S. KATO
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CONSTRUCTION OF n-END CATENOIDS WITH PRESCRIBED FLUX

SHIN KATO

1. Introduction

Let $X: \hat{C}-\{q_1, \cdots, q_n\} \to R^3$ be an *n-end catenoid*, that is, a complete minimal surface of genus 0 having n catenoid ends at q_i's, where $\hat{C}:=C\cup\{\infty\}$. Let $G: \hat{C}-\{q_1, \cdots, q_n\} \to S^2$ be its *Gauss map* which can be extended naturally on $\hat{C}$, and let $w(q_i)$ denote the *weight* of the end q_i, that is, the similitude ratio of the asymptotic catenoid of the end q_i to the standard catenoid ($g=-z$, $\eta=-z^{-2}dz$). Remark that $w(q_i)$ takes negative value if the orientation of the end q_i differs from that of the standard catenoid, and that $w(q_i)$ vanishes if the end q_i is a flat end or is removed. The vector $w(q_i)G(q_i)$ is called the *flux vector* of the end q_i and, it follows from the *flux formula* (cf. e.g. [2]) that $\sum_{i=1}^n w(q_i)G(q_i)=0$. Now, conversely, we consider the following

PROBLEM. *Given n unit vectors $v_1, \cdots, v_n$ in R^3 and n non-zero real numbers $a_1, \cdots, a_n$ satisfying $\sum_{i=1}^n a_i v_i=0$, is there an n-end catenoid $X: \hat{C}-\{q_1, \cdots, q_n\} \to R^3$ such that $G(q_i)=v_i$ and $w(q_i)=a_i$?*

In this paper, we study the problem in the case when q_i coincides with $\sigma(v_i)$ for each i, where $\sigma: S^2 \to \hat{C}$ is the stereographic projection from the north pole. Our main result is stated as follows.

THEOREM. *Let $v_1, \cdots, v_n$ be unit vectors in R^3, and $a_1, \cdots, a_n$ non-zero real numbers satisfying $\sum_{i=1}^n a_i v_i=0$. Set $p_i:=\sigma(v_i)$ and*

$$F_i(z):=\frac{\bar{p}_i z+1}{z-p_i}.$$

Suppose there are complex numbers $b_1, \cdots, b_n$ satisfying

$$b_i \sum_{j\in N_i} b_j=a_i \qquad i=1, \cdots, n, \tag{1.1}$$

$$\sum_{j\in N_i} b_j F_i(p_j)=0 \qquad i=1, \cdots, n \tag{1.2}$$

and $\sum_{i=1}^n b_i\neq 0$, where $N_i:=\{j\in N \mid 1\leq j\leq n,\ j\neq i\}$. Then there exists an n-end

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catenoid $X:\hat{C}-\{p_1, \cdots, p_n\}\to \boldsymbol{R}^3$ *such that* $G(p_i)=v_i$ *and* $w(p_i)=a_i$.

In Section 2, we prove this theorem by giving explicit representation for the solution surface.

Except for the (2-end) catenoid, examples of n-end catenoids were first introduced by Jorge-Meeks [3]. These are the case with $p_i=\zeta_n^i$ and $a_i\equiv 1$, where ζ_n is a primitive root of the equation $z^n=1$. (Throughout this paper, we keep this notation.) Subsequently, Karcher [4] constructed some new 4-end catenoids, and Lopez [5] classified all of the 3-end catenoids. In each of these examples, $v_1, \cdots, v_n$ lies on the same great circle in S^2.

Recently, Xu [9], Rossman [7] and Umehara-Yamada [8] constructed polyhedrally symmetric n-end catenoids and some less symmetric ones, in each of which $v_1, \cdots, v_n$ do not lie on the same great circle in S^2. For this purpose, Xu used directly the Enneper-Weierstrass representation. On the other hand, Rossman employed the conjugate surface method and constructed also higher genus examples (see also Berglund-Rossman [1]). Umehara and Yamada constructed polyhedrally symmetric ones as limits of those corresponding CMC-c surfaces in $\boldsymbol{H}^3(-c^2)$.

Each example of n-end catenoids in Jorge-Meeks [3] and Xu [9] has the ends of the same weight, and it is easy to observe that they are all described by the following special case of our theorem.

COROLLARY. *Let* $v_1, \cdots, v_n$ *be unit vectors satisfying* $\sum_{i=1}^n v_i=0$, *and* p_i, F_i *and* N_i *as in Theorem. If*

$$\sum_{j\in N_i} F_i(p_j)=0 \qquad i=1, \cdots, n\,,$$

then there exists an n-end catenoid $X:\hat{C}-\{p_1, \cdots, p_n\}\to \boldsymbol{R}^3$ *such that* $G(p_i)=v_i$ *and* $w(p_i)\equiv 1$.

Finally, we emphasize that almost all of the known examples can be constructed by our theorem. In Section 3, we discuss this and also give far more new examples of families of n-end catenoids having ends of 2, 3 or 4 different weights.

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2. Proof of Theorem

In this section, we prove our main theorem. First, we recall the following famous and significant

ENNEPER-WEIERSTRASS REPRESENTATION. (cf. [6]) *Let* Σ *be a Riemann* *surface, g a meromorphic function on Σ, and η a holomorphic 1-form on Σ. Define a map $X: \Sigma \to \boldsymbol{R}^3$ by*

$$X(z)=\mathrm{Re}\int^{z}(1-g^2,\ \sqrt{-1}(1+g^2),\ 2g)\eta\ .$$

If

$$\mathrm{Re}\int_C(1-g^2,\ \sqrt{-1}(1+g^2),\ 2g)\eta=0 \tag{2.1}$$

for any closed curve C on Σ, then X is a conformal minimal branched immersion whose Gauss map is $\sigma^{-1}\circ g$. Moreover, the induced metric of Σ is given by

$$ds^2=(1+|g|^2)^2|\eta|^2\ .$$

Proof of Theorem. First, we assume $p_i\neq\infty$ for any i. Set

$$f(z):=\sum_{i=1}^{n}\frac{b_i}{z-p_i},\qquad \beta:=\sum_{i=1}^{n}b_i\ , \tag{2.2}$$

and

$$g(z):=z-\frac{\beta}{f(z)},\qquad \eta:=-\{f(z)\}^2dz\ .$$

We will show that the surface $X: \hat{\boldsymbol{C}}-\{q_1, \cdots, q_n\}\to\boldsymbol{R}^3$ represented by these data is an n-end catenoid we want to construct.

Let (v_{i1}, v_{i2}, v_{i3}) be the orthogonal coordinate of the vector v_i. Then, by using the assumptions (1.1) and (1.2), we have, for any i,

$$\begin{aligned}\mathrm{Res}_{z=p_i}\{-(1-g^2)f^2\}&=2b_i\sum_{j\in N_i}b_j\frac{p_ip_j-1}{p_i-p_j}\\&=-2\left\{b_i(\beta-b_i)\frac{p_i+\bar{p}_i}{|p_i|^2+1}+\frac{b_i(p_i^2-1)}{|p_i|^2+1}\sum_{j\in N_i}b_jF_i(p_j)\right\}\\&=-2a_iv_{i1}\in\boldsymbol{R}\ ,\end{aligned}$$

$$\begin{aligned}\mathrm{Res}_{z=p_i}\{-\sqrt{-1}(1+g^2)f^2\}&=\frac{2b_i}{\sqrt{-1}}\sum_{j\in N_i}b_j\frac{p_ip_j+1}{p_i-p_j}\\&=-\frac{2}{\sqrt{-1}}\left\{b_i(\beta-b_i)\frac{p_i-\bar{p}_i}{|p_i|^2+1}+\frac{b_i(p_i^2+1)}{|p_i|^2+1}\sum_{j\in N_i}b_jF_i(p_j)\right\}\\&=-2a_iv_{i2}\in\boldsymbol{R}\ ,\end{aligned}$$

$$\begin{aligned}\mathrm{Res}_{z=p_i}\{-2gf^2\}&=-2b_i\sum_{j\in N_i}b_j\frac{p_i+p_j}{p_i-p_j}\\&=-2\left\{b_i(\beta-b_i)\frac{|p_i|^2-1}{|p_i|^2+1}-\frac{2b_ip_i}{|p_i|^2+1}\sum_{j\in N_i}b_jF_i(p_j)\right\}\\&=-2a_iv_{i3}\in\boldsymbol{R}\ .\end{aligned}$$

Hence the condition (2.1) holds, and the surface X is well-defined. Moreover, since $\beta \neq 0$, the induced metric

$$ds^2 = (|f|^2 + |zf - \beta|^2)^2 |dz|^2$$

is non-degenerate. By simple calculation, we get the following expansions around p_i.

$$g(z) = p_i - \frac{\beta - b_i}{b_i}(z - p_i) + O((z - p_i)^2),$$

$$\eta = \left\{ -\frac{b_i^2}{(z - p_i)^2} + O\left(\frac{1}{z - p_i}\right) \right\} dz .$$

Therefore, for any i, the surface X has a catenoid end at p_i such that $G(p_i) = \sigma^{-1} \circ g(p_i) = v_i$ and

$$w(p_i) = -\frac{\beta - b_i}{b_i}(-b_i^2) = b_i(\beta - b_i) = a_i .$$

On the other hand, it is easy to see that, even if $p_i = \infty$ for some i, the assertion of Theorem and the data (2.2) are valid in the sense that

$$F_i(p_j) = \frac{\bar{p}_i p_j + 1}{p_j - p_i} = \frac{\infty p_j + 1}{p_j - \infty} = -p_j ,$$

$$F_j(p_i) = \frac{\bar{p}_j p_i + 1}{p_i - p_j} = \frac{\bar{p}_j \infty + 1}{\infty - p_j} = \bar{p}_j ,$$

$$\frac{b_i}{z - p_i} = \frac{b_i}{z - \infty} = 0 .$$

q. e. d.

Proof of Corollary. Apply Theorem to the case when $a_i \equiv 1$ and $b_i \equiv 1/\sqrt{n-1}$. q. e. d.

Remark 2.1. By the proof of Theorem, we can observe the flux formula from another point of view. Namely we see that

$$\sum_{i=1}^{n} a_i v_{i1} = \sum_{i=1}^{n} \mathrm{Res}_{z=p_i} \left\{ \frac{(1-g^2)f^2}{2} \right\} = -\sum_{i,j=1;\, i \neq j}^{n} b_i b_j \frac{p_i p_j - 1}{p_i - p_j} = 0 ,$$

$$\sum_{i=1}^{n} a_i v_{i2} = \sum_{i=1}^{n} \mathrm{Res}_{z=p_i} \left\{ \frac{\sqrt{-1}(1+g^2)f^2}{2} \right\} = -\frac{1}{\sqrt{-1}} \sum_{i,j=1;\, i \neq j}^{n} b_i b_j \frac{p_i p_j + 1}{p_i - p_j} = 0 ,$$

$$\sum_{i=1}^{n} a_i v_{i3} = \sum_{i=1}^{n} \mathrm{Res}_{z=p_i} \{ g f^2 \} = -\sum_{i,j=1;\, i \neq j}^{n} b_i b_j \frac{p_i + p_j}{p_i - p_j} = 0 .$$

3. Examples

First, we remark that the linear transformation $F_i(z)$ defined in Section 1 is identified with an isometry of the unit sphere $S^2=\sigma^{-1}(\hat{C})$ such that $F_i(p_i)=\infty$ and $F_i(-1/\bar{p}_i)=0$. Therefore, if the subset $\{v_j\}_{j=1}^k$ of $S^2-\{\sigma^{-1}(p_i)\}$ is invariant under the action of some nontrivial subgroup of $SO(3)$ which fixes $\sigma^{-1}(p_i)$, then clearly $\sum_{j=1}^k F_i(\sigma(v_j))=0$ (cf. Xu [9, Lemma 4.6]). By this observation, we get the following example without any more computation.

Example 3.1. (Families of polyhedrally symmetric minimal surfaces) Let P be a regular polyhedron inscribed to the unit sphere S^2 in $\boldsymbol{R}^3$, $\{v_j\}_{j=1}^k$ the set of the vertices of P, $\{v_j'\}_{j=1}^{k'}$ the set of the centers of the edges of P, and $\{v_j''\}_{j=1}^{k''}$ the set of the varycenters of the faces of P. It is well-known that

$$(k, k', k'')=\begin{cases} (4, 6, 4) & \text{if } P \text{ is a regular tetrahedron,} \\ (8, 12, 6) & \text{if } P \text{ is a cube,} \\ (6, 12, 8) & \text{if } P \text{ is a regular octahedron,} \\ (20, 30, 12) & \text{if } P \text{ is a regular dodecahedron,} \\ (12, 30, 20) & \text{if } P \text{ is a regular icosahedron.} \end{cases}$$

Set $p_j:=\sigma(v_j)$, $p_j':=\sigma(v_j'/|v_j'|)$, and $p_j'':=\sigma(v_j''/|v_j''|)$. For any real numbers b, b' and b'', define a surface $X_{(b,b',b'')}:\hat{\boldsymbol{C}}-\{p_1, \cdots, p_k, p_1', \cdots, p_{k'}', p_1'', \cdots, p_{k''}''\}\to\boldsymbol{R}^3$ by the data

$$f(z):=b\sum_{j=1}^{k}\frac{1}{z-p_j}+b'\sum_{j=1}^{k'}\frac{1}{z-p_j'}+b''\sum_{j=1}^{k''}\frac{1}{z-p_j''}, \quad \beta:=kb+k'b'+k''b''.$$

Then $\{X_{(b,b',b'')}\}$ is a 3-parameter family of minimal surfaces which are invariant under the action of the polyhedral group Γ_P corresponding to P. For a generic (b, b', b''), $X_{(b,b',b'')}$ has $k+k'+k''$ catenoid ends whose weights take 3 different values. More precisely, by using Lemma A.1 in Appendix, we see that, *for any positive numbers a, a' and a'', there exists a Γ_P-invariant $(k+k'+k'')$-end catenoid $X_{(b,b',b'')}$ such that*

$$g(p_j)=p_j, \quad w(p_j)=a \quad j=1, \cdots, k, \tag{3.1}$$

$$g(p_j')=p_j', \quad w(p_j')=a' \quad j=1, \cdots, k', \tag{3.2}$$

$$g(p_j'')=p_j'', \quad w(p_j'')=a'' \quad j=1, \cdots, k''. \tag{3.3}$$

When one of b, b' and b'' vanishes, since k, k' or k'' ends are removed, it has $k'+k''$, $k+k''$ or $k+k'$ ends. By using Lemma A.2, we see that, *for any non-zero real numbers a and a', there exists a Γ_P-invariant $(k+k')$-end catenoid $X_{(b,b',0)}:\hat{\boldsymbol{C}}-\{p_1, \cdots, p_k, p_1', \cdots, p_{k'}'\}\to\boldsymbol{R}^3$ satisfying the conditions* (3.1) *and* (3.2). Indeed, in the construction above, we may choose purely imaginary numbers b,

b' and b'' in place of real numbers, and, by Lemma A.2, we get $X_{(b,b',0)}$ as above. Of course, the same assertion holds also in the case with $b'=0$ or $b=0$.

Xu [9], Rossman [7] and Umehara-Yamada [8] studied the special cases of this type when two of b, b' and b'' vanish.

The dihedral version of this type is the following

Example 3.2. (Families of D_k-invariant minimal surfaces) Let k be an integer greater than 1. For any real numbers b, b' and b'', define a surface $X_{(b,b',b'')}: \hat{\boldsymbol{C}}-\{1, \zeta_{2k}, \cdots, \zeta_{2k}^{2k-1}, \infty, 0\} \to \boldsymbol{R}^3$ by the data

$$f(z):=b\frac{kz^{k-1}}{z^k-1}+b'\frac{kz^{k-1}}{z^k+1}+\frac{b''}{z}, \qquad \beta:=k(b+b')+2b''.$$

Then $\{X_{(b,b',b'')}\}$ is a 3-parameter family of D_k-invariant minimal surfaces. For a generic (b, b', b''), $X_{(b,b',b'')}$ has $2k+2$ catenoid ends whose weights take 3 different values. More precisely, by using Lemma A.1, we see that, *for any positive numbers a, a' and a'', there exists a D_k-invariant $(2k+2)$-end catenoid $X_{(b,b',b'')}$ such that*

$$g(\zeta_k^j)=\zeta_k^j, \qquad w(\zeta_k^j)=a \quad j=0, \cdots, k-1, \tag{3.4}$$

$$g(\zeta_{2k}^{2j-1})=\zeta_{2k}^{2j-1}, \qquad w(\zeta_{2k}^{2j-1})=a' \quad j=1, \cdots, k, \tag{3.5}$$

$$g(\infty)=\infty, \qquad g(0)=0, \qquad w(\infty)=w(0)=a''. \tag{3.6}$$

When b'' (resp. b')$=0$, it has $2k$ (resp. $k+2$) ends and the similar result as above also holds. It was partially obtained by Karcher [4] ($k=2$), Xu [9] and Rossman [7] ($k\geqq 3$). More generally, by Lemma A.2 and the same consideration as in Example 3.1, we see that, *for any non-zero real numbers a and a', there exists a D_k-invariant $2k$-end catenoid $X_{(b,b',0)}: \hat{\boldsymbol{C}}-\{1, \zeta_{2k}, \cdots, \zeta_{2k}^{2k-1}\} \to \boldsymbol{R}^3$ satisfying the conditions* (3.4) *and* (3.5), and that, *for any non-zero real numbers a and a'', there exists a D_k-invariant $(k+2)$-end catenoid $X_{(b,0,b'')}: \hat{\boldsymbol{C}}-\{1, \zeta_k, \cdots, \zeta_k^{k-1}, \infty, 0\} \to \boldsymbol{R}^3$ satisfying the conditions* (3.4) *and* (3.6).

When $b'=b''=0$, we get the examples in Jorge-Meeks [3].

By the consideration in Examples 3.1-2, we can observe that there are essentially different n-end catenoids with the same data $v_1, \cdots, v_n, a_1, \cdots, a_n$. For example, in Example 3.2, applying Lemma A.2 for $(k, k')=(k, 2)$, we see that, for any non-zero real numbers a and a'' such that $k^2a\neq 4a''$, there exist two D_k-invariant $(k+2)$-end catenoids $X_{(b_\pm,0,b''_\pm)}$ satisfying the conditions (3.4) and (3.6). Since the metric of $X_{(b,0,b'')}$ is given by

$$\frac{[|(kb+b'')z^k-b''|^2+|z\{(kb+b'')-b''z^k\}|^2]^2}{|z(z^k-1)|^4}|dz|^2$$

and $|b''_+|\neq|b''_-|$ if $k^2a\neq 2a''$, $X_{(b_+,0,b''_+)}$ and $X_{(b_-,0,b''_-)}$ are not isometric with each other for generic a and a''.

We can deform the surfaces in Example 3.2 to the following

Example 3.3. (Families of C_k-invariant minimal surfaces) Let k be an integer greater than 1. For any non-zero real number p and real numbers b and b', define the surface $X_{(p,b,b')}: \hat{\boldsymbol{C}}-\{p, p\zeta_k, \cdots, p\zeta_k^{k-1}, \infty, 0\} \to \boldsymbol{R}^3$ by the data

$$f(z):=b\frac{kz^{k-1}}{z^k-p^k}+\frac{b''}{z}, \qquad \beta:=kb+b'+b'',$$

where $b'':=(k-1)(p^2-1)b/2+p^2b'$. Then $\{X_{(p,b,b')}\}$ is a 3-parameter family of C_k-invariant minimal surfaces.

Indeed it is clear that the condition (1.2) holds with $p_i=\infty$ and 0, and we have only to check it with $p_i=p\zeta_k^l$ ($l=0, 1, \cdots, k-1$). By direct computation,

$$\begin{aligned} & b\sum_{j=1}^{k-1}\frac{p\zeta_k^{-l}p\zeta_k^{l+j}+1}{p\zeta_k^{l+j}-p\zeta_k^l}+b'\frac{p\zeta_k^{-l}\infty+1}{\infty-p\zeta_k^l}+b''\frac{p\zeta_k^{-l}0+1}{0-p\zeta_k^l} \\ &=b\sum_{j=1}^{k-1}\frac{p^2\zeta_k^j+1}{p\zeta_k^l(\zeta_k^j-1)}+b'\frac{p}{\zeta_k^l}-b''\frac{1}{p\zeta_k^l} \\ &=\frac{1}{p\zeta_k^l}\left\{b\sum_{j=1}^{k-1}\frac{p^2\zeta_k^j+1}{\zeta_k^j-1}+b'p^2-b''\right\} \\ &=\frac{1}{p\zeta_k^l}\left\{\frac{(k-1)(p^2-1)b}{2}+p^2b'-b''\right\} \\ &=0. \end{aligned}$$

Hence the surface $X_{(p,b,b')}$ is well-defined.

For a generic (p, b, b'), $X_{(p,b,b')}$ has $k+2$ catenoid ends whose weights take 3 different values. More precisely, by using Lemma A.3, we see that, *for any non-zero real number p and positive numbers a, a' and a'' satisfying $a\sum_{j=0}^{k-1}\sigma^{-1}(p\zeta_k^j)+a'\sigma^{-1}(\infty)+a''\sigma^{-1}(0)=0$ (i.e. $ak(p^2-1)/(p^2+1)+a'-a''=0$), there exists a C_k-invariant $(k+2)$-end catenoid $X_{(p,b,b')}$ such that*

$$g(p\zeta_k^j)=\zeta_k^j, \qquad w(p\zeta_k^j)=a \quad j=0, \cdots, k-1, \tag{3.7}$$

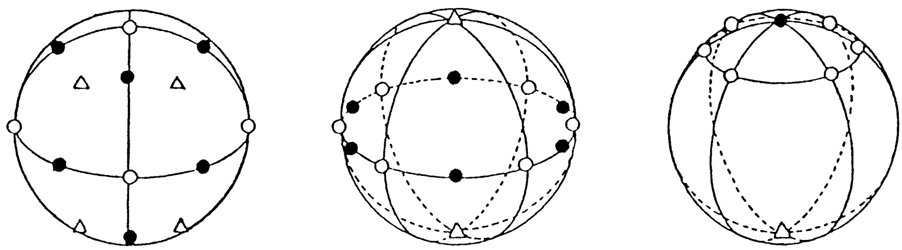
$$g(\infty)=\infty, \qquad w(\infty)=a', \tag{3.8}$$

$$g(0)=0, \qquad w(0)=a''. \tag{3.9}$$

Karcher [4] constructed this example with $k=2$.

When $b'=0$, since the end ∞ is removed, it has $k+1$ ends. Xu [9] constructed this example with $(p, b')=(\sqrt{(k+1)/(k-1)}, 0)$. However, more generally, by using Lemma A.2, we see that, *for any non-zero real number $p\neq\pm1$ and non-zero real numbers a and a'' satisfying $a\sum_{j=0}^{k-1}\sigma^{-1}(p\zeta_k^j)+a''\sigma^{-1}(0)=0$ (i.e. $ak(p^2-1)/(p^2+1)-a''=0$), there exists a C_k-invariant $(k+1)$-end catenoid $X_{(p,b,0)}$: $\hat{\boldsymbol{C}}-\{p, p\zeta_k, \cdots, p\zeta_k^{k-1}, 0\}\to\boldsymbol{R}^3$ satisfying the conditions* (3.7) *and* (3.9).

See Figure 3.1 for the arrangements of the ends. We point to the positions of the ends by the symbols ○, ● and △. Any two ends of different symbols may have weights different from each other.



Example 3.1 — P is a regular octahedron; Example 3.2 — $k=6$; Example 3.3 — $k=6$

Figure 3.1.

By the similar computation as in Example 3.3, we get various examples of families of n-end catenoids, e. g. as follows.

Example 3.4. (Example 3.2 plus $2k$ ends) Let k be an integer greater than 1. For any non-zero complex number ξ such that $\xi^{2k}\neq 1$, and any real numbers b, b', b'' and b''' satisfying

$$(3.10)\qquad 2k\{(\xi^k+|\xi|^2)(\xi^k+1)b+(\xi^k-|\xi|^2)(\xi^k-1)b'\}$$
$$=2(|\xi|^2-1)(\xi^{2k}-1)b''+\{(k-1)(|\xi|^2\xi^{2k}+1)-(3k-1)(\xi^{2k}+|\xi|^2)\}b''',$$

define a surface $X_{(\xi,b,b',b'',b''')}:\hat{\boldsymbol{C}}-\{1,\zeta_{2k},\cdots,\zeta_{2k}^{2k-1},\infty,0,\xi^{\pm1},\xi^{\pm1}\zeta_k,\cdots,\xi^{\pm1}\zeta_k^{k-1}\}\to\boldsymbol{R}^3$ by the data

$$f(z):=b\frac{kz^{k-1}}{z^k-1}+b'\frac{kz^{k-1}}{z^k+1}+\frac{b''}{z}+b'''\Big(\frac{kz^{k-1}}{z^k-\xi^k}+\frac{kz^{k-1}}{z^k-\xi^{-k}}\Big),\quad \beta:=k(b+b'+2b''')+2b''.$$

Since we assume (3.10), the condition (1.2) in Theorem is satisfied and $\{X_{(\xi,b,b',b'',b''')}\}$ is a family of D_k-invariant minimal surfaces. For a generic (ξ, b, b', b'', b''') satisfying the condition (3.10), $X_{(\xi,b,b',b'',b''')}$ has $4k+2$ catenoid ends whose weights take 4 different values. However, when some of b, b' and b'' vanish, it has $4k$, $3k+2$, $3k$, $2k+2$ or $2k$ ends.

Here we claim that, for any ξ, there are infinitely many $(b, b', b'', b''')\in\boldsymbol{R}^4$ satisfying the condition (3.10).

CASE 1. When $\xi^k\in\boldsymbol{R}$ i. e. $\xi=p\zeta_{2k}^j$ for some non-zero real number p $(\neq\pm1)$ and some integer j, the condition (3.10) is satisfied if and only if the following condition holds:

$$2kp^2[\{p^{k-2}+(-1)^j\}\{p^k+(-1)^j\}b+\{p^{k-2}-(-1)^j\}\{p^k-(-1)^j\}b']$$
$$=2(p^2-1)(p^{2k}-1)b''+\{(k-1)(p^{2k+2}+1)-(3k-1)p^2(p^{2k-2}+1)\}b'''.$$

Hence the claim above is justified. In this case, $X_{(\xi,b,b',b'',b''')}$ has the symmetry of a regular k-angular prism.

CASE 2. If we set $p:=|\xi|$ and $r:=\operatorname{Re}\xi^k/|\xi|^k$, then, by using the equality $\xi^{2k}=2rp^k\xi^k-p^{2k}$, we can rewrite the condition (3.10) as follows.

$$\begin{aligned}&2k[\{(2rp^k+p^2+1)\xi^k-p^2(p^{2k-2}-1)\}b+\{(2rp^k-p^2-1)\xi^k-p^2(p^{2k-2}-1)\}b']\\&\quad=2(p^2-1)\{2rp^k\xi^k-(p^{2k}+1)\}b''\\&\qquad+[(k-1)\{2rp^{k+2}\xi^k-(p^{2k+2}-1)\}-(3k-1)\{2rp^k\xi^k-p^2(p^{2k-2}-1)\}]b'''.\end{aligned}$$

Therefore, when $\xi^k\notin\boldsymbol{R}$, the condition (3.10) is satisfied if and only if both of the following conditions hold:

$$\begin{aligned}(3.11)\qquad&2kp^2(p^{2k-2}-1)(b+b')\\&\quad=2(p^2-1)(p^{2k}+1)b''+\{(k-1)(p^{2k+2}-1)-(3k-1)p^2(p^{2k-2}-1)\}b''';\end{aligned}$$

$$\begin{aligned}(3.12)\qquad&k\{(2rp^k+p^2+1)b+(2rp^k-p^2-1)b'\}\\&\quad=rp^k[2(p^2-1)b''+\{(k-1)p^2-(3k-1)\}b'''].\end{aligned}$$

Hence, also in this case, the claim above is justified. Let us observe this case more concretely.

(1) When $\xi^k\in\sqrt{-1}\boldsymbol{R}$ i.e. $\xi=p\zeta_{4k}^{2j-1}$ for some non-zero real number p and some integer j, the condition (3.10) is satisfied if and only if both of the following conditions hold:

$$\begin{aligned}&4kp^2(p^{2k-2}-1)b\\&\quad=2(p^2-1)(p^{2k}+1)b''+\{(k-1)(p^{2k+2}-1)-(3k-1)p^2(p^{2k-2}-1)\}b''';\\&b=b'.\end{aligned}$$

In this case, $X_{(\xi,b,b',b'',b''')}$ has the symmetry of a regular k-angular antiprism.

(2) When $\xi^{2k}\notin\boldsymbol{R}$ i.e. $\xi^k\notin\boldsymbol{R}\cup\sqrt{-1}\boldsymbol{R}$, the condition (3.10) is satisfied if and only if both of the following conditions hold:

$$\begin{aligned}&2(k-1)rp^k(p^2-1)(b+b')+\{(k-1)(p^{2k+2}-1)-(3k-1)p^2(p^{2k-2}-1)\}(b-b')\\&\quad=4rp^k(p^2-1)b'';\\&(2rp^k+p^{2k}+1)b+(2rp^k-p^{2k}-1)b'+2rp^kb'''=0.\end{aligned}$$

In this case, if $|\xi|\neq1$ and $b'''\neq0$, then $X_{(\xi,b,b',b'',b''')}$ has neither the symmetry of a regular k-angular prism nor the symmetry of a regular k-angular antiprism.

(3) When $|\xi|=1$, the condition (3.11) is automatically satisfied. Therefore the condition (3.10) is satisfied if and only if the following condition holds:

$$(r+1)b+(r-1)b'+rb'''=0.$$

In this case, we can choose b'' independently.

See Figure 3.2 for the arrangements of the additional ends $\xi^{\pm 1}\zeta_k^j$ in each case above.

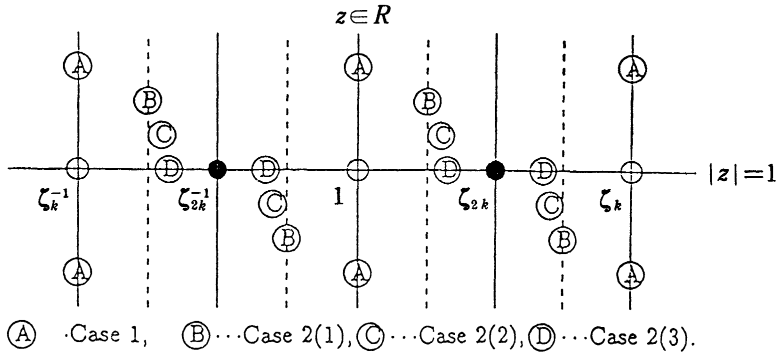


Figure 3.2.

Now we will describe a less symmetric

Example 3.5. (All of the 3-end catenoids) For any non-zero real numbers p and p' such that $(p-p')(pp'+1)\neq 0$, and any non-zero real or purely imaginary number b, define a surface $X_{(p,p',b)}: \hat{C}-\{p, p', 0\}\to R^3$ by the data

$$f(z):=b\left\{\frac{p(p-p')}{z-p}+\frac{p'(p'-p)}{z-p'}+\frac{pp'(pp'+1)}{z}\right\},\quad \beta:=b(p^2p'^2+p^2+p'^2-pp').$$

This representation gives the affirmative answer to our problem in the case when $n=3$ and v_i's are different from each other. Conversely, all of the 3-end catenoids are described by this. Lopez [5] recently proved this result by somewhat different but essentially the same representation.

In the case $n\geqq 4$, our problem is still open.

Appendix

Here we prove three lemmas which were used in the previous section.

LEMMA A.1. *Let n be an integer greater than* 1, *and $a_1, \cdots, a_n$ positive numbers. If there are two indices i_1 and i_2 such that $a_{i_1}=a_{i_2}=\max_{1\leq i\leq n} a_i$, then there are positive numbers, $b_1, \cdots, b_n$ satisfying*

$$(1.1)\qquad b_i \textstyle\sum_{j\in N_i} b_j=a_i \qquad i=1, \cdots, n\,.$$

In particular, if $a_i=a_j$ for some i and j, then we can choose b_i and b_j as the same value.

Proof. We may assume $i_1=1$ and $i_2=2$ without loss of generality. Set

$$\varphi(t):=(n-2)t-\sum_{i=1}^{n}\sqrt{t^2-4a_i}\,.$$

It is clear that $\varphi(t)$ is a continuous function on $[2\sqrt{a_1}, +\infty)$ and $\lim_{t\to+\infty}\varphi(t)<0$. On the other hand, by the assumption, we have

$$\begin{aligned}\varphi(2\sqrt{a_1})&=2(n-2)\sqrt{a_1}-2\sum_{i=1}^{n}\sqrt{a_1-a_i}\\&=2\sum_{i=3}^{n}(\sqrt{a_1}-\sqrt{a_1-a_i})\\&\geqq 0\,.\end{aligned}$$

Hence, by the intermediate value theorem, there is a positive number $\tau\geqq 2\sqrt{a_1}$ such that $\varphi(\tau)=0$. Set

$$b_i:=\frac{\tau-\sqrt{\tau^2-4a_i}}{2}\qquad i=1,\cdots,n\,.$$

Then, for any i, b_i is positive and

$$\begin{aligned}b_i\sum_{j\in N_i}b_j&=\frac{\tau-\sqrt{\tau^2-4a_i}}{2}\sum_{j\in N_i}\frac{\tau-\sqrt{\tau^2-4a_j}}{2}\\&=\frac{\tau-\sqrt{\tau^2-4a_i}}{2}\times\frac{(n-1)\tau-\{(n-2)\tau-\sqrt{\tau^2-4a_i}\}}{2}\\&=a_i\,,\end{aligned}$$

namely, $b_1,\cdots,b_n$ satisfy the condition (1.1). q.e.d.

LEMMA A.2. *Let k be an integer greater than 1, k' a positive integer, and a and a' non-zero real numbers. Then there is at least one (b, b') which belongs to either $\boldsymbol{R}^2$ or $(\sqrt{-1}\boldsymbol{R})^2$ and satisfies*

$$\text{(A.1)}\qquad\begin{cases}b\{(k-1)b+k'b'\}=a\,,\\ b'\{kb+(k'-1)b'\}=a'\end{cases}$$

and $kb+k'b'\neq 0$, if $k'>1$, or if $k'=1$ and $a'\neq ka$, k^2a.

Proof. When $k'>1$, solving the system of quadric equations (A.1), we get a solution $(b, b')=(b_\pm, b'_\pm)$, where

$$b_\pm:=\sqrt{\frac{k^2a^2-k'^2a'^2-D\pm 2k'a'\sqrt{D}}{4(k-1)(k+k'-1)a'}}\,,\qquad b'_\pm:=\frac{a-(k-1)b_\pm^2}{k'b_\pm}$$

and

$$D:=k^2a^2+2(kk'-2k-2k'+2)aa'+k'^2a'^2\,.$$

By the assumption and direct computation, we have $D>0$ and $k^2a^2-k'^2a'^2-D \pm 2k'a'\sqrt{D}\neq 0$. Therefore, we can easily see that $(b_\pm, b'_\pm)$ belongs to either $\boldsymbol{R}^2$ or $(\sqrt{-1}\boldsymbol{R})^2$. Moreover, since

$$kb_\pm+k'b'_\pm=\frac{a+b_\pm^2}{b_\pm}=\frac{a'+b'^2_\pm}{b'_\pm},$$

we have $kb_+ + k'b'_+\neq 0$ or $kb_- + k'b'_-\neq 0$. Indeed, if $k^2a\neq k'^2a'$ (resp. $k^2a=k'^2a'$), then each (resp. one) of $(b_\pm, b'_\pm)$ satisfies $kb_\pm+k'b'_\pm\neq 0$.

On the other hand, when $k'=1$ and $a'\neq ka$, k^2a, solving the system of quadric equations (A.1), we get a solution

$$(b, b'):=\left(\sqrt{\frac{ka-a'}{k(k-1)}}, \frac{a'}{kb}\right).$$

We can easily see that (b, b') belongs to either $\boldsymbol{R}^2$ or $(\sqrt{-1}\boldsymbol{R})^2$. Moreover, we have

$$kb+b'=\frac{k^2a-a'}{k(k-1)b}\neq 0.$$

q. e. d.

LEMMA A.3. *Let k be an integer greater than* 1, *and a, a' and a'' positive numbers. If* $\max\{a, a''\}\leqq a'<ka+a''$ *or* $\max\{a, a'\}\leqq a''<ka+a'$, *then there are positive numbers b, b' and b'' satisfying*

$$\text{(A.2)}\qquad \begin{cases} b\{(k-1)b+b'+b''\}=a\,, \\ b'(kb+b'')=a'\,, \\ b''(kb+b')=a''\,. \end{cases}$$

Proof. Solving the system of quadric equations (A.2), we get a solution (b, b', b''), where

$$b:=\sqrt{\frac{\sqrt{D}-k(a'+a''-a)}{k(k^2-1)}},$$

$$b':=\frac{k^2a+(2k+1)a'-a''-\sqrt{D}}{2k(k+1)b},$$

$$b'':=\frac{k^2a+(2k+1)a''-a'-\sqrt{D}}{2k(k+1)b}$$

and

$$D:=k^4a^2+a'^2+a''^2-2k^2(a'+a'')a+2(2k^2-1)a'a''\,.$$

By the assumption and direct computation, we have $D>0$ and

$$0<k(a'+a''-a)<\sqrt{D}<\min\{k^2a+(2k+1)a'-a'',\ k^2a+(2k+1)a''-a'\}.$$

Therefore, we can easily see that b, b' and b'' are positive numbers. q. e. d.

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DEPARTMENT OF MATHEMATICS
NARA WOMEN'S UNIVERSITY
NARA 630, JAPAN

Present Address:
DEPARTMENT OF MATHEMATICS
OSAKA UNIVERSITY
TOYONAKA, OSAKA 560, JAPAN