac{-\beta_1 e^H}{b_1 b_2 b_3} (\alpha - b_1)(\alpha - b_2)(\alpha - b_3)$$

with the exception of $\alpha = 0$, b_4 and b_5, that is, there is no lacunary value of the second kind excepting $\alpha = 0$, b_4 and b_5. Now we have

$$F(z, \alpha) = \frac{-\beta_1 e^H}{b_1 b_2 b_3} (\alpha - b_1)(\alpha - b_2)(\alpha - b_3) + \alpha P(\alpha) ,$$

where

$$\begin{aligned} P(\alpha) = \alpha^3 &+ \alpha^2 (x - y + z - b_1 - b_2 - b_3) \\ &+ \alpha(-(b_2 + b_3)x + (b_1 + b_3)y - (b_1 + b_2)z + b_1 b_2 + b_1 b_3 + b_2 b_3) \\ &+ b_2 b_3 x - b_1 b_3 y + b_1 b_2 z - b_1 b_2 b_3 . \end{aligned}$$

Hence we have three cases:

(a) $P(\alpha) = \alpha(\alpha - b_4)(\alpha - b_5)$,

(b) $P(\alpha) = (\alpha - b_4)^2 (\alpha - b_5)$,

(c) $P(\alpha) = (\alpha - b_4)(\alpha - b_5)^2$.

CASE (a). Then we have

$$x-y+z-b_1-b_2-b_3=-b_4-b_5\ ,$$

$$-(b_2+b_3)x+(b_1+b_3)y-(b_1+b_2)z+b_1b_2+b_1b_3+b_2b_3=b_4b_5\ ,$$

$$b_2b_3x-b_1b_3y+b_1b_2z=b_1b_2b_3\ .$$

Hence

$$\begin{cases} S_1=\dfrac{\beta_1}{b_1b_2b_3}e^H+b_4+b_5\ , \\ S_2=\dfrac{(b_1+b_2+b_3)}{b_1b_2b_3}\beta_1e^H+b_4b_5\ , \\ S_3=\dfrac{b_1b_2+b_1b_3+b_2b_3}{b_1b_2b_3}\beta_1e^H\ , \\ S_4=\beta_1e^H\ . \end{cases}$$

This surface is denoted by R_7^*.

CASE (b). Then we have

$$x-y+z-b_1-b_2-b_3=-2b_4-b_5\ ,$$

$$-(b_2+b_3)x+(b_1+b_3)y-(b_1+b_2)z+b_1b_2+b_1b_3+b_2b_3=b_4^2+2b_4b_5\ ,$$

$$b_2b_3x-b_1b_3y+b_1b_2z=b_1b_2b_3-b_4^2b_5\ .$$

Hence

$$\begin{cases} S_1=\dfrac{\beta_1}{b_1b_2b_3}e^H+2b_4+b_5\ , \\ S_2=\dfrac{(b_1+b_2+b_3)}{b_1b_2b_3}\beta_1e^H+b_4^2+2b_4b_5\ , \\ S_3=\dfrac{(b_1b_2+b_1b_3+b_2b_3)}{b_1b_2b_3}\beta_1e^H+b_4^2b_5\ , \\ S_4=\beta_1e^H\ . \end{cases}$$

We denote this surface by R_8^*.

CASE (c). Then we have

$$x-y+z-b_1-b_2-b_3=-b_4-2b_5\ ,$$

$$-(b_2+b_3)x+(b_1+b_3)y-(b_1+b_2)z+b_1b_2+b_1b_3+b_2b_3=2b_4b_5+b_5^2\ ,$$

$$b_2b_3x-b_1b_3y+b_1b_2z=b_1b_2b_3-b_4b_5^2\ .$$

Hence

$$\begin{cases} S_1=\dfrac{\beta_1 e^H}{b_1b_2b_3}+b_4+2b_5\,, \\ S_2=\dfrac{(b_1+b_2+b_3)\beta_1}{b_1b_2b_3}e^H+2b_4b_5+b_5^2\,, \\ S_3=\dfrac{(b_1b_2+b_1b_3+b_2b_3)\beta_1}{b_1b_2b_3}e^H+b_4b_5^2\,, \\ S_4=\beta_1 e^H\,. \end{cases}$$

This surface is denoted by R_9^*.

CASE (iv). We have $S_4=\beta_1 e^{H_1}$ and

$$\begin{cases} b_1^4-S_1b_1^3+S_2b_1^2-S_3b_1+\beta_1 e^{H_1}=c_1\,, \\ b_2^4-S_1b_2^3+S_2b_2^2-S_3b_2+\beta_1 e^{H_1}=c_2\,, \\ b_3^4-S_1b_3^3+S_2b_3^2-S_3b_3+\beta_1 e^{H_1}=\beta_2 e^{H_2}\,, \\ b_4^4-S_1b_4^3+S_2b_4^2-S_3b_4+\beta_1 e^{H_1}=\beta_3 e^{H_3}\,, \\ b_5^4-S_1b_5^3+S_2b_5^2-S_3b_5+\beta_1 e^{H_1}=\beta_4 e^{H_4}\,. \end{cases}$$

Then from the first three equations we have

$$S_1=\frac{\beta_1 e^{H_1}}{b_1b_2b_3}-\frac{\beta_2 e^{H_2}}{b_3(b_1-b_3)(b_2-b_3)}-x+y+b_1+b_2+b_3\,,$$

$$S_2=\frac{(b_1+b_2+b_3)}{b_1b_2b_3}\beta_1 e^{H_1}-\frac{(b_1+b_2)\beta_2 e^{H_2}}{b_3(b_1-b_3)(b_2-b_3)}-(b_2+b_3)x$$
$$+(b_1+b_3)y+b_1b_2+b_1b_3+b_2b_3\,,$$

$$S_3=\frac{b_1b_2+b_1b_3+b_2b_3}{b_1b_2b_3}\beta_1 e^{H_1}-\frac{b_1b_2\beta_2 e^{H_2}}{b_3(b_1-b_3)(b_2-b_3)}$$
$$-b_2b_3x+b_1b_3y+b_1b_2b_3\,,$$

$$S_4=\beta_1 e^{H_1}$$

with $xb_1(b_1-b_2)(b_1-b_3)=c_1$ and $yb_2(b_1-b_2)(b_2-b_3)=c_2$. Substituting these into remaining two equations and using Borel's unicity theorem we have

$$H_1=H_2=H_3=H_4\ (\equiv H)\,,$$

$$\frac{\beta_1}{b_1b_2b_3b_4}-\frac{\beta_2}{b_3(b_1-b_3)(b_2-b_3)(b_4-b_3)}+\frac{\beta_3}{b_4(b_4-b_1)(b_4-b_2)(b_4-b_3)}=0\,,$$

$$\frac{\beta_1}{b_1b_2b_3b_4}-\frac{\beta_2}{b_3(b_1-b_3)(b_2-b_3)(b_5-b_3)}+\frac{\beta_3}{b_5(b_5-b_1)(b_5-b_2)(b_5-b_3)}=0\,,$$

$$\frac{x}{b_4-b_1}-\frac{y}{b_4-b_2}+1=0$$

and

$$\frac{x}{b_5-b_1}-\frac{y}{b_5-b_2}+1=0\,.$$

Let us consider $F(z, \alpha)=\alpha^4-S_1\alpha^3+S_2\alpha^2-S_3\alpha+S_4$. Then

$$F(z, \alpha)=e^H(A\alpha+B)(\alpha-b_1)(\alpha-b_2)+\alpha(\alpha-b_3)P(\alpha)\,,$$

where A, B are constants:

$$A=\frac{\beta_2}{b_3(b_1-b_3)(b_2-b_3)}-\frac{\beta_1}{b_1b_2b_3} \quad \text{and} \quad B=\frac{\beta_1b_3}{b_1b_2b_3}\,,$$

and $P(\alpha)$ is equal to

$$\alpha^2-(b_1+b_2-x+y)\alpha+b_1b_2-b_2x+b_1y\,.$$

$P(\alpha)$ satisfies $P(b_4)=P(b_5)=0$. Hence $P(\alpha)=(\alpha-b_4)(\alpha-b_5)$. Therefore

$$b_1+b_2-x+y=b_4+b_5$$

and

$$b_1b_2-b_2x+b_1y=b_4b_5\,.$$

We impose a condition that $A\alpha+B$ does not vanish excepting $\alpha=b_1$ and $\alpha=b_2$. Here B does not vanish. If $B=0$, then

$$F(z, 0)=e^H A\alpha(\alpha-b_1)(\alpha-b_2)+\alpha(\alpha-b_3)(\alpha-b_4)(\alpha-b_5)\,.$$

Hence $F(z, 0)=0$, which is absurd. Therefore we have three possible cases:

(a) $A=0$, (b) $A\alpha+B=A(\alpha-b_1)$, (c) $A\alpha+B=A(\alpha-b_2)$.

CASE (a). Then

$$\frac{\beta_2}{b_3(b_1-b_3)(b_2-b_3)}=\frac{\beta_1}{b_1b_2b_3}\,.$$

Hence we have

$$\begin{cases} S_1=b_3+b_4+b_5\,, \\ S_2=\dfrac{1}{b_1b_2}\beta_1e^H+b_3b_4+b_3b_5+b_4b_5\,, \\ S_3=\dfrac{b_1+b_2}{b_1b_2}\beta_1e^H+b_3b_4b_5\,, \\ S_4=\beta_1e^H\,. \end{cases}$$

We denote this surface by $R_{10}{}^*$.

CASE (b). Then $A=-\beta_1/b_1^2b_2$. Hence

$$\frac{\beta_1}{b_1b_2b_3}-\frac{\beta_2}{b_3(b_1-b_3)(b_2-b_3)}=\frac{\beta_1}{b_1^2b_2}.$$

Further we have

$$\frac{b_1+b_2+b_3}{b_1b_2b_3}\beta_1-\frac{(b_1+b_2)\beta_2}{b_3(b_1-b_3)(b_2-b_3)}=\frac{2b_1+b_2}{b_1^2b_2}\beta_1$$

and

$$\frac{b_1b_2+b_1b_3+b_2b_3}{b_1b_2b_3}\beta_1-\frac{b_1b_2\beta_2}{b_3(b_1-b_3)(b_2-b_3)}=\frac{b_1^2+2b_1b_2}{b_1^2b_2}\beta_1 .$$

Therefore we have

$$\begin{cases} S_1=\dfrac{\beta_1}{b_1^2b_2}e^H+b_3+b_4+b_5 , \\ S_2=\dfrac{2b_1+b_2}{b_1^2b_2}\beta_1e^H+b_3b_4+b_3b_5+b_4b_5 , \\ S_3=\dfrac{b_1^2+2b_1b_2}{b_1^2b_2}\beta_1e^H+b_3b_4b_5 , \\ S_4=\beta_1e^H . \end{cases}$$

We denote this surface by R_{11}^*.

CASE (c). Then we have similarly

$$\begin{cases} S_1=\dfrac{\beta_1}{b_1b_2^2}e^H+b_3+b_4+b_5 , \\ S_2=\dfrac{b_1+2b_2}{b_1b_2^2}\beta_1e^H+b_3b_4+b_3b_5+b_4b_5 , \\ S_3=\dfrac{2b_1b_2+b_2^2}{b_1b_2^2}\beta_1e^H+b_3b_4b_5 , \\ S_4=\beta_1e^H . \end{cases}$$

We denote this surface by R_{12}^*.

We now have listed up twelve surfaces R_j^* ($j=1, 2, \cdots, 12$), which satisfy $P(y)=7$. However we prove that there are only three different surfaces among R_j^* ($j=1, 2, \cdots, 12$), when the same e^H is used.

Let us put $F(z, y)\equiv y^4-S_1y^3+S_2y^2-S_3y+S_4$ and $G(z, Y)\equiv Y^4-T_1Y^3+T_2Y^2-T_3Y+T_4$. If there is a suitable linear transformation $y=\alpha Y+\beta$, for which $F(z, \alpha Y+\beta)=\alpha^4G(z, Y)$, then two surfaces defined by $F(z, y)=0$ and $G(z, Y)=0$ are called the same surface or conformally equivalent with each other and this fact is denoted by $\sim$. Evidently

$$T_1=\frac{1}{\alpha}(S_1-4\beta),$$

$$T_2=\frac{1}{\alpha^2}(S_2-3\beta S_1+6\beta^2),$$

$$T_3=\frac{1}{\alpha^3}(S_3-2\beta S_2+3\beta^2 S_1-4\beta^3),$$

and

$$T_4=\frac{1}{\alpha^4}(S_4-\beta S_3+\beta^2 S_2-\beta^3 S_1+\beta^4).$$

Now we put

$$\alpha B_1+\beta=0, \qquad \beta=b_3$$

$$\alpha B_2=b_1-b_3,$$

$$\alpha B_3=b_2-b_3,$$

$$\alpha B_4=b_4-b_3,$$

$$\alpha B_5=b_5-b_3.$$

It is easy to prove that $R_1^*\sim R_7^*$, $R_2^*\sim R_8^*$ and $R_3^*\sim R_9^*$. Next we put

$$\alpha B_1+\beta=0, \qquad \beta=b_2$$

$$\alpha B_2=b_1-b_2,$$

$$\alpha B_3=b_3-b_2,$$

$$\alpha B_4=b_4-b_2,$$

$$\alpha B_5=b_5-b_2.$$

Again it is easy to prove that $R_4^*\sim R_{10}^*$, $R_5^*\sim R_{11}^*$ and $R_6^*\sim R_{12}^*$. Next we put

$$\alpha B_1=-b_1, \qquad \beta=b_1$$

$$\alpha B_2=b_2-b_1,$$

$$\alpha B_3=b_4-b_1,$$

$$\alpha B_4=b_5-b_1,$$

$$\alpha B_5=b_3-b_1.$$

Then we have $R_1^*\sim R_3^*$. Similarly we can prove that $R_2^*\sim R_1^*$. Next we put

$$\alpha B_5=-b_5, \qquad \beta=b_5$$

$$\alpha B_4=b_4-b_5\,,$$

$$\alpha B_3=b_3-b_5\,,$$

$$\alpha B_1=b_2-b_5\,,$$

$$\alpha B_2=b_1-b_5\,.$$

Then we can prove that $R_{11}^*\sim R_{12}^*$.

Therefore we may pick up R_4^*, R_7^*, R_6^* as three representatives of these twelve surfaces. Other representative may be selected several times.

§5. Discriminants of R_4^*, R_6^* and R_7^*

Firstly we consider the case R_4^*. The surface R_4^* is defined by

$$y^4-S_1y^3+S_2y^2-S_3y+S_4=0$$

with

$$\begin{cases} S_1=y_1\,, \\ S_2=y_0e^H+y_2\,, \\ S_3=b_1y_0e^H+y_3\,, \\ S_4=y_4\,. \end{cases}$$

Here

$$y_1=b_2+b_3+b_4+b_5\,,\qquad y_2=b_2b_3+b_2b_4+b_2b_5+b_3b_4+b_3b_5+b_4b_5\,,$$

$$y_3=b_2b_3b_4+b_2b_3b_5+b_2b_4b_5+b_3b_4b_5\,,\qquad y_4=b_2b_3b_4b_5\,.$$

Discriminant Δ is given by

$$-27M^4+144LM^2N-128L^2N^2+256N^3-4L^3M^2+16L^4N\,,$$

where

$$L=-\frac{3}{8}S_1^2+S_2\,,$$

$$M=-\frac{1}{8}S_1^3+\frac{1}{2}S_1S_2-S_3\,,$$

$$N=-\frac{3}{256}S_1^4+\frac{1}{16}S_1^2S_2-\frac{1}{4}S_1S_3+S_4\,.$$

For simplicity's sake we put $y_0e^H=X$. Then

$$L=X+\alpha_1\,,$$

$$M=\beta_0X+\beta_1\,,$$

$$N=\gamma_0X+\gamma_1\,,$$

where

$$\alpha_1=y_2-\frac{3}{8}y_1^2, \qquad \beta_0=\frac{1}{2}y_1-b_1, \qquad \beta_1=-\frac{1}{8}y_1^3+\frac{1}{2}y_1y_2-y_3,$$

$$\gamma_0=\frac{1}{16}y_1^2-\frac{1}{4}b_1y_1, \qquad \gamma_1=-\frac{3}{256}y_1^4+\frac{1}{16}y_1^2y_2-\frac{1}{4}y_1y_3+y_4 .$$

Then

$$\Delta=-4b_1^2y_0^5e^{5H}+A_4y_0^4e^{4H}+A_3y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H+A_0$$

with a non-zero constant A_0. Why is $A_0\neq 0$? Suppose $A_0=0$. Firstly we have $4T(r, y)=(1+o(1))T(r, e^H)$ for R_4^*. Now by an analogue of the proof of Ullrich-Selberg's ramification theorem [6], [7].

$$4N(r, R_4^*)\leqq N(r, 0, \Delta)$$

$$\leqq 4(1+o(1))T(r, e^H) .$$

Hence

$$N(r, R_4^*)\leqq 4(1+o(1))T(r, y) .$$

Thus

$$\varepsilon=\varliminf_{r\to\infty}\frac{N(r, R_4^*)}{T(r, y)}\leqq 4 .$$

Therefore by [6]

$$\sum\delta(w_\nu)\leqq 2+\varepsilon\leqq 6 .$$

But $7\leqq\sum\delta(w_\nu)$. This is a contradiction. The surface R_6^* is defined by

$$y_4-S_1y^3+S_2y^2-S_3y+S_4=0$$

with

$$\begin{cases} S_1=X+y_1 , \qquad X=y_0e^H \\ S_2=2b_1X+y_2 , \\ S_3=b_1^2X+y_3 , \\ S_4=y_4 . \end{cases}$$

Here

$$y_1=b_2+b_3+b_4+b_5 , \qquad y_2=b_2b_3+b_2b_4+b_2b_5+b_3b_4+b_3b_5+b_4b_5 ,$$

$$y_3=b_2b_3b_4+b_2b_3b_5+b_2b_4b_5+b_3b_4b_5 , \qquad y_4=b_2b_3b_4b_5 .$$

Now

$$L=-\frac{3}{8}S_1^2+S_2=-\left(\frac{3}{8}X^2+\alpha_1X+\alpha_2\right)$$

$$M=-\frac{1}{8}S_1^3+\frac{1}{2}S_1S_2-S_3=-\left(\frac{1}{8}X^3+\beta_1X^2+\beta_2X+\beta_3\right),$$

$$N=-\frac{3}{256}S_1^4+\frac{1}{16}S_1^2S_2-\frac{1}{4}S_1S_3+S_4$$

$$=-\left(\frac{3}{256}X^4+\gamma_1X^3+\gamma_2X^2+\gamma_3X+\gamma_4\right),$$

where

$$\alpha_1=\frac{3}{4}y_1-2b_1, \qquad \alpha_2=\frac{3}{8}y_1^2-y_2.$$

$$\beta_1=\frac{3}{8}y_1-b_1, \qquad \beta_2=\frac{3}{8}y_1^2-\frac{1}{2}y_2-b_1y_1+b_1^2,$$

$$\beta_3=\frac{1}{8}y_1^3-\frac{1}{2}y_1y_2+y_3,$$

$$\gamma_1=\frac{3}{64}y_1-\frac{b_1}{8}, \qquad \gamma_2=\frac{9}{128}y_1^2-\frac{1}{4}b_1y_1-\frac{1}{16}y_2+\frac{1}{4}b_1^2,$$

$$\gamma_3=\frac{3}{64}y_1^3-\frac{1}{8}y_1y_2-\frac{1}{8}b_1y_1^2+\frac{1}{4}b_1^2y_1+\frac{1}{4}y_3,$$

$$\gamma_4=\frac{3}{256}y_1^4-\frac{1}{16}y_1^2y_2+\frac{1}{4}y_1y_3-y_4.$$

Then we have $2\beta_1=\alpha_1$, $16\gamma_1=\alpha_1$ and $\alpha_2=4\beta_2-16\gamma_2$. Hence Δ is of at most six degree of X. Now the coefficient of X^6 is just

$$-\frac{27}{16}(\beta_3-4\gamma_3)^2+\frac{9\alpha_1}{2}(\beta_2-8\gamma_2)(\beta_3-4\gamma_3)$$

$$+\alpha_1^3(\beta_3-4\gamma_3)+4(\beta_2-8\gamma_2)^3+\alpha_1^2(\beta_2-8\gamma_2)^2.$$

See § 3. In the present case we have

$$\beta_3-4\gamma_3=-\frac{1}{16}y_1(y_1-4b_1)^2\equiv-y_1\left(\frac{y_1}{4}-b_1\right)^2$$

and

$$\beta_2-8\gamma_2=-\frac{3}{16}y_1^2+b_1y_1-b_1^2.$$

Hence the coefficient of X^6 of Δ is equal to

$$-\frac{27}{16}y_1^2\left(\frac{y_1}{4}-b_1\right)^4+\frac{9}{2}\left(\frac{3}{4}y_1-2b_1\right)\left(\frac{3}{16}y_1^2-b_1y_1+b_1^2\right)y_1\left(\frac{y_1}{4}-b_1\right)^2$$

$$-\left(\frac{3}{4}y_1-2b_1\right)^3y_1\left(\frac{y_1}{4}-b_1\right)^2-4\left(\frac{3}{16}y_1^2-b_1y_1+b_1^2\right)^3$$

$$+\left(\frac{3}{4}y_1-2b_1\right)^2\left(\frac{3}{16}y_1^2-b_1y_1+b_1^2\right)^2$$

$$=\left(\frac{y_1}{4}-b_1\right)^2\Big[-\frac{27}{16}y_1^2\left(\frac{y_1}{4}-b_1\right)^2+\frac{9}{2}\left(\frac{3}{4}y_1-2b_1\right)\left(\frac{3}{4}y_1-b_1\right)\left(\frac{1}{4}y_1-b_1\right)y_1$$

$$-y_1\left(\frac{3}{4}y_1-2b_1\right)^3-4\left(\frac{3}{4}y_1-b_1\right)^3\left(\frac{1}{4}y_1-b_1\right)$$

$$+\left(\frac{3}{4}y_1-2b_1\right)^2\left(\frac{3}{4}y_1-b_1\right)^2\Big]$$

$$=0 .$$

Therefore

$$\Delta=A_5y_0{}^5e^{5H}+A_4y_0{}^4e^{4H}+A_3y_0{}^3e^{3H}+A_2y_0{}^2e^{2H}+A_1y_0e^H+A_0$$

with $A_0\cdot A_5\neq 0$.

We shall now consider the case $R_7{}^*$. The surface $R_7{}^*$ is defined by $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ with

$$\begin{cases} S_1=y_0e^H+y_1\equiv X+y_1 , \\ S_2=x_1X+y_2 , \\ S_3=x_2X , \\ S_4=x_3X , \end{cases}$$

where $y_1=b_4+b_5$, $y_2=b_4b_5$, $x_1=b_1+b_2+b_3$, $x_2=b_1b_2+b_1b_3+b_2b_3$ and $x_3=b_1b_2b_3$. Then

$$L=-\frac{3}{8}S_1^2+S_2=-\left(\frac{3}{8}X^2+\alpha_1X+\alpha_2\right),$$

$$M=-\frac{1}{8}S_1^3+\frac{1}{2}S_1S_2-S_3=-\left(\frac{1}{8}X^3+\beta_1X^2+\beta_2X+\beta_3\right),$$

$$N=-\frac{3}{256}S_1^4+\frac{1}{16}S_1^2S_2-\frac{1}{4}S_1S_3+S_4$$

$$=-\left(\frac{3}{256}X^4+\gamma_1X^3+\gamma_2X^2+\gamma_3X+\gamma_4\right)$$

with

$$\alpha_1=\frac{3}{4}y_1-x_1 , \qquad \alpha_2=\frac{3}{8}y_1^2-y_2 ,$$

$$\beta_1=\frac{3}{8}y_1-\frac{1}{2}x_1 , \qquad \beta_2=\frac{3}{8}y_1^2-\frac{1}{2}x_1y_1-\frac{1}{2}y_2+x_2 ,$$

$$\beta_3=\frac{1}{8}y_1^3-\frac{1}{2}y_1y_2 .$$

$$\gamma_1=\frac{3}{64}y_1-\frac{1}{16}x_1, \qquad \gamma_2=\frac{9}{128}y_1^2-\frac{1}{8}x_1y_1-\frac{1}{16}y_2+\frac{1}{4}x_2,$$

$$\gamma_3=\frac{3}{64}y_1^3-\frac{1}{16}x_1y_1^2-\frac{1}{8}y_1y_2+\frac{1}{4}y_1x_2-x_3,$$

$$\gamma_4=\frac{3}{256}y_1^4-\frac{1}{16}y_1^2y_2.$$

Evidently we have $2\beta_1=\alpha_1$, $16\gamma_1=\alpha_1$ and $\alpha_2=4\beta_2-16\gamma_2$. Hence the discriminant Δ is at most six degree with respect to y_0e^H. Let us consider the constant term of Δ, which is equal to

$$-27\beta_3^4+144\alpha_2\beta_3^2\gamma_4-128\alpha_2^2\gamma_4^2-256\gamma_4^3+4\alpha_2^3\beta_3^2-16\alpha_2^4\gamma_4.$$

Hence we have

$$-27\Big(\frac{1}{8}y_1^3-\frac{1}{2}y_1y_2\Big)^4-128\Big(\frac{3}{8}y_1^2-y_2\Big)^2\Big(\frac{3}{256}y_1^4-\frac{1}{16}y_1^2y_2\Big)^2$$

$$+144\Big(\frac{3}{8}y_1^2-y_2\Big)\Big(\frac{3}{256}y_1^4-\frac{1}{16}y_1^2y_2\Big)\Big(\frac{1}{8}y_1^3-\frac{1}{2}y_1y_2\Big)^2$$

$$-256\Big(\frac{3}{256}y_1^4-\frac{1}{16}y_1^2y_2\Big)^3+4\Big(\frac{3}{8}y_1^2-y_2\Big)^3\Big(\frac{1}{8}y_1^3-\frac{1}{2}y_1y_2\Big)^2$$

$$-16\Big(\frac{3}{8}y_1^2-y_2\Big)^4\Big(\frac{3}{256}y_1^4-\frac{1}{16}y_1^2y_2\Big).$$

Then this is equal to the following expression:

$$y_1\Big[-\frac{27}{16}\Big(\frac{1}{4}y_1^2-y_2\Big)^4+\frac{9}{4}\Big(\frac{3}{8}y_1^2-y_2\Big)\Big(\frac{3}{16}y_1^2-y_2\Big)\Big(\frac{1}{4}y_1^2-y_2\Big)^2$$

$$-\frac{1}{2}\Big(\frac{3}{8}y_1^2-y_2\Big)^2\Big(\frac{3}{16}y_1^2-y_2\Big)^2-\frac{1}{16}y_1^2\Big(\frac{3}{16}y_1^2-y_2\Big)^3$$

$$-\frac{1}{16}\Big(\frac{3}{8}y_1^2-y_2\Big)^3\Big(\frac{1}{8}y_1^2-y_2\Big)\Big].$$

which is identically equal to 0. Hence the discriminant Δ of R_7^* has the form:

$$A_6y_0^6e^{6H}+A_5y_0^5e^{5H}+A_4y_0^4e^{4H}+A_3y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H$$

with non-zero constants A_1, A_6.

§ 6. A lemma

It is necessary to give an explicit proof of the following

LEMMA. *Let R be the Riemann surface R_4^* defined by*

$$y^4-S_1y^3+S_2y^2-S_3y+S_4=0$$

with

$$\begin{cases} S_1=x_1\,, \\ S_2=y_0e^H+x_2\,, \\ S_3=b_1y_0e^H+x_3\,, \\ S_4=x_4\,, \end{cases}$$

where $x_1=b_2+b_3+b_4+b_5$, $x_2=b_2b_3+b_2b_4+b_2b_5+b_3b_4+b_3b_5+b_4b_5$, $x_3=b_2b_3b_4+b_2b_3b_5+b_2b_4b_5+b_3b_4b_5$, $x_4=b_2b_3b_4b_5$, *Let F be a regular function on R_4^*. Then F is representable as*

$$F=f_1+f_2y+f_3y^2+f_4y^3\,,$$

where f_1, f_2, f_3 and f_4 are meromorphic functions in $|z|<\infty$, all of which are regular at any points z satisfying $H'(z)\neq 0$.

Proof. Let z_0 be a point satisfying $H'(z)\neq 0$. Let us put $t=z-z_0$. We should consider several cases.

1). There are two points of R_4^* on z_0 and both points are branch points. Then there are two different branches of y. And

$$y_1=A_0+A_1t^{p/2}+A_2t^{(p+1)/2}+\cdots\,,$$

$$y_2=B_0+B_1t^{q/2}+B_2t^{(q+1)/2}+\cdots\,.$$

2). There are two points of R_4^* on z_0 and only one is a branch point. Then

$$y_1=A_0+A_1t^{p/3}+A_2t^{(p+1)/3}+\cdots$$

and

$$y_2=B_0+B_1t^q+B_2t^{q+1}+\cdots\,.$$

3). There are three points of R_4^* on z_0. Then

$$y_1=A_0+A_1t^{p/2}+A_2t^{(p+1)/2}+\cdots\,,$$

$$y_2=B_0+B_1t^q+B_2t^{q+1}+\cdots\,,$$

$$y_3=C_0+C_1t^r+C_2t^{r+1}+\cdots\,.$$

4). There is only one point of R_4^* on z_0. Then

$$y_1=A_0+A_1t^{p/4}+A_2t^{(p+1)/4}+\cdots\,.$$

5). There are four points of R_4^* on z_0. Then

$$y_1=A_0+A_1t^p+\cdots\,,$$

$$y_2=B_0+B_1t^q+\cdots$$

$$y_3=C_0+C_1t^r+\cdots,$$

$$y_4=D_0+D_1t^s+\cdots.$$

Since $H'(z_0)\neq 0$, we have

$$e^{H(z)}=e^{H(z_0)}(1+d_1t+d_2t^2+\cdots), \qquad d_1\neq 0.$$

CASE 1). Suppose that $p\geqq 3$. Then

$$y_1=A_0+A_1t^{p/2}+\cdots$$

$$y_1^2=A_0^2+2A_0A_1t^{p/2}+\cdots$$

$$y_1^3=A_0^3+3A_0^2A_1t^{p/2}+\cdots$$

and

$$y_1^4=A_0^4+4A_0^3A_1t^{p/2}+\cdots.$$

Hence by $y_1^4-x_1y_1^3+(y_0e^H+x_2)y_1^2-(b_1y_0e^H+x_3)y_1+x_4=0$ we have

$$y_0e^{H(z_0)}d_1A_0^2-b_1y_0e^{H(z_0)}d_1A_0=0.$$

Therefore

$$A_0(A_0-b_1)d_1y_0e^{H(z_0)}=0,$$

that is, either $A_0=0$ or $A_0=b_1$. On the other hand

$$A_0^4-x_1A_0^3+(y_0e^{H(z_0)}+x_2)A_0^2-(b_1y_0e^{H(z_0)}+x_3)A_0+x_4=0.$$

If $A_0=0$, then $x_4=0$. But $x_4=b_2b_3b_4b_5\neq 0$. This is absurd. If $A_0=b_1$, then

$$\begin{aligned}&b_1^4-x_1b_1^3+(y_0e^{H(z_0)}+x_2)b_1^2-(b_1y_0e^{H(z_0)}+x_3)b_1+x_4\\&=A_0^4-x_1A_0^3+(y_0e^{H(z_0)}+x_2)A_0^2-(b_1y_0e^{H(z_0)}+x_3)A_0+x_4\\&=0.\end{aligned}$$

This contradicts that b_1 is a lacunary value of y. Hence $1\leqq p\leqq 2$. Similarly $1\leqq q\leqq 2$. Similarly we can prove the following facts: In case 2) we have $1\leqq p\leqq 3$, $q=1$ and in case 3) $1\leqq p\leqq 2$, $q=1$, $r=1$ and in case 4) $1\leqq p\leqq 4$ and in case 5) $p=q=r=s=1$.

CASE 1), Suppose that $y_1=A_0+A_2t+\cdots+A_s^*t^{s/2}+\cdots$ with the smallest odd s such that $A_s^*\neq 0$ and $s\geqq 3$. Then

$$A_0^4-x_1A_0^3+(y_0e^{H(z_0)}+x_2)A_0^2-(b_1y_0e^{H(z_0)}+x_3)A_0+x_4=0,$$

$$\{4A_0^3-3x_1A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}A_2$$
$$+y_0e^{H(z_0)}d_1A_0(A_0-b_1)=0$$

and

$$4A_0^3A_s^*-x_13A_0^3A_s^*+(y_0e^{H(z_0)}+x_2)2A_0A_s^*-(b_1y_0e^{H(z_0)}+x_3)A_s^*=0.$$

Hence by $A_s^*\neq 0$ we have

$$4A_0^3-3x_1A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)=0\,.$$

Therefore

$$A_0(A_0-b_1)=0\,,$$

which is absurd. Hence we have

$$y_1=A_0+A_1t^{1/2}+A_2t+A_3t^{3/2}+\cdots$$

and

$$y_2=B_0+B_1t^{1/2}+B_2t+B_3t^{3/2}+\cdots\,.$$

In case 2) we can prove that either

$$y_1=A_0+A_1t^{1/3}+A_2t^{2/3}+A_3t+\cdots$$

or

$$y_1=A_0+A_2t^{2/3}+A_3t+A_4t^{4/3}+\cdots$$

with

$$y_2=B_0+B_1t+B_2t^2+\cdots\,.$$

In case 3) we have

$$y_1=A_0+A_1t^{1/2}+A_2t+A_3t^{3/2}+\cdots$$

and in case 4) we have either

$$y_1=A_0+A_1t^{1/4}+A_2t^{2/4}+A_3t^{3/4}+A_4t+\cdots$$

or

$$y_1=A_0+A_2t^{2/4}+A_3t^{3/4}+A_4t+\cdots$$

or

$$y_1=A_0+A_3t^{3/4}+A_4t+A_5t^{5/4}+\cdots\,.$$

Firstly we consider case 4). Suppose that

$$y_1=A_0+A_1t^{1/4}+A_2t^{1/2}+A_3t^{3/4}+A_4t+\cdots\,.$$

Let us put

$$f_1=\frac{\alpha_n}{t^n}+\cdots,\quad f_2=\frac{\beta_n}{t^n}+\cdots,\quad f_3=\frac{\gamma_n}{t^n}+\cdots,\quad f_4=\frac{\delta_n}{t^n}+\cdots\,.$$

Then

$$F=f_1+f_2y_1+f_3y_1^2+f_4y_1^3$$

is pole-free. Hence

$$\alpha_n+\beta_nA_0+\gamma_nA_0^2+\delta_nA_0^3=0\,,$$

$$\beta_nA_1+\gamma_n2A_0A_1+\delta_n3A_0^2A_1=0\,,$$

$$\beta_nA_2+\gamma_n(2A_0A_2+A_1^2)+\delta_n(3A_0^2A_2+3A_0A_1^2)=0$$

and

$$\beta_n A_3+\gamma_n(2A_0A_3+2A_1A_2)+\delta_n(3A_0^2A_3+6A_0A_1A_2+A_1^3)=0\,.$$

$A_1\neq 0$ implies $\beta_n+\gamma_n 2A_0+\delta_n 3A_0^2=0$ and hence

$$(\gamma_n+\delta_n 3A_0)A_1^2=0\,.$$

Therefore $\gamma_n+\delta_n 3A_0=0$. This gives $\delta_n A_1^3=0$, that is, $\delta_n=0$. Hence $\gamma_n=\beta_n=\alpha_n=0$, which is absurd. Hence we may put $A_1=0$. Then

$$y_1=A_0+A_2t^{1/2}+A_4t+\cdots+A_st^{s/4}+\cdots+A_{s+2}t^{(s+2)/4}+\cdots$$

with the smallest odd $s>1$ for which $A_s\neq 0$. By

$$y_1^4-x_1y_1^3+(y_0e^H+x_2)y_1^2-(b_1y_0e^H+x_3)y_1+x_4=0$$

we have

$$\{4A_0^3-x_1 3A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}A_2=0\,,$$

$$4A_0^3A_4-x_1 3A_0^2A_4+(y_0e^{H(z_0)}+x_2)2A_0A_4-(b_1y_0e^{H(z_0)}+x_3)A_4$$
$$+\{6A_0^2-x_1 3A_0+y_0e^{H(z_0)}+x_2\}A_2^2+y_0e^{H(z_0)}d_1A_0(A_0-b_1)=0$$

and

$$\{4A_0^3-x_1 3A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}A_{s+2}$$
$$+\{6A_0^2-x_1 3A_0+y_0e^{H(z_0)}+x_2\}2A_2A_s=0\,.$$

Since $A_2\neq 0$ and $A_s\neq 0$,

$$6A_0^2-x_1 3A_0+y_0e^{H(z_0)}+x_2=0$$

and hence

$$d_1A_0(A_0-b_1)=0\,,$$

which is again a contradiction. Hence we may put $A_2=0$. Then

$$y_1=A_0+A_3t^{3/4}+A_4t+\cdots\,.$$

In this case we have

$$A_0^4-x_1A_0^3+(y_0e^{H(z_0)}+x_2)A_0^2-(b_1y_0e^{H(z_0)}+x_3)A_0+x_4=0\,,$$

$$\{4A_0^3-x_1 3A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}A_3=0$$

and

$$\{4A_0^3-x_1 3A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}A_4$$
$$+y_0e^{H(z_0)}d_1A_0(A_0-b_1)=0\,.$$

By $A_3\neq 0$, the coefficient of A $=0$. Hence $A_0(A_0-b_1)=0$, which is a contradiction. Hence case 4) does not occur.

Now we consider case 5). Then $F_j=f_1+f_2y_j+f_3y_j^2+f_4y_j^3$ are pole-free for $j=1, 2, 3, 4$. Hence

$$\begin{cases} \alpha_n+\beta_n A_0+\gamma_n A_0^2+\delta_n A_0^3=0\,, \\ \alpha_n+\beta_n B_0+\gamma_n B_0^2+\delta_n B_0^3=0\,, \\ \alpha_n+\beta_n C_0+\gamma_n C_0^2+\delta_n C_0^3=0\,, \\ \alpha_n+\beta_n D_0+\gamma_n D_0^2+\delta_n D_0^3=0\,. \end{cases}$$

Then $A_0=B_0$ or $\beta_n+\gamma_n(A_0+B_0)+\delta_n(A_0^2+A_0B_0+B_0^2)=0$ and $A_0=C_0$ or $\beta_n+\gamma_n(A_0+C_0)+\delta_n(A_0^2+A_0C_0+C_0^2)=0$ and $A_0=D_0$ or $\beta_n+\gamma_n(A_0+D_0)+\delta_n(A_0^2+A_0D_0+D_0^2)=0$. If $A_0\neq B_0$, $A_0\neq C_0$, $A_0\neq D_0$, then

$$B_0=C_0 \quad \text{or} \quad \gamma_n+\delta_n(A_0+B_0+C_0)=0$$

and

$$B_0=D_0 \quad \text{or} \quad \gamma_n+\delta_n(A_0+B_0+D_0)=0\,.$$

If further $B_0\neq C_0$, $B_0\neq D_0$, then $\delta_n(C_0-D_0)=0$. Hence either $C_0=D_0$ or $\delta_n=0$. If $\delta_n=0$, then $\gamma_n=\beta_n=\alpha_n=0$, which is absurd. Hence $C_0=D_0$. Therefore we may assume that $A_0=B_0$. By the definition of R_4^* we have

$$A_0^4-x_1A_0^3+(y_0e^{H(z_0)}+x_2)A_0^2-(b_1y_0e^{H(z_0)}+x_3)A_0+x_4=0\,,$$

$$\{4A_0^3-x_13A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}A_1$$
$$+y_0e^{H(z_0)}d_1A_0(A_0-b_1)=0\,.$$

If $4A_0^3-x_13A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)=0$, then $A_0(A_0-b_1)=0$, which is absurd. Hence $4A_0^3-x_13A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\neq 0$. Thus we have

$$\{4A_0^3-x_13A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}(A_1-B_1)=0\,,$$

which gives $A_1=B_1$. Similarly, if put $y_1=A_0+A_1t+A_2t^2+\cdots+A_nt^n+\cdots$, then

$$A_n(4A_0^3-x_13A_0^2+(y_0e^{H(z_0)}+x_2)2A_2-(b_1y_0e^{H(z_0)}+x_3))$$
$$+P(A_0,\cdots,A_{n-1})=0\,,$$

where $P(A_2,\cdots,A_{n-1})$ is a polynomial of $A_0,\cdots,A_{n-1}$. Hence we have $A_n=B_n$. Therefore $y_1\equiv y_2$, which is absurd.

CASE 2). If $y_1=A_0+A_1t^{1/3}+A_2t^{2/3}+A_3t+\cdots$ and $y_2=B_0+B_1t+B_2t^2+\cdots$, then by the pole-freeness of $F_j=f_1+f_2y_j+f_3y_j^2+f_4y_j^3$ we have

$$\alpha_n+\beta_nA_0+\gamma_nA_0^2+\delta_nA_0^3=0$$

$$(\beta_n+\gamma_n2A_0+\delta_n3A_0^2)A_1=0$$

and

$$(\beta_n+\gamma_n2A_0+\delta_n3A_0^2)A_2+(\gamma_n+\delta_n3A_0)A_1^2=0\,.$$

Hence $A_1\neq 0$ implies that $\beta_n+\gamma_n 2A_0+\delta_n 3A_0^2=0$ and $\gamma_n+\delta_n 3A_0=0$. Further we have

$$\alpha_n+\beta_n B_0+\gamma_n B_0^2+\delta_n B_0^3=0 .$$

Hence

$$(\beta_n+\gamma_n(A_0+B_0)+\delta_n(A_0^2+A_0B_0+B_0^2))(A_0-B_0)=0 .$$

If $A_0\neq B_0$, then

$$\beta_n+\gamma_n(A_0+B_0)+\delta_n(A_0^2+A_0B_0+B_0^2)=0 .$$

By $\beta_n+\gamma_n 2A_0+\delta_n 3A_0^2=0$ we have

$$(B_0-A_0)\{\gamma_n+\delta_n(2A_0+B_0)\}=0 ,$$

that is,

$$\gamma_n+\delta_n(2A_0+B_0)=0 .$$

By $\gamma_n+\delta_n 3A_0=0$ we have $\delta_n(B_0-A_0)=0$, that is, $\delta_n=0$. Then successively $\gamma_n=\beta_n=\alpha_n=0$, which is absurd. Hence $A_0=B_0$.

Substituting $y_1=A_0+A_1t^{1/3}+A_2t^{2/3}+\cdots$ into the defining equation of R_4^* we have

$$A_0^4-x_1A_0^3+(y_0e^{H(z_0)}+x_2)A_0^2-(b_1y_0e^{H(z_0)}+x_3)A_0+x_4=0 ,$$

$$\{4A_0^3-x_1 3A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}A_1=0$$

and

$$4A_0^3A_2+6A_0^2A_1^2-x_1(3A_0^2A_2+3A_0A_1^2)+(y_0e^{H(z_0)}+x_2)(2A_0A_2+A_1^2)$$
$$-(b_1y_0e^{H(z_0)}+x_3)A_2=0 .$$

Hence

$$6A_0^2-x_1 3A_0+y_0e^{H(z_0)}+x_2=0 .$$

On the other hand by $y_2=B_0+B_1t+B_2t^2+\cdots$ we have

$$\{4B_0^3-x_1 3B_0^2+(y_0e^{H(z_0)}+x_2)2B_0-(b_1y_0e^{H(z_0)}+x_3)\}B_2$$
$$+\{6B_0^2-x_1 3B_0+y_0e^{H(z_0)}+x_2\}B_1^2+y_0e^{H(z_0)}d_1B_0(B_0-b_1)=0 .$$

Since $A_0=B_0$, the coefficients of B_2 and B_1^2 are equal to zero. Therefore $A_0(A_0-b_1)=0$, which is absurd.

If $y_1=A_0+A_2t^{2/3}+A_3t+\cdots$, then by the defining equation of R_4^* we have

$$A_0^4-x_1A_0^3+(y_0e^{H(z_0)}+x_2)A_0^2-(b_1y_0e^{H(z_0)}+x_3)A_0+x_4=0 ,$$

$$\{4A_0^3-x_1 3A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}A_2=0$$

and

$$\{4A_0^3-x_1 3A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}A_3$$
$$+y_0e^{H(z_0)}d_1(A_0-b_1)A_0=0 .$$

Since $A_2 \neq 0$, we have $(A_0-b_1)A_0=0$, which is absurd.

CASE 3). In this case we have

$$\begin{cases} y_1=A_0+A_1t^{1/2}+A_2t+\cdots, \\ y_2=B_0+B_1t+\cdots, \\ y_3=C_0+C_1t+\cdots. \end{cases}$$

$F_j=f_1+f_2y_j+f_3y_j^2+f_4y_j^3$ is pole-free for $j=1, 2, 3$. Hence

$$\alpha_n+\beta_nA_0+\gamma_nA_0^2+\delta_nA_0^3=0,$$

$$(\beta_n+\gamma_n2A_0+\delta_n3A_0^2)A_1=0, \qquad A_1 \neq 0$$

$$\alpha_n+\beta_nB_0+\gamma_nB_0^2+\delta_nB_0^3=0,$$

and

$$\alpha_n+\beta_nC_0+\gamma_nC_0^2+\delta_nC_0^3=0.$$

Therefore

$$A_0=B_0 \quad \text{or} \quad \beta_n+\gamma_n(A_0+B_0)+\delta_n(A_0^2+A_0B_0+B_0^2)=0$$

and

$$A_0=C_0 \quad \text{or} \quad \beta_n+\gamma_n(A_0+C_0)+\delta_n(A_0^2+A_0C_0+C_0^2)=0.$$

If $A_0 \neq B_0$, then $\gamma_n+\delta_n(2A_0+B_0)=0$. If $A_0 \neq C_0$, then $\gamma_n+\delta_n(2A_0+C_0)=0$. Hence $(B_0-C_0)\delta_n=0$. If $B_0 \neq C_0$, then $\delta_n=0$ and $\gamma_n=\beta_n=\alpha_n=0$, which is absurd. Hence $B_0=C_0$. If this is the case, then we can conclude $y_2 \equiv y_3$ as in Case 5). Hence we may suppose that $A_0=B_0$. By making use of the equation of surface R_4^*, we have

$$A_0^4-x_1A_0^3+(y_0e^{H(z_0)}+x_2)A_0^2-(b_1y_0e^{H(z_0)}+x_3)A_0+x_4=0,$$

$$4A_0^3-x_13A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)=0,$$

and

$$\{4B_0^3-x_13B_0^2+(y_0e^{H(z_0)}+x_2)2B_0-(b_1y_0e^{H(z_0)}+x_3)\}B_1$$
$$+y_0e^{H(z_0)}d_1(B_0-b_1)B_0=0.$$

By $A_0=B_0$ we have

$$A_0(A_0-b_1)=0,$$

which is absurd.

CASE 1). In this case we may put

$$y_1=A_0+A_1t^{1/2}+A_2t+\cdots,$$

$$y_2=B_0+B_1t^{1/2}+B_2t+\cdots.$$

Since $F_k=f_1+f_2y_k+f_3y_k+f_4y_k^3$ $(k=1, 2)$ are pole-free, we have

$$\alpha_n+\beta_n A_0+\gamma_n A_0^2+\delta_n A_0^3=0\,,$$

$$\beta_n+\gamma_n 2A_0+\delta_n 3A_0^2=0$$

and

$$\alpha_n+\beta_n B_0+\gamma_n B_0^2+\delta_n B_0^3=0\,,$$

$$\beta_n+\gamma_n 2B_0+\delta_n 3B_0^2=0\,.$$

Hence we have

$$\beta_n+\gamma_n(A_0+B_0)+\delta_n(A_0^2+A_0B_0+B_0^2)=0\,,$$

if $A_0\neq B_0$. Hence $\gamma_n+\delta_n(2A_0+B_0)=0$. Similarly we have $\gamma_n+\delta_n(A_0+2B_0)=0$ if $A_0\neq B_0$. Hence $\delta_n=0$ and successively $\gamma_n=0$, $\beta_n=0$ and $\alpha_n=0$, which is absurd. Therefore $A_0=B_0$.

Anyway we have

$$y_1=A_0+A_1t^{1/2}+A_2t+A_3t^{3/2}+A_4t^2+A_5t^{5/2}+\cdots$$

and

$$y_2=A_0+B_1t^{1/2}+B_2t+B_3t^{3/2}+B_4t^2+B_5t^{5/2}+\cdots\,.$$

Substituting these into the defining equation of $R_4{}^*$ we have

$$A_0^4-x_1A_0^3+(y_0e^{H(z_0)}+x_2)A_0^2-(b_1y_0e^{H(z_0)}+x_3)A_0+x_4=0\,,$$

$$4A_0^3-x_13A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)=0\,,$$

$$\{4A_0^3-x_13A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}\,A_2$$

$$+\{6A_0^2-x_13A_0+(y_0e^{H(z_0)}+x_2)\}\,A_1^2+y_0e^{H(z_0)}d_1A_0(A_0-b_1)=0\,.$$

Hence we have

$$\{6A_0^2-x_13A_0+y_0e^{H(z_0)}+x_2\}\,A_1^2=y_0e^{H(z_0)}d_1A_0(b_1-A_0)\,.$$

Since $A_0(b_1-A_0)\neq 0$, we have

$$6A_0^2-x_13A_0+y_0e^{H(z_0)}+x_2\neq 0\,.$$

Therefore

$$\{6A_0^2-x_13A_0+y_0e^{H(z_0)}+x_2\}\,(A_1-B_1)(A_1+B_1)=0\,,$$

that is, either $A_1=B_1$ or $A_1=-B_1$. Further

$$\{4A_0^3-x_13A_0^2+(y_0e^{H(z_0)}+x_2)2A_0-(b_1y_0e^{H(z_0)}+x_3)\}\,A_3$$

$$+\{6A_0^2-x_13A_0+y_0e^{H(z_0)}+x_2\}\,2A_1A_2+(4A_0-x_1)A_1^3$$

$$-b_1y_0e^{H(z_0)}d_1A_1+y_0e^{H(z_0)}d_12A_0A_1=0\,.$$

Hence

$$\{6A_0^2-x_1 3A_0+y_0e^{H(z_0)}+x_2\}2A_1A_2+(4A_0-x_1)A_1^3$$
$$=y_0e^{H(z_0)}d_1(b_1-2A_0)A_1 .$$

Thus we have

$$\{6A_0^2-x_1 3A_0+y_0e^{H(z_0)}+x_2\}2(A_2-B_2)=0 ,$$

that is, $A_2=B_2$. Similarly we have

$$4A_0^3A_4+6A_0^2 2A_1A_3+4A_0 3A_1^2A_2+A_1^4-x_1(3A_0^2A_4+3A_0 2A_1A_3+3A_1^2A_2)$$
$$+(y_0e^{H(z_0)}+x_2)(2A_0A_4+2A_1A_3)+y_0e^{H(z_0)}(d_1 2A_0A_2+d_2A_0^2)$$
$$-(b_1y_0e^{H(z_0)}+x_3)A_4-b_1y_0e^{H(z_0)}(d_1A_2+d_2A_0)=0 .$$

Thus

$$\{6A_0^2-x_1 3A_0+y_0e^{H(z_0)}+x_2\}2A_1A_3$$
$$=(x_1-4A_0)3A_1^2A_2-A_1^4+y_0e^{H(z_0)}\{(b_1-2A_0)d_1A_2+(b_1-A_0)d_2A_0\} .$$

For y_2 we have a similar relation. Hence

$$\{6A_0^2-x_1 3A_0+y_0e^{H(z_0)}+x_2\}2(A_1A_3-B_1B_3)=0 .$$

Therefore $A_3=B_3$ if $A_1=B_1$ and $A_3=-B_3$ if $A_1=-B_1$. Quite similarly we have

$$\{6A_0^2-x_1 3A_0+y_0e^{H(z_0)}+x_2\}(2A_1A_4+2A_2A_3)$$
$$=(x_1-4A_0)(3A_1^3A_3+3A_1A_2^2)-4A_1^3A_2$$
$$+y_0e^{H(z_0)}\{b_1d_1A_3+b_1d_2A_1-d_1(2A_0A_3+2A_1A_2)-d_2 2A_0A_1\}$$

and a similar relation for $B_0=A_0$, B_1, $B_2=A_2$ and B_3 with $B_1B_3=A_1A_3$. Then we have

$$A_4+A_2\frac{A_3}{A_1}=B_4+B_2\frac{B_3}{B_1} .$$

that is, $A_4=B_4$. This method of proof goes through by induction and finally we arrive at

$$A_{2n}=B_{2n} , \qquad A_1A_{2n-1}=B_1B_{2n-1} .$$

If $A_j=B_j$ for all j, then $y_1\equiv y_2$, which is absurd. If $A_j=B_j$ for all even j and $A_j=-B_j$ for all odd j, then

$$y_2(t)=\sum_{j=0}^{\infty}A_{2j}t^{(2j)/2}-\sum_{j=0}^{\infty}A_{2j+1}t^{(2j+1)/2}$$
$$=\sum_{j=0}^{\infty}A_{2j}(te^{2\pi i})^{(2j)/2}+\sum_{j=0}^{\infty}A_{2j+1}(te^{2\pi i})^{(2j+1)/2}$$
$$=y_1(te^{2\pi i}) .$$

Hence y_1, y_2 are the same branch with a different representation. Therefore there are only two sheets over $|t|<t_0$. This is a contradiction.

We can prove quite similarly that corresponding lemmas for the surfaces X_1, R_6^* and R_7^* do hold. Since $X_2\sim X_1$, $R_1^*\sim R_2^*\sim R_3^*\sim R_7^*\sim R_8^*\sim R_9^*$, $R_4^*\sim R_{10}^*$, $R_5^*\sim R_6^*\sim R_{11}^*\sim R_{12}^*$, when the same e^H is commonly used, it is sufficient to prove lemmas for representatives R_4^*, X_1, R_6^* and R_7^*, respectively.

§7. Transformation formula of discriminants

The following method of proof of transformation formula of discriminants is suggested by Referee of our previous paper [4]. We now make use of his suggestion with thanks. Starting from a surface R

$$y^4-S_1y^3+S_2y^2-S_3y+S_4=0\,,$$

we have the representation of discriminant Δ as

$$\{(y_1-y_2)(y_1-y_3)(y_1-y_4)(y_2-y_3)(y_2-y_4)(y_3-y_4)\}^2.$$

Let F be a regular function on R. Then F can be written as

$$F=f_1+f_2y+f_3y^2+f_4y^3$$

as in lemma in § 6. F satisfies

$$F^4-U_1F^3+U_2F^2-U_3F+U_4=0\,.$$

The discriminant D of this surface is given by

$$\{(F_1-F_2)(F_1-F_3)(F_1-F_4)(F_2-F_3)(F_2-F_4)(F_3-F_4)\}^2.$$

Here $F_j=f_1+f_2y_j+f_3y_j^2+f_4y_j^3$ for $j=1, 2, 3, 4$. Then

$$F_j-F_k=(y_j-y_k)\{f_2+f_3(y_j+y_k)+f_4(y_j^2+y_jy_k+y_k^2)\}\,.$$

Hence

$$D=\Delta\cdot G^2\,,$$

where

$$\begin{aligned}G=&\{f_2+f_3(y_1+y_2)+f_4(y_1^2+y_1y_2+y_2^2)\}\{f_2+f_3(y_3+y_4)+f_4(y_3^2+y_3y_4+y_4^2)\}\\&\{f_2+f_3(y_1+y_3)+f_4(y_1^2+y_1y_3+y_3^2)\}\{f_2+f_3(y_2+y_4)+f_4(y_2^2+y_2y_4+y_4^2)\}\\&\{f_2+f_3(y_1+y_4)+f_4(y_1^2+y_1y_4+y_4^2)\}\{f_2+f_3(y_2+y_3)+f_4(y_2^2+y_2y_3+y_3^2)\}\,.\end{aligned}$$

Now G is a homogeneous polynomial of sixth degree of f_2, f_3, f_4 with suitable symmetric polynomial coefficients of y_1, y_2, y_3, y_4. Therefore every coefficient is a polynomial of S_1, S_2, S_3 and S_4. Here $S_1=y_1+y_2+y_3+y_4$, $S_2=y_1y_2+y_1y_3+y_1y_4+y_2y_3+y_2y_4+y_3y_4$, $S_3=y_1y_2y_3+y_1y_2y_4+y_1y_3y_4+y_2y_3y_4$ and $S_4=y_1y_2y_3y_4$.

Hence G may have poles at z_0 at which $H'(z_0)=0$.

Now we introduce a new assumption that $H(z)$ is a polynomial. From now on we consider the problem under this finiteness assumption.

Let R be the surface R_4^*: $y^4-S_1y^3+S_2y^2-S_3y+S_4=0$ with $S_1=y_1$, $S_2=y_0e^H+y_2$, $S_3=b_1y_0e^H+y_3$ and $S_4=y_4$, where $y_1=b_2+b_3+b_4+b_5$, $y_2=b_2b_3+b_2b_4+b_2b_5+b_3b_4+b_3b_5+b_4b_5$, $y_3=b_2b_3b_4+b_2b_3b_5+b_2b_4b_5+b_3b_4b_5$ and $y_4=b_2b_3b_4$. Then $P(y)=7$. Suppose that $P(R_4^*)=8$. Then there is a non-constant regular function F on R_4^* such that $P(F)=8$ and

$$F=f_1+f_2y+f_3y^2+f_4y^3\,,$$

where f_1, f_2, f_3, f_4 are meromorphic in $|z|<\infty$ and regular excepting at most at points satisfying $H'=0$. We may assume that F defines the surface X_1. Hence

$$F^4-U_1F^3+U_2F^2-U_3F+U_4=0$$

with

$$U_1=x_0e^L+x_1\,,$$

$$U_2=(a_1+a_2)x_0e^L+x_2\,,$$

$$U_3=a_1a_2x_0e^L+x_3$$

and

$$U_4=x_4\,,$$

where $x_1=a_3+a_4+a_5+a_6$, $x_2=a_3a_4+a_3a_5+a_3a_6+a_4a_5+a_4a_6+a_5a_6$, $x_3=a_3a_4a_5+a_3a_4a_6+a_3a_5a_6+a_4a_5a_6$ and $x_4=a_3a_4a_5a_6$. Discriminants of R_4^* and X_1 are denoted by Δ and D, respectively. Then we have

$$D=\Delta\cdot G^2\,.$$

Evidently the number of poles of G is finite. Let us put

$$D=A_6(x_0e^L-\gamma_1)(x_0e^L-\gamma_2)(x_0e^L-\gamma_3)(x_0e^L-\gamma_4)(x_0e^L-\gamma_5)(x_0e^L-\gamma_6)$$

and

$$\Delta=-4b_1^4(y_0e^H-\delta_1)(y_0e^H-\delta_2)(y_0e^H-\delta_3)(y_0e^H-\delta_4)(y_0e^H-\delta_5)\,.$$

CASE 1). The counting function of simple zeros of Δ satisfies

$$N_2(r,\,0,\,\Delta)\sim 5T(r,\,e^H)\,,$$

that is, $\delta_i\neq\delta_j$ for $i\neq j$. Then

$$N_2(r,\,0,\,\Delta)=N_2(r,\,0,\,D)\sim m\cdot T(r,\,e^L)$$

with $m=1, 2, 3, 4, 6$. Then L should be a polynomial, whose degree coincides with the one of H. In this case we return back y from F. Then we have

$$\Delta=D\cdot I^2\,.$$

The number of poles of I is finite again. This shows that the zeros of G is finite in number. Hence

$$D=\Delta\cdot\beta^2\cdot e^{2M}$$

with a rational function β and with an entire function M, $M(0)=0$. In this case $\gamma_i\neq\gamma_j$ for $i\neq j$.

Case 2). $N_2(r, 0, \Delta)\sim 3T(r, e^H)$, that is, δ_1, δ_2, δ_3, δ_4 are different and $\delta_4=\delta_5$. Then

$$N_2(r, 0, \Delta)=N_2(r, 0, D)\sim m\cdot T(r, e^L)$$

with $m=1, 2, 3, 4, 6$. Then L should be a polynomial, whose degree coincides with the one of H. Again we can return back y from F. Then $\Delta=D\cdot I^2$, where I has only finitely many poles. Hence G has only finitely many zeros. Cases $m=1$ and 3 donot occur. Suppose that $m=2$ or $m=4$. Then the counting function of multiple zeros of Δ satisfies

$$N_0(r, 0, \Delta)=N_0(r, 0, D),$$

where $N_0(r, 0, \Delta)=N(r, 0, \Delta)-N_2(r, 0, \Delta)$. However

$$N_0(r, 0, \Delta)\sim 2m(r, e^H)$$

and

$$N_0(r, 0, D)\sim 4m(r, e^L) \qquad \text{if } m=2,$$

$$N_0(r, 0, D)\sim 2m(r, e^L) \qquad \text{if } m=4.$$

However

$$3m(r, e^H)\sim N_2(r, 0, \Delta)=N_2(r, 0, D)\sim 2m(r, e^L) \qquad \text{if } m=2$$

and

$$\sim 4m(r, e^L) \qquad \text{if } m=4.$$

These give a contradiction.

CASE 3). $N_2(r, 0, \Delta)\sim 2T(r, e^H)$, that is, δ_1, δ_2, δ_3 are different and $\delta_3=\delta_4=\delta_5$. $N_2(r, 0, D)\sim m\cdot T(r, e^L)$ with $m=1, 2, 3, 4, 6$. Then L should be a polynomial. We can return back y from F. Then $\Delta=D\cdot I^2$, where I has only finitely many poles. In any case $m=1$ or $m=2$ or $m=3$ or $m=4$ or $m=6$ gives a contradiction.

CASE 4). $N_2(r, 0, \Delta)\sim T(r, e^H)$, that is, $\delta_1\neq\delta_2$ and $\delta_2=\delta_3=\delta_4=\delta_5$ or δ_1, δ_2, δ_3 are different and $\delta_2=\delta_4$, $\delta_3=\delta_5$. $N_2(r, 0, D)\sim m\cdot T(r, e^L)$ with $m=1, 2, 3, 4, 6$. Then L should be a polynomial. We can return back y from F. Then $\Delta=D\cdot I^2$, where I has only finitely many poles. In any case $m=1$ or $m=2$ or $m=3$ or $m=4$ or $m=6$ gives a contradiction.

CASE 5). Δ does not have any simple zero. Then we arrive at a contradiction easily.

Therefore we have

$$D=\Delta\cdot\beta^{2}\cdot e^{2M}$$

with a rational function β and D, Δ must have only simple factors.

We have proved the above relation for the surface R_4^*. For R_6^* and R_7^* we can prove the same fact.

§ 8. Theorems

We shall prove the following

THEOREM 1. *Let R_4^* be the Riemann surface. Assume that its discriminant $\Delta_{R_4^*}$ satisfies*

$$\Delta_{R_4^*}=-4b_1^2y_0^5e^{5H}+A_4y_0^4e^{4H}+A_3y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^{H}+A_0\,,$$

where at least one of A_j $(j=1, 2, 3, 4)$ does not vanish. Then $P(R_4^*)=7$, if H is a polynomial.

Proof. Suppose that $P(R_4^*)=8$. Then on R_4^* there is a regular function F for which $P(F)=8$. Suppose that F defines the surface X_1. (We may assume so, since $X_2\sim X_1$.) Then

$$D=\Delta_{R_4^*}\cdot\beta^{2}\cdot e^{2M}\,,$$

which is just the following identity:

$$B_6x_0^6e^{6L}+B_5x_0^5e^{5L}+B_4x_0^4e^{4L}+B_3x_0^3e^{3L}+B_2x_0^2e^{2L}+B_1x_0e^{L}+B_0$$
$$=(-4b_1^2y_0^5e^{5H}+A_4y_0^4e^{4H}+A_3y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^{H}+A_0)\beta^2e^{2M}.$$

Now we shall make use of Borel's unicity theorem. In this case we have

$$6T(r, e^L)\sim N_2(r, 0, D)=N_2(r, 0, \Delta_{R_4^*})\sim 5T(r, e^H).$$

Hence

$$T(r, e^H)\sim\frac{6}{5}T(r, e^L).$$

This relation makes our discussion simpler. Firstly assume that $M\equiv 0$. Then there remains only one possibility: $6L=5H$, $B_0=\beta^2A_0$, $B_6x_0^6=-4b_1^2\beta^2y_0^5$ and $B_5=B_4=B_3=B_2=B_1=A_4=A_3=A_2=A_1=0$, which contradicts our assumption: at least one of A_j, $j=1, 2, 3, 4$ does not vanish. Hence we have the desired result.

Assume that $M\not\equiv 0$. $5H+2M=0$ and $6L=-5H$, $B_0=-4b_1^2\beta^2y_0^5$, $B_6x_0^6=\beta^2A_0$, $B_5=B_4=B_3=B_2=B_1=A_4=A_3=A_2=A_1=0$, which contradicts our assumption: at least one of A_j, $j=1, 2, 3, 4$ does not vanish. Hence we have the desired result.

THEOREM 2. *Let R_6^* be the Riemann surface, whose discriminant $\Delta_{R_6^*}$ is*

$$\Delta_{R_6^*}=A_5y_0^5e^{5H}+A_4y_0^4e^{4H}+A_3y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H+A_0$$

with non-zero constants A_0 *and* A_5. *Suppose that at least one of* A_j $(j=1, 2, 3, 4)$ *does not vanish. Then* $P(R_6^*)=7$, *if* H *is a polynomial.*

Proof is similar as in Theorem 1. So we shall omit it.

THEOREM 3. *Let* R_7^* *be the Riemann surface, whose discriminant* $\Delta_{R_7^*}$ *is*

$$\Delta_{R_7^*}=A_6y_0^6e^{6H}+A_5y_0^5e^{5H}+A_4y_0^4e^{4H}+A_3y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H$$

with non-zero constants A_1 *and* A_6. *Suppose that at least one of* A_j $(j=2, 3, 4, 5)$ *does not vanish. Then* $P(R_7^*)=7$, *if* H *is a polynomial.*

Proof of Theorem 3. Suppose that $P(R_7^*)=8$. Then on R_7^* there is a regular function F for which $P(F)=8$. Suppose that F defines the surface X_1. Then similarly

$$D=\Delta_{R_7^*}\cdot\beta^2\cdot e^{2M}.$$

This is just the following identity:

$$\begin{aligned}&B_6x_0^6e^{6L}+B_5x_0^5e^{5L}+B_4x_0^4e^{4L}+B_3x_0^3e^{3L}+B_2x_0^2e^{2L}+B_1x_0e^L+B_0\\&=(A_6y_0^6e^{6H}+A_5y_0^5e^{5H}+A_4y_0^4e^{4H}+A_3y_0^3e^{3H}+A_2y_0^2e^{2H}+A_1y_0e^H)\beta^2e^{2M}.\end{aligned}$$

In this case we have

$$6T(r, e^L)\sim N_2(r, 0, D)=N_2(r, 0, \Delta_{R_7^*})\sim 5T(r, e^H).$$

Hence

$$T(r, e^H)\sim\frac{6}{5}T(r, e^L).$$

There are only two possible cases: $2M+H=0$ or $2M+6H=0$. If $2M=-H$, then $B_0=A_1\beta^2y_0$, $x_0^6B_6=A_6y_0^6\beta^2$ and $B_5=B_4=B_3=B_2=B_1=A_5=A_4=A_3=A_2=0$ and $6L=5H$. If $2M=-6H$, then $B_0=A_6y_0^6\beta^2$, $B_6x_0^6=A_1y_0\beta^2$, $6L=-5H$ and $B_5=B_4=B_3=B_2=B_1=A_5=A_4=A_3=A_2=0$. In any cases we have a contradiction: $A_j=0$ for $j=2, 3, 4, 5$. Thus we have the desired result.

In the above we list up three theorems which correspond three representatives R_4^*, R_6^* and R_7^*. Theorems are almost similar for other surfaces. We shall omit their formulations. (We can make use of similar transformation $Y=\alpha y+\beta$. Then the discriminant is transformed into constant times of a discriminant. Hence the non-vanishing property of coefficients of discriminant is preserved.)

REFERENCES

[1] K. NIINO AND M. OZAWA, Deficiencies of an entire algebroid functions, Kōdai Math. Sem. Rep., **22** (1970), 98-113.
[2] M. OZAWA, On complex analytic mappings, Kōdai Math. Sem. Rep., **17** (1965), 93-102.
[3] M. OZAWA AND K. NIINO, Surveys of analytic mappings between two Riemann surfaces, Analytic Function Theory of One Complex Variable, Pitman Research Notes, **212**, 1989, 226-252.
[4] M. OZAWA AND K. SAWADA, Three-sheeted algebroid surfaces whose Picard constants are five, Kodai Math. J., **17** (1994), 101-124.
[5] G. RÉMOUNDOS, Extension aux Fonctions Algébroïdes Multiformes du Théorème de M. Picard et de ses Généralisations, Mém. Sci. Math., **23**, Gauthier-Villars, Paris, 1927.
[6] H. SELBERG, Algebroide Funktionen und Umkehrfunktionen Abelscher Integrale, Avh. Norske Vid. Akad. Oslo, **8** (1934), 1-72.
[7] E. ULLRICH, Über den Einfluss der Verzweigtheit einer Algebroide auf ihre Wertverteilung, J. Reine Angew. Math., **169** (1931), 198-220.
[8] G. VALIRON, Sur la dérivée de fonctions algebroïdes, Bull. Soc. Math. France, **59** (1931), 17-39.

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