G. CHEN AND X. ZOU
KODAI MATH. J.
18 (1995), 75—85

RIGIDITY OF COMPACT SUBMANIFOLDS IN A UNIT SPHERE

GUANGHUA CHEN AND XIAORONG ZOU

Abstract

In this paper, we prove a Pinching theorem for compact submanifolds with non-zero parallel mean curvature, which improve the Pinching constant in [5]. For lower dimensional compact submanifolds we obtain a strong result. Meanwhile, we study the Pinching problem for the sectional curvatures of minimals submanifolds, and obtain the best Pinching constant so far.

§ 1. Introduction

Let M^n be a smooth compact n-dimensional Riemann manifold immersed in a unit sphere S^{n+p} of dimension $(n+p)$, and let S be the square of the length of the second fundamental form. S. T. Yau [5] proved that if

$$S \leq \frac{n}{3+\sqrt{n}-(p-1)^{-1}} \tag{1.1}$$

everywhere on M, then M^n lies in a totally geodesic S^{n+1}. An estimate of the value for S next to $\frac{n}{3+\sqrt{n}-(p-1)^{-1}}$ should be of intetest. We give the best Pinching constant so far in Theorem 1.

On the other hand, we know from [2] that if M is a minimal submanifold in S^{n+p} and $S \leqq (2/3)n$, then either M is totally geodesic or M is the Verovese surface in S^4. For submanifolds of lower dimension which have non-zero parallel mean curvatures, we have similar results written as Theorem 2.

Simons [4] proved that if the average of the sectional curvatures of a compact minimal submanifold in S^{n+p} is greater than $1-\frac{1}{(n-1)(2-(1/n))}$, then it must be totally geodesic. Later, S. T. Yau [5] proved that if the sectional curvatures of the submanifold are greater than $(p-1/2p-1)$, then the same conclusion holds. This paper will give an improvement of the Pinching constant which will be explained in our Theorem 3.

Now, our main results are showed as follows:

Received August 25, 1993.

THEOREM 1. *Let M^n be an n-dimensional compact submanifold in S^{n+p} with non-zero parallel mean curvature. If either of the following conditions is satisfied, then M^n lies in a totally geodesic S^{n+1}:*

$$S \leqq \min\left\{\frac{2}{3}n, \frac{2n}{1+\sqrt{\frac{n}{2}}}\right\}, \qquad p>2 \text{ and } n \neq 8, \tag{1.2}$$

$$S \leqq \min\left\{\frac{n}{2-\frac{1}{p-1}}, \frac{2n}{1+\sqrt{\frac{n}{2}}}\right\}, \qquad p>1 \text{ and } (n, p) \neq (8, 3). \tag{1.3}$$

THEOREM 2. *Let M^n be an n-dimensional ($2 \leqq n \leqq 7$) compact submanifold in S^{n+p} with $p>2$ which has non-zero parallel mean curvature. If $S \leqq (2/3)n$, then M^n is totally umbilical.*

THEOREM 3. *Let M^n be a compact minimal submanifold in the sphere S^{n+p} with $p>1$. Suppose the sectional curvatures of M^n are everywhere not less than $(1/2-1/3p)$. Then either M^n is a totally geodesic sphere or the Verovese surface in S^4.*

Remark 1. It is clear that the constants in (1.2) and (1.3) are some improvements of the constant in (1.1).

Remark 2. It is easy to see that the Pinching constant for the sectional curvature in Theorem 3 is always less than $(p-1/2p-1)$, once $p>1$.

§ 2. Preliminaries

Let M^n be a compact n-dimensional submanifold of unit sphere S^{n+p}. We choose a local field of adapted orthonormal frames $e_1, \cdots, e_{n+p}$ in S^{n+p} such that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. We shall make use of the following convention on the ranges of indices:

$$1 \leqq A, B, C, \cdots \leqq n+p; \qquad 1 \leqq i, j, k, \cdots \leqq n;$$
$$n+1 \leqq \alpha, \beta, \gamma, \cdots \leqq n+p,$$

and we shall agree that repeated indices are summed over the respective ranges with respect to the frame field of S^{n+p} chosen above. Let $\omega_1, \cdots, \omega_{n+p}$ be the field of dual frames. Then the structure equations of S^{n+p} are given by

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B \qquad \omega_{AB} + \omega_{BA} = 0 \tag{2.1}$$
$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \Phi_{AB}$$

$$\Phi_{AB}=\frac{1}{2}\sum_{C,D}K_{ABCD}\omega_C\wedge\omega_D \tag{2.2}$$

$$K_{ABCD}+K_{ABDC}=0\,.$$

We restrict these forms to M. Then we have

$$\omega_\alpha=0 \tag{2.3}$$

$$\omega_{\alpha i}=\sum_j h_{ij}^\alpha\omega_j\,,\qquad h_{ij}^\alpha=h_{ij}^\alpha \tag{2.4}$$

$$d\omega_i=-\sum_j\omega_{ij}\wedge\omega_j\,,\qquad \omega_{ij}+\omega_{ji}=0 \tag{2.5}$$

$$d\omega_{ij}=-\sum_k\omega_{ik}\wedge\omega_{kj}+\Omega_{ij}$$

$$\Omega_{ij}=\frac{1}{2}\sum_{k,l}R_{ijkl}\omega_k\wedge\omega_l \tag{2.6}$$

$$R_{ijkl}=K_{ijkl}+\sum_\alpha(h_{ik}^\alpha h_{jl}^\alpha-h_{il}^\alpha h_{jk}^\alpha) \tag{2.7}$$

$$d\omega_{\alpha\beta}=-\sum_\gamma\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}+\Omega_{\alpha\beta} \tag{2.8}$$

$$\Omega_{\alpha\beta}=\frac{1}{2}\sum_{k,l}R_{\alpha\beta kl}\omega_k\wedge\omega_l$$

$$R_{\alpha\beta kl}=K_{\alpha\beta kl}+\sum_i(h_{ik}^\alpha h_{il}^\beta-h_{il}^\alpha h_{ik}^\beta)\,. \tag{2.9}$$

Let $B=\sum_{i,j,\alpha}h_{ij}^\alpha\omega_i\omega_j e_\alpha$ be the second fundamental form of M, and $S=\|B\|^2=\sum_{i,j,\alpha}(h_{ij}^\alpha)^2$ be the square length of B. We denote H_α the matrix (h_{ij}^α). We call $\eta=(1/n)\sum_\alpha\operatorname{tr}H_\alpha e_\alpha$ the mean curvature vector, and its length of called the mean curvature, i. e. $H=\|\eta\|$.

An immersion is said to be minimal if

$$\operatorname{tr}H_\alpha=0\,,\qquad n+1\leqq\alpha\leqq n+p\,. \tag{2.10}$$

If the vector η is not zero and parellel in the normal bundle of M, letting $e_{n+1}=\eta/\|\eta\|$, we obtain that H is a non-zero constant and

$$\operatorname{tr}H_\alpha=0,\quad \alpha\neq n+1,\qquad \operatorname{tr}H_{n+1}=nH \tag{2.11}$$

$$\sum_i h_{iijk}^\alpha=0\,,\qquad n+1\leqq\alpha\leqq n+p\,, \tag{2.12}$$

where h_{ijkl}^α is defined as in [5].

$$\omega_{\alpha n+1}=0\,,$$

$$H_\alpha H_{n+1}=H_{n+1}H_\alpha\,. \tag{2.13}$$

We define Δh_{ij}^α by

$$\Delta h_{ij}^{\alpha}=\sum_{k} h_{ijkk}^{\alpha}\,.$$

From [5], we have

$$\Delta h_{ij}^{\beta}=\sum_{k} h_{kkij}^{\beta}+\sum_{k}\Big(\sum_{m} h_{mk}^{\beta} R_{mijk}+\sum_{m} h_{im}^{\beta} R_{mkjk}+\sum_{\alpha\neq n+1} h_{ki}^{\alpha} R_{\alpha\beta jk}\Big). \tag{2.14}$$

§ 3. Proof of Theorems

LEMMA 1. *Suppose $b_1, b_2, \cdots, b_n$ are n real numbers such that $\sum_{i=1}^{n} b_i=0$. Then we have*

$$2\sum_{i=1}^{n} b_i^4 \leqq \Big(\sum_{i=1}^{n} b_i^2\Big)^2. \tag{3.1}$$

In particular if $n=2$, the equality holds.

Proof. The proof is based on induction on n as follows.

(i) When $n=2$, the inequality (3.1) holds obviously.

(ii) Suppose the inequality (3.1) holds for $n=m-1$. Then we only need to established the inequality (3.1) for $n=m$.

For fixed i and j such that $1\leqq i<j\leqq m$, from our assumption, we have

$$2\Big[\sum_{\substack{1\leq p\leq m\\ p\neq i,j}} b_p^4+(b_i+b_j)^4\Big]\leqq\Big[\sum_{\substack{1\leq p\leq m\\ p\neq i,j}} b_p^2+(b_i+b_j)^2\Big]^2$$

$$=\Big(\sum_{p=1}^{m} b_p^2\Big)^2+4b_ib_j\sum_{p=1}^{m} b_p^2+4b_i^2b_j^2\,.$$

So

$$2\sum_{p=1}^{m} b_p^4\leqq\Big(\sum_{p=1}^{m} b_p^2\Big)^2+4b_ib_j\sum_{p=1}^{m} b_p^2-8b_i^2b_j^2-8b_ib_j^3-8b_i^3b_j\,. \tag{3.2}$$

By summing up (3.2) over index i, j $(1\leqq i, j\leqq m)$, we have

$$2C_m^2\sum_{p=1}^{m} b_p^4\leqq C_m^2\Big(\sum_{p=1}^{m} b_p^2\Big)^2+4\sum_{i<j} b_ib_j\sum_{p=1}^{m} b_p^2-8\sum_{i<j} b_i^2b_j^2-8\sum_{i<j} b_ib_j^3-8\sum_{i<j} b_i^3b_j$$

which implies from the symetry of i and j,

$$4C_m^2\sum_{p=1}^{m} b_p^4\leqq 2C_m^2\Big(\sum_{p=1}^{m} b_p^2\Big)^2+4\sum_{i\neq j} b_ib_j\sum_{p=1}^{m} b_p^2-8\sum_{i\neq j} b_i^2b_j^2-16\sum_{i\neq j} b_ib_j^3$$

$$=(2C_m^2-12)\Big(\sum_{p=1}^{m} b_p^2\Big)^2+24\sum_{p=1}^{m} b_p^4\,.$$

So we have (3.1) for $n=m$. Q. E. D.

LEMMA 2. *Suppose $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$ are $2n$ real numbers satisfying*

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1$$

$$\sum_{i=1}^{n} b_i = 0 .$$

Then we have

$$\sum_{1 \leq i, j \leq n} a_i a_j (b_i - b_j)^2 \leq 1 + \sqrt{\frac{n}{2}} . \tag{3.3}$$

Proof. The inequality (3.3) follows if we can prove:

$$\sum_{1 \leq i, j \leq n} a_i a_j (b_i^2 + b_j^2) \leq 1 + \sqrt{\frac{n}{2}} .$$

Let $a_{ij} = b_i^2 + b_j^2$, so we get a symmetric matrix $A = (a_{ij})$ and

$$\sum_{1 \leq i, j \leq n} a_i a_j (b_i^2 + b_j^2) = (a_1 \quad a_2 \quad \cdots \quad a_n) A \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} .$$

It is easy to see

$$A = \begin{pmatrix} b_1^2 & 1 \\ b_2^2 & 1 \\ \vdots & \vdots \\ b_n^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ b_1^2 & b_2^2 & \cdots & b_n^2 \end{pmatrix}$$

which means rank $A \leqq 2$. Therefore what we need to prove is

$$\max\{\lambda_1, \lambda_2\} \leqq 1 + \sqrt{\frac{n}{2}} \tag{3.4}$$

where λ_1, λ_2 are two possiblely non-zero eigenvalues of A.

On the other hand, it is well known that

$$\lambda_1 + \lambda_2 = \operatorname{tr} A = 2 \sum_{i=1}^{n} b_i^2 = 2$$

$$\lambda_1 \lambda_2 = \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{vmatrix} = - \sum_{i<j} (b_i^2 - b_j^2)^2$$

$$= -\frac{1}{2} \sum_{i,j} (b_i^2 - b_j^2)^2 = -n \sum_{i=1}^{n} b_i^4 + 1 .$$

So we obtain from Lemma 1 that

$$\lambda_1 = 1 + \sqrt{n \sum_{i=1}^{n} b_i^4} \leqq 1 + \sqrt{\frac{n}{2}}$$

$$\lambda_2=1-\sqrt{n\sum_{i=1}^{n} b_i^4}\leqq 1\,.$$

So (3.4) is correct and therefore we complete the proof of Lemma 2.

From Lemma 1 and Lemma 2, we can easily obtain

LEMMA 3. *Suppose $a_1, \cdots, a_n ; b_1, \cdots, b_n$ are real numbers, and $\sum_i b_i=0$, then*

$$\sum_{i,j} a_i a_j (b_i-b_j)^2 \leqq \left(1+\sqrt{\frac{n}{2}}\right)\left(\sum_i a_i^2\right)\left(\sum_j b_j^2\right).$$

In particular, if $n=2$, then the equality holds iff $a_1=a_2$ or $b_1=0$.

Since the following inequality holds:

$$\left|\sum_{i,j} a_i a_j (b_i-b_j)^2\right| \leqq \sum_{i,j} |a_i|\,|a_j|(b_i-b_j)^2\,.$$

So we can easily obtain:

LEMMA 4. *Suppose $a_1, \cdots, a_n ; b_1, \cdots, b_n$ are real numbers, and $\sum_i b_i=0$. Then*

$$\left|\sum_{i,j} a_i a_j (b_i-b_j)^2\right| \leqq \left(1+\sqrt{\frac{n}{2}}\right)\left(\sum_i a_i^2\right)\left(\sum_j b_j^2\right).$$

In particular, if $n=2$, then the equality holds iff $a_1=a_2$ or $b_1=0$.

In [2], Li proved that

LEMMA 5. *Suppose $A_1, \cdots, A_p$ are symmetric with $p\geqq 2$, denoting*

$$S_{\alpha\beta}=\mathrm{tr}\, A_\alpha A_\beta^t, \quad N(A_\alpha)=S_\alpha=S_{\alpha\alpha}, \quad S=\sum_{\alpha=1}^{p} S_\alpha\,.$$

Then we have

$$\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha)+\sum_{\alpha,\beta} S_{\alpha\beta}^2 \leqq \frac{3}{2} S^2\,, \tag{3.5}$$

In the equality holds, then at most two of the matrices are non-zero, and these two matrice can be transformed simultaneously by an orthogonal matrix in scalar multiples of $\tilde{A}$ and $\tilde{B}$ respectively where

$$\tilde{A}=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \tilde{B}=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

LEMMA 6. *Let M^n be a compact hypersurface of the unit sphere S^{n+1} with* *non-zero constant mean curvature H. If $S<2\sqrt{n-1}$ for $n>2$ and $S<2+4H^2$ for $n=2$, then M^n is totally umbilical.*

Proof. It is proved in [3], if

$$S<nH^2+\left(\sqrt{\frac{n^3H^2}{4(n-1)}+n}-\frac{1}{2}\sqrt{\frac{(n-2)^2n}{n-1}}H\right)^2 \tag{3.6}$$

$$=\frac{n^3}{2(n-1)}H^2+n-\frac{n(n-2)}{2(n-1)}H\sqrt{n^2H^2+4(n-1)}$$

M has to be totally umbilical. We denote

$$g(H)=\frac{n^3}{2(n-1)}H^2+n-\frac{n(n-2)}{2(n-1)}H\sqrt{n^2H^2+4(n-1)}$$

and considering the minimum of $g(H)$, we can deduce that the minimum of $g(H)$ is $2\sqrt{n-1}$ for $n>2$. Thus we complete the proof of Lemma 6.

Now we begin the proof of Theorem 1.

Based on (2.12), the equation (2.19) gives

$$\Delta h_{ij}^{\beta}=\sum_{k,m}h_{km}^{\beta}R_{mijk}+\sum_{k,m}h_{mi}^{\beta}R_{mkjk}-\sum_{k,\alpha\neq n+1}h_{ki}^{\alpha}R_{\beta\alpha jk},\qquad \beta\neq n+1 \tag{3.7}$$

The Gauss equation (2.7) and the Ricci equation (2.9) then imply

$$\begin{aligned}\Delta h_{ij}^{\beta}=&\sum_{\alpha,k,m}h_{km}^{\beta}h_{mj}^{\alpha}h_{ik}^{\alpha}-\sum_{\alpha,k,m}h_{km}^{\beta}h_{mk}^{\alpha}h_{ij}^{\alpha}+\sum_{\alpha,k,m}h_{mi}^{\beta}h_{mj}^{\alpha}h_{kk}^{\alpha}\\&-\sum_{\alpha,k,m}h_{mi}^{\beta}h_{mk}^{\alpha}h_{kj}^{\alpha}+nh_{ij}^{\beta}-\sum_{\alpha\neq n+1,k,m}h_{ki}^{\alpha}h_{mj}^{\beta}h_{mk}^{\alpha}+\sum_{\substack{\alpha\neq n+1\\k,m}}h_{ki}^{\alpha}h_{mk}^{\beta}h_{mj}^{\alpha},\\&\beta\neq n+1\,.\end{aligned} \tag{3.8}$$

So we can give the following equality immediately

$$\begin{aligned}\sum_{\substack{\beta\neq n+1\\i,j}}h_{ij}^{\beta}\Delta h_{ij}^{\beta}=&\sum_{\beta\neq n+1}\operatorname{tr}(H_{n+1}H_{\beta})^2-\sum_{\beta\neq n+1}[\operatorname{tr}(H_{n+1}H_{\beta})]^2\\&+nH\sum_{\beta\neq n+1}\operatorname{tr}(H_{n+1}H_{\beta}^2)-\sum_{\beta\neq n+1}\operatorname{tr}(H_{n+1}^2H_{\beta}^2)+n\sum_{\substack{\beta\neq n+1\\i,j}}(h_{ij}^{\beta})^2\\&+\sum_{\alpha,\beta\neq n+1}\operatorname{tr}(H_{\alpha}H_{\beta}-H_{\beta}H_{\alpha})^2-\sum_{\alpha,\beta\neq n+1}(\operatorname{tr}(H_{\alpha}H_{\beta}))^2\,.\end{aligned} \tag{3.9}$$

Following (2.13), Lemma 4 and Lemma 5, one can then prove

$$\begin{aligned}\sum_{\substack{\beta\neq n+1\\i,j}}h_{ij}^{\beta}\Delta h_{ij}^{\beta}\geqq& nH\sum_{\beta\neq n+1}\operatorname{tr}(H_{n+1}H_{\beta}^2)-\sum_{\beta\neq n+1}[\operatorname{tr}(H_{n+1}H_{\beta})]^2\\&+n\sum_{\substack{\beta\neq n+1\\i,j}}(h_{ij}^{\beta})^2-\frac{3}{2}\Big[\sum_{\substack{\beta\neq n+1\\i,j}}(h_{ij}^{\beta})^2\Big]^2\,.\end{aligned} \tag{3.10}$$

Now fix a vector e_β $(\beta\neq n+1)$. From (2.11) and (2.13), H_{n+1} and H_β are diagonalized simultaneously. Then we have

$$\begin{aligned}(3.11)\qquad & nH\,\mathrm{tr}\,(H_{n+1}H_\beta^2)-[\mathrm{tr}\,(H_{n+1}H_\beta)]^2\\ &=\sum_{i,j} h_{ii}^{n+1}h_{jj}^{n+1}(h_{jj}^\beta)^2-(\sum_i h_{ii}^{n+1}h_{ii}^\beta)^2\\ &=\sum_{i,j}[h_{ii}^{n+1}h_{jj}^{n+1}(h_{jj}^\beta)^2-h_{ii}^{n+1}h_{ii}^\beta h_{jj}^{n+1}h_{jj}^\beta]\\ &=\frac{1}{2}\sum_{i,j} h_{ii}^{n+1}h_{jj}^{n+1}(h_{ii}^\beta-h_{jj}^\beta)^2\,.\end{aligned}$$

Notice that (2.11), from Lemma 4 we have

$$\begin{aligned}(3.12)\qquad & nH\,\mathrm{tr}\,(H_{n+1}H_\beta^2)-[\mathrm{tr}\,(H_{n+1}H_\beta)]^2\\ &\geqq -\frac{1}{2}\Big(1+\sqrt{\frac{n}{2}}\Big)\Big(\sum_i (h_{ii}^{n+1})^2\Big)\Big(\sum_j (h_{jj}^\beta)^2\Big)\\ &\geqq -\frac{1}{2}\Big(1+\sqrt{\frac{n}{2}}\Big)\Big(\sum_{i,j} (h_{ij}^{n++1})^2\Big)\Big(\sum_{i,j} (h_{ij}^\beta)^2\Big)\,.\end{aligned}$$

Substituting (3.12) into (3.10), we can straightly see that

$$\begin{aligned}(3.13)\qquad \sum_{\substack{\beta\neq n+1\\ i,j}} h_{ij}^\beta\Delta h_{ij}^\beta &\geqq \sum_{\substack{\beta\neq n+1\\ i,j}}(h_{ij}^\beta)^2\Big[n-\frac{1}{2}\Big(1+\sqrt{\frac{n}{2}}\Big)\sum_{i,j}(h_{ij}^{n+1})^2-\frac{3}{2}\sum_{\substack{\beta\neq n+1\\ i,j}}(h_{ij}^\beta)^2\Big]\\ &\geqq \sum_{\substack{\beta\neq n+1\\ i,j}}(h_{ij}^\beta)^2(n-MS)\end{aligned}$$

where $M=\max\{1/2(1+\sqrt{n/2}),\,3/2\}$. If $S\leqq n/M$, we have

$$(3.14)\qquad \frac{1}{2}\Delta\sum_{\substack{\beta\neq n+1\\ i,j}}(h_{ij}^\beta)^2=\sum_{\substack{\beta\neq n+1\\ i,j,k}}(h_{ijk}^\beta)^2+\sum_{\substack{\beta\neq n+1\\ i,j}}h_{ij}^\beta\Delta h_{ij}^\beta\geqq 0\,.$$

So, it follows that $\sum_{\substack{\beta\neq n+1\\ i,j}}(h_{ij}^\beta)^2$ is constant by the Hopf maximum principle. Then (3.14) becomes equality, and the right hand side of (3.14) must be zero. In particular

$$\sum_{\substack{\beta\neq n+1\\ i,j}}(h_{ij}^\beta)^2(n-MS)=0\,.$$

If

$$\sum_{\substack{\beta\neq n+1\\ i,j}}(h_{ij}^\beta)^2=0 \qquad (*)$$

it is well known that M lies in a totally geodesic S^{n+1} from Theorem 1 of [5]. We will prove that the equality (*) always holds. Now we assume $\sum(h_{ij}^\beta)^2\neq 0$. Then we have the following two cases.

CASE (1). Where $n>8$. It is easy to deduce the equlity (*) holds.

CASE (2). Where $n<8$. Combining with Lemma 4, we obtain $\sum_{i,j}(h_{ij}^{n+1})^2$

$=0$ and $H=0$.

The case (1) is a contradiction to the assumption and the result deduced in the case (2) is contradictory to the hypothesis of $H\neq 0$.

On the other hand, by Simons' approach in [1], we can substitute the number 3/2 above by $(2-(1/p-1))$, using the same argument as above and Theorem 1 of [5], we can also prove the equality (*) holds.

This completes the proof of Theorem 1.

The proof of Theorem 2 is based on Theorem 1 and Lemma 6. In fact when $2\leqq n\leqq 7$, we have

$$\min\left\{\frac{2}{3}n,\ \frac{2n}{1+\sqrt{\frac{n}{2}}}\right\}=\frac{2}{3}n\,.$$

By Theorem 1, if $S\leqq(2/3)n$, it follows that the codimension is reduced to 1, i.e. M lies in S^{n+1}. The square of the length of the second fundamental form as a hypersurface still equals S. Lemma 6 tell us that M has to be totally umbilical.

Let us now turn to the proof of Theorem 3. It follows from (2.14) and (2.7) that

$$\sum_{i,j,\alpha} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}=\sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{mk}^{\alpha}R_{mijk}+\sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{im}^{\alpha}R_{mkjk} -\sum_{\alpha,\beta,i,j,k,l} h_{ij}^{\alpha}h_{ki}^{\beta}(h_{lj}^{\alpha}h_{lk}^{\beta}-h_{lk}^{\alpha}h_{lj}^{\beta})\,. \tag{3.15}$$

The first two terms together on the right hand side of (3.15) is equal to

$$\sum_{\alpha,\beta}\mathrm{tr}\,(H_\alpha^2H_\beta^2)-\sum_{\alpha,\beta}\mathrm{tr}\,(H_\alpha H_\beta)^2-\sum_{\alpha,\beta}(\mathrm{tr}\,H_\alpha H_\beta)^2 +\sum_{\alpha,\beta}(\mathrm{tr}\,H_\beta)[\mathrm{tr}\,(H_\alpha^2H_\beta)]-\sum_{\alpha}(\mathrm{tr}\,H_\alpha)^2+nS\,. \tag{3.16}$$

Hence for any real number a, we can get

$$\begin{aligned}\sum_{i,j,\alpha} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}=&(1+a)\sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{mk}^{\alpha}R_{mijk}+(1+a)\sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{im}^{\alpha}R_{mkjk}\\ &-(1-a)\sum_{\alpha,\beta}\mathrm{tr}\,(H_\alpha^2H_\beta^2)+(1-a)\sum_{\alpha,\beta}\mathrm{tr}\,(H_\alpha H_\beta)^2+a\sum_{\alpha,\beta}(\mathrm{tr}\,H_\alpha H_\beta)^2\\ &-a\sum_{\alpha,\beta}(\mathrm{tr}\,H_\beta)[\mathrm{tr}\,(H_\alpha^2H_\beta)]+a\sum_{\alpha}(\mathrm{tr}\,H_\alpha)^2-naS\,.\end{aligned} \tag{3.17}$$

For a fixed α, let α_i be the eigenvalues of the matrix H_α. Then

$$\sum_{i,j,k,m} h_{ij}^{\alpha}h_{mk}^{\alpha}R_{mijk}+\sum_{i,j,k,m} h_{ij}^{\alpha}h_{im}^{\alpha}R_{mkjk} =\frac{1}{2}\sum_{i,j}(\alpha_i-\alpha_j)^2R_{ijij} \tag{3.18}$$

$$\geqq \frac{1}{2}\sum_{i,j}(\alpha_i-\alpha_j)^2 K_M$$

$$=nK_M\sum_{i,j}(h_{ij}^{\alpha})^2$$

where K_M denote the function which assigns to each point of M the infinimum of the sectional curvature of M at that point.

We can choose adapted frame $e_{n+1}, \cdots, e_{n+p}$, so that matrix $(S_{\alpha\beta})=(\mathrm{tr}\,(H_\alpha H_\beta))$ is diagonalized, i. e.

(3.19) $$S_{\alpha\beta}=S_\alpha\delta_{\alpha\beta}\,.$$

It is easy to see

(3.20) $$\sum_\alpha(\mathrm{tr}\,H_\alpha^2)^2\geqq\frac{1}{p}S\,.$$

From Lemma 5, we have

(3.21) $$\sum_{\alpha,\beta}\mathrm{tr}\,(H_\alpha^2H_\beta^2)-\sum_{\alpha,\beta}\mathrm{tr}\,(H_\alpha H_\beta)^2$$

$$=\frac{1}{2}\sum_{\alpha,\beta}N(H_\alpha H_\beta-H_\beta H_\alpha)$$

$$\leqq\frac{1}{2}\left(\frac{3}{2}S^2-\sum_\alpha S_\alpha^2\right)$$

$$=\frac{3}{4}S^2-\frac{1}{2}\sum_\alpha(\mathrm{tr}\,H_\alpha^2)^2$$

and the equality holds if and only if at most two matrices H_α and H_β are not zero, and these two matrices can be transformed simultaneously by an orthogonal matrix into multiples of $\tilde{A}$ and $\tilde{B}$ as in Lemma 5 respectively.

Hence from (3.18)-(3.21), by taking $0\leqq a\leqq 1$ in (3.17), we obtain

(3.22) $$\sum_{i,j,\alpha}h_{ij}^\alpha\Delta h_{ij}^\alpha\geqq(1+a)nK_MS-(1-a)\frac{3}{4}S^4+\frac{1+a}{2}\frac{S^2}{p}-naS\,.$$

If $a=1-(4/3p+2)$, the right hand side of (3.21) is

$$\frac{nS}{3p+2}[6pK_M-(3p-2)]\,.$$

It the hypothesis of Theorem 3 is satisfied, then

$$\sum_{\alpha,i,j}h_{ij}^\alpha\Delta h_{ij}^\alpha\geqq 0$$

and hence (3.21) and (3.22) become equalities, $S\equiv$cont. So suppose $S\neq 0$. Then

(3.23) $$K_M=\frac{3p-2}{6p}\,.$$

Substituting (3.23) into (10.1) in [5], we obtain immediately $S \leqq (2/3)n$.

By the hypothesis $S \neq 0$ and [2], we know S must be $(2/3)n$, and M is a Veronese surface in S^4.

Now it is the end of the proof of Theorem 3.

REFERENCES

[1] S. S. CHERN, M. DO CARMO AND S. KOBAYASHI, Minimal submanifolds of a sphere with second fundamental form of constant length in Shiing-Shen Chern Selected Papers, 393-409, Springer Verlag, 1978.

[2] L. ANMIN AND L. JIMIN, Advanced in Math. (China), **20** (1991), No. 3, p. 375.

[3] M. OKUMURA, Hypersurfaces and a Pinching problem on the second fundamental tensor, Amer. J. Math., **96** (1974), 207-213.

[4] J. SIMONS, Minimal verieties in Riemannian manifolds, Ann. Math., **88** (1968), 62-105.

[5] S.T. YAU, Submanifolds with constant mean curvature, (I), Amer. J. Math., **96** (1974), 346-366; (II), **97** (1975), 76-100.

DEPARTMENT OF THE GIFTED YOUNG
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
HEFEI, ANHUI, 230026
P.R. CHINA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
HEFEI, ANHUI, 230026
P.R. CHINA