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A NOTE ON EQUIVALENCE CLASSES OF PLATS

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1. Introduction

This note is summary of the author's paper [7]. By connecting strings in pairs on the top and the bottom of a braid (Figure 1), we obtain a plat (Figure 2).

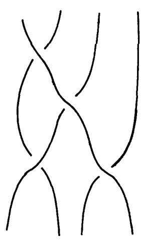

Figure 1

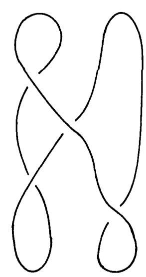

Figure 2

If it comes from a braid with 2n-strands, it is called a 2n-plat. Two plats are said to be equivalent if there is a homeomorphism h which carries upper and lower halves of $\mathbf{R}^3$ to themselves and a plat to the other plat (Figure 3).

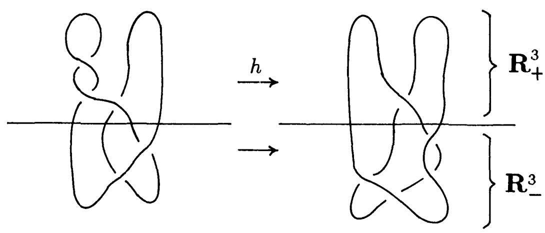

Figure 3

One of the main problems on plats is to decide when two plats are equivalent. In this note we assign a matrix to a plat and show that if two plats are equivalent then assigned matrices are equivalent in our sense. Furthermore we produce a numerical invariant from the matrices. Generally it needs a large amount of calculation to obtain the invariant. But it is quite elementary and can be done by an electric computer.

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2. Definitions

Let $\mathbf{R}^3_+ = \{(x,y,z) \in \mathbf{R}^3 | z \geqq 0\}, \mathbf{R}^3_- = \{(x,y,z) \in \mathbf{R}^3 | z \leqq 0\}$ and $\rho : \mathbf{R}^3_+ \to \mathbf{R}^3_-$ be defined by $\rho(x,y,z) = (x,y,-z)$. Let $A = A_1 \cup A_2 \cup \ldots \cup A_n$ be a union of arcs which are properly embedded in $\mathbf{R}^3_+$ as shown in Figure 4.

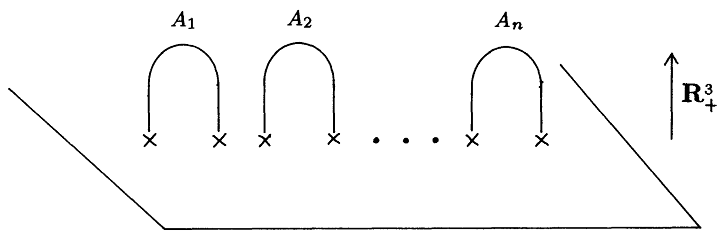

Figure 4

It is assumed that the arcs are unlinked and unknotted. Let $A' = A'_1 \cup A'_2 \cup \ldots A'_n$ be defined by $A'_i = \rho(A_i)$ for any $i = 1, \ldots, n$. Then A' is the union of the arcs in $\mathbf{R}^3_-$.

DEFINITION 1. Let B_{2n} be the braid group of $2n$-strands. Here in this note we regard an element of B_{2n} as an isotopy class of an orientation preserving homeomorphism $\varphi : (\partial\mathbf{R}^3_+, \partial A) \to (\partial\mathbf{R}^3_+, \partial A)$.

DEFINITION 2. Let $\varphi : (\partial\mathbf{R}^3_+, \partial A) \to (\partial\mathbf{R}^3_+, \partial A) = (\partial\mathbf{R}^3_-, \partial A)$ be a homeomorphism. The identification space $(\mathbf{R}^3_+, A) \cup_\varphi (\mathbf{R}^3_-, A')$ can be identified with $\mathbf{R}^3$ containing $L = A \cup_\varphi A'$ as a link. In this situation we call $(\mathbf{R}^3_+, A) \cup_\varphi (\mathbf{R}^3_-, A')$ a plat representation of the link L. We also call an isotopy class $[\varphi]$ a plat(or $2n$-plat). Hence we call an element of B_{2n} a plat in this note.

Next we give definition of equivalence of plats. Two definitions will be considered. One is required to be orientation preserving while the other is not.

DEFINITION 3. Two plats $[\varphi]$ and $[\psi]$ are said (orientation preservingly) equivalent if there is an orientation preserving homeomorphism $h : \mathbf{R}^3_+ \cup_\varphi \mathbf{R}^3_- \to \mathbf{R}^3_+ \cup_\psi \mathbf{R}^3_-$ which satisfies $h(\mathbf{R}^3_\pm) = \mathbf{R}^3_\pm, h(A) = A$ and $h(A') = A'$. In particular if h preserves orientations of A and A' it is said to be orientation preservingly equivalent.

Next we define a subgroup K_{2n} of B_{2n}.

DEFINITION 4. An element $[\varphi]$ of B_{2n} belongs to K_{2n} if and only if there is a homeomorphism $\hat{\varphi} : (\mathbf{R}^3_+, A) \to (\mathbf{R}^3_+, A)$ such that $\hat{\varphi} | \partial\mathbf{R}^3_+ = \varphi$.

Under the definitions we obtain the following lemma.

LEMMA 1. *Two plats $[\varphi], [\psi] \in B_{2n}$ are equivalent if and only if there are $g_1, g_2 \in K_{2n}$ which satisfy $\varphi = g_2 \circ \psi \circ g_1$.*

3. Jacobian matrices

Any plat $\varphi : (\partial \mathbf{R}^3_+, \partial A) \to (\partial \mathbf{R}^3_+, \partial A)$ induces an isomorphism $\varphi_* : \pi_1(\partial \mathbf{R}^3_+ - \partial A) \to \pi_1(\partial \mathbf{R}^3_+ - \partial A)$. The group $\pi_1(\partial \mathbf{R}^3_+ - \partial A)$ is identified with a free group of rank $2n$ by taking $a_1, \ldots, a_n, b_1, \ldots, b_n$ in Figure 5 as a generating system.

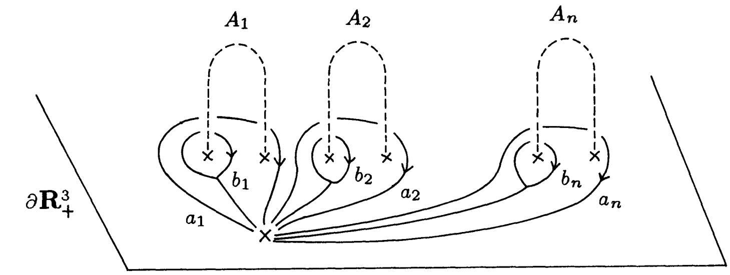

Figure 5

Then $\varphi : (\partial \mathbf{R}^3_+, \partial A) \to (\partial \mathbf{R}^3_+, \partial A)$ induces an isomorphism $\varphi_* : F(a_1, \ldots, a_n, b_1, \ldots, b_n) \to F(a_1, \ldots, a_n, b_1, \ldots, b_n)$. Let G denote an infinite cyclic group $\mathbf{Z}$ when we work in oriented case and a cyclic group $\mathbf{Z}_2$ of order 2 in general case. The group G will be presented as $G =< t|\emptyset >$ for oriented case while $G =< t|t^2 = 1 >$ for general case. Let $\alpha : F = F(a_1, \ldots, a_n, b_1, \ldots, b_n) \to G$ be a homomorphism defined by $\alpha(a_i) = 1, \alpha(b_i) = t$ for $i = 1, \ldots, n$. We denote $^- : \mathbf{Z}G \to \mathbf{Z}G$ an involution defined by $\overline{(\Sigma n_i t^i)} = \Sigma n_i t^{-i}$. Hereafter we work on orientation preserving equivalence and simply call it equivalence. But it will be easy to modify statements to adapt to non orientation preserving case.

Next we define Jacobian matrix $J\varphi_*$ which is associated with a plat φ

$$J\varphi_* = \begin{pmatrix} (\frac{\partial \varphi(a_i)}{\partial a_j})_{i,j=1,\ldots,n} & (\frac{\partial \varphi(a_i)}{\partial b_j})_{i,j=1,\ldots,n} \\ (\frac{\partial \varphi(b_i)}{\partial a_j})_{i,j=1,\ldots,n} & (\frac{\partial \varphi(b_i)}{\partial b_j})_{i,j=1,\ldots,n} \end{pmatrix}^\alpha$$

Here a matrix $(c_{ij})^\alpha$ is denoted by $(\alpha(c_{ij}))$. The symbol $\frac{\partial}{\partial x}$ denotes free derivative of Fox [5]. We denote four submatrices of $J\varphi_*$ by A, B, C and D as follows:

$$J\varphi_* = \begin{pmatrix} C & A \\ D & B \end{pmatrix}$$

where the four matrices are size of $n \times n$. The submatrix $\begin{pmatrix} A \\ B \end{pmatrix}$ of $J\varphi_*$ is important for our purpose. We denote it by $R\varphi_*$. Namely $R\varphi_* = \begin{pmatrix} A \\ B \end{pmatrix}$.

4. The main results

THEOREM 1. *Suppose that two plats $[\varphi], [\psi] \in B_{2n}$ are equivalent. Let $R\varphi_* = \begin{pmatrix} A \\ B \end{pmatrix}$ and $R\psi_* = \begin{pmatrix} A' \\ B' \end{pmatrix}$. Then there are $n \times n$ unimodular matrices U, G and an $n \times n$ matrix W which satisfy*

$$\begin{pmatrix} U & 0 \\ W & {}^*U^{-1} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} G = \begin{pmatrix} A' \\ B' \end{pmatrix}$$

where, for $U = (u_{ij}), {}^*U$ denotes ${}^t\overline{U}$, that is ${}^*U = (\overline{u_{ij}})$.

To prove the theorem the following is a key.

LEMMA 2. *If φ belongs to K_{2n}, then $J\varphi_*$ has a form $\begin{pmatrix} U & 0 \\ W & {}^*U^{-1} \end{pmatrix}$ where U and W are as in Theorem 1.*

We consider equivalence relation between matrices like $R\varphi_*$.

DEFINITION 5. Two matrices $R\varphi_* = \begin{pmatrix} A \\ B \end{pmatrix}$ and $R\psi_* = \begin{pmatrix} A' \\ B' \end{pmatrix}$ are said equivalent if and only if there are $n \times n$ unimodular matrices U, G and an $n \times n$ matrix W which satisfy

$$\begin{pmatrix} U & 0 \\ W & {}^*U^{-1} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} G = \begin{pmatrix} A' \\ B' \end{pmatrix}.$$

In term of this equivalence we can restate Theorem 1 as two plats φ and ψ are equivalent only if $R\varphi_*$ and $R\psi_*$ are equivalent. We choose a nice representative from an equivalence class of $R\varphi_*$.

THEOREM 2. *Let φ be a plat and let $R\varphi_* = \begin{pmatrix} A \\ B \end{pmatrix}$. Then $\begin{pmatrix} A \\ B \end{pmatrix}$ is equivalent to a matrix which has a form*

$$\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \\ 1 & * \\ 0 & B_{22} \end{pmatrix}$$

where A_{22} and B_{22} are $(n-1) \times (n-1)$ matrices while A and B are $n \times n$ matrices.

We call a matrix which has the form as in Theorem 2 a reduced form of $R\varphi_*$.

THEOREM 3. *Let φ and ψ be plat representation of knots. Suppose that φ and ψ are equivalent. Let*

$$R\varphi_* \sim \begin{pmatrix} 0 & 0 \\ 0 & A_{22} \\ 1 & * \\ 0 & B_{22} \end{pmatrix}, \quad R\psi_* \sim \begin{pmatrix} 0 & 0 \\ 0 & A'_{22} \\ 1 & * \\ 0 & B'_{22} \end{pmatrix}.$$

Then there are unimodular matrices U and G and a matrix W which satisfy

(i) $$A'_{22} = UA_{22}G$$

(ii) $$B'_{22} = WAG + {}^*U^{-1}B_{22}G.$$

Remark 1. The first equation (i) of Theorem 3 shows $A' = UAG^{-1}$. This means the matrices A and A' are similar. Let us substitute G in the second equation (ii) of Theorem 3 by $A'^{-1}UA$. Then we have

$$(*) \qquad {}^*W + BA^{-1} = {}^*UB'A'^{-1}U.$$

The entries of the matrix BA^{-1} are elements of the quotient field, say F, of $\mathbf{Z}G$. Then BA^{-1} can be regarded as a bilinear form from $\mathbf{Z}G \times \mathbf{Z}G \to F/\mathbf{Z}G$. So is $B'A'^{-1}$. The equation $(*)$ means the bilinear form $BA^{-1} : \mathbf{Z}G \times \mathbf{Z}G \to F/\mathbf{Z}G$ and $B'A'^{-1} : \mathbf{Z}G \times \mathbf{Z}G \to F/\mathbf{Z}G$ are isomorphic. Hence we proved that an isomorphism class of BA^{-1} is an invariant of the equivalence class of $\begin{pmatrix} A \\ B \end{pmatrix}$.

If we substitute t by -1 then Theorem 3 holds for non orientable case.

COROLLARY. *For a matrix* $\begin{pmatrix} A \\ B \end{pmatrix}$, $gcd_{i,j=1,\dots,n}(a_{ij})$ *and* $detB$ *mod* $gcd_{i,j=1,\dots,n}(a_{ij})$ *are, up to multiplication of units, invariants of an equivalence class of* $\begin{pmatrix} A \\ B \end{pmatrix}$.

Remark 2. Corollary above holds for non orientation preserving case if we substitute t by -1.

5. Examples

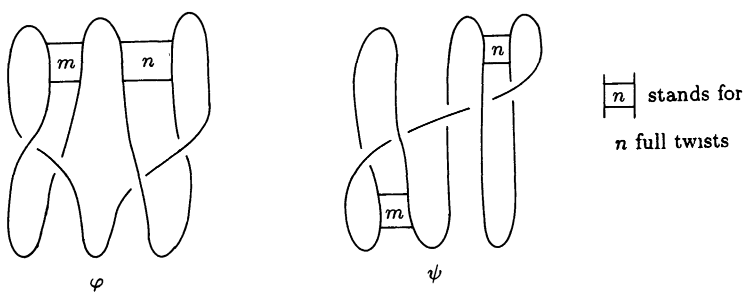

Figure 6

Let us consider the following two plats φ and ψ (Figure 6).

Here we denote $\gcd(4m-1, 4n-1)$ by d and suppose that d is neither 1 nor 3. To simplify calculation we substitute t with -1. Then

$$(R\varphi_*)^{t=-1} = \begin{pmatrix} A \\ B \end{pmatrix}^{t=-1} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-4m & 0 \\ 0 & 0 & 1-4n \\ 1 & * & * \\ 0 & 1-2m & 0 \\ 0 & -1+4m & 1+4n \end{pmatrix}$$

and

$$(R\psi_*)^{t=-1} = \begin{pmatrix} A' \\ B' \end{pmatrix}^{t=-1} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-4m & 0 \\ 0 & 0 & 1-4n \\ 1 & * & * \\ 0 & 1+4m & 0 \\ 0 & -1+4m & 1+4n \end{pmatrix}.$$

We have $\det B'_{22} - \det B_{22} = (1+4n)(1+4m-1+2m) = 6m(4n-1+2) \equiv 3 \bmod d$ and $\gcd_{i,j=1,\dots,n}(a_{ij}) = d$. Since d is neither 1 or 3, $3 \not\equiv 0 \bmod d$. Thus $\det B_{22} \not\equiv \det B'_{22} \bmod \gcd_{i,j=1,\dots,n}(a_{ij})$. This implies that $\begin{pmatrix} A \\ B \end{pmatrix}$ and $\begin{pmatrix} A' \\ B' \end{pmatrix}$ are not equivalent. In conclusion we can say that φ and ψ are not equivalent by Theorem 1. Note that these two plats are equivalent as knots.

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