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ON MINIMAL SURFACES WITH THE RICCI CONDITION IN SPACE FORMS

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0. Introduction

A 2-dimensional Riemannian metric ds^2 is said to satisfy the Ricci condition with respect to c if its Gaussian curvature K satisfies $K < c$ and the new metric $d\hat{s}^2 = \sqrt{c-K}ds^2$ is flat.

Let $X^N(c)$ denote the N-dimensional simply connected space form of constant curvature c, and in particular, let $\mathbf{R}^N = X^N(0)$. The induced metric ds^2 on a minimal surface in $X^3(c)$ satisfies the Ricci condition with respect to c except at points where the Gaussian curvature $= c$. Conversely, assume that a Riemannian metric ds^2 on a 2-dimensional simply connected manifold M satisfies the Ricci condition with respect to c. Then there exists a smooth 2π-periodic family of isometric minimal immersions $f_\theta : (M, ds^2) \to X^3(c)$; $\theta \in \mathbf{R}$, which is called the associated family. Moreover, up to congruences, the maps f_θ; $0 \leqq \theta < \pi$ represent all local isometric minimal immersions of (M, ds^2) into $X^3(c)$ (see [5]). So, the Ricci condition with respect to c is an intrinsic characterization of minimal surfaces in $X^3(c)$.

Here we consider the following problem, which may be seen as a kind of rigidity problem.

PROBLEM. *Classify those minimal surfaces in $X^N(c)$ whose induced metrics satisfy the Ricci condition with respect to c, or equivalently, classify those minimal surfaces in $X^N(c)$ which are locally isometric to minimal surfaces in $X^3(c)$.*

A submanifold in $X^N(c)$ is said to lie fully in $X^N(c)$ if it does not lie in a totally geodesic submanifold of $X^N(c)$. Let $S(N,c)$ denote the set of all Riemannian structures of minimal surfaces lying fully in $X^N(c)$. Then the problem is to determine the intersection of $S(3,c)$ and $S(N,c)$.

1. Examples

In this section, we give examples of minimal surfaces in $X^N(c)$ which do not lie in a totally geodesic $X^3(c)$ and whose induced metrics satisfy the Ricci condition with respect to c. The following three types of examples are known.

Example 1 ([6]). Let $f_\theta : (M, ds^2) \to \mathbf{R}^3$; $\theta \in \mathbf{R}$ be the associated family of isometric minimal immersions of a 2-dimensional Riemannian manifold (M, ds^2) into

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$\mathbf{R}^3$. Then we can construct an isometric minimal immersion $f : (M, ds^2) \to \mathbf{R}^6$ by setting

$$f = f_\theta \cos\varphi \oplus f_{\theta+\pi/2} \sin\varphi, \tag{1}$$

where the symbol $\oplus$ denotes the direct sum with respect to an orthogonal decomposition $\mathbf{R}^6 = \mathbf{R}^3 \oplus \mathbf{R}^3$. The metric induced by f is ds^2, which satisfies the Ricci condition with respect to 0 except at points where the Gaussian curvature $= 0$. Furthermore, in general, $f(M)$ lies fully in $\mathbf{R}^6$ if $\varphi \neq 0 \pmod{\pi/2}$.

Example 2 ([6]). Let $c > 0$. Let $f_\theta : (M, ds^2) \to X^3(c)$ ($\subset \mathbf{R}^4$); $\theta \in \mathbf{R}$ be the associated family of isometric minimal immersions of a 2-dimensional Riemannian manifold (M, ds^2) into $X^3(c)$. Then we can construct an isometric minimal immersion $f : (M, ds^2) \to X^{4m+3}(c)$ ($\subset \mathbf{R}^{4m+4}$) by setting

$$f = a_0 f_{\theta_0} \oplus \cdots \oplus a_m f_{\theta_m}, \tag{2}$$

where $\sum_{i=1}^m a_i^2 = 1$, $0 \leqq \theta_0 < \theta_1 < \cdots < \theta_m < \pi$, each f_{θ_i} is viewed as an $\mathbf{R}^4$-valued function with $|f_{\theta_i}| = 1/\sqrt{c}$, and the symbol $\oplus$ denotes the direct sum with respect to an orthogonal decomposition $\mathbf{R}^{4m+4} = \mathbf{R}^4 \oplus \cdots \oplus \mathbf{R}^4$. The metric induced by f is ds^2, which satisfies the Ricci condition with respect to c except at points where the Gaussian curvature $= c$. Furthermore, in general, $f(M)$ lies fully in $X^{4m+3}(c)$.

Example 3 ([1] and [4]). Every 2-dimensional flat metric automatically satisfies the Ricci condition with respect to $c > 0$, and there are flat minimal surfaces lying fully in $X^{2n+1}(c)$ where $c > 0$.

2. Known results

In the Euclidean case where $c = 0$, Lawson solved the problem completely as follows.

THEOREM 1 ([6] AND [7, CHAPTER IV]). *Let $f : M \to \mathbf{R}^N$ be a minimal immersion of a 2-dimensional manifold M into $\mathbf{R}^N$. Suppose that the induced metric ds^2 satisfies the Ricci condition with respect to* 0 *except at isolated points where the Gaussian curvature* $= 0$. *Then either (i) $f(M)$ lies in a totally geodesic $\mathbf{R}^3$, or (ii) $f(M)$ lies fully in a totally geodesic $\mathbf{R}^6$ and f is of the form of* (1) *in Example* 1 *for* $\varphi \neq 0 \pmod{\pi/2}$.

Remark 1. Theorem 1 says that $S(3,0)$ and $S(N,0)$ are disjoint if $N = 4$, $N = 5$ or $N \geqq 7$. Theorem 1 says also that $S(3,0)$ is included in $S(6,0)$ through Example 1.

Concerning the spherical case where $c > 0$, Lawson posed the following conjecture.

CONJECTURE ([6]). *Let $f : M \to X^N(c)$ be a minimal immersion of a 2-dimensional manifold M into $X^N(c)$ where $c > 0$. Suppose that the induced metric ds^2 satisfies the Ricci condition with respect to c except at isolated points where the Gaussian curvature $= c$. Then f must be of the form of* (2) *in Example* 2.

As a matter of fact, there are easy counter-examples to this conjecture (cf. Example 3). So one should consider the conjecture for non-flat minimal surfaces. In [8], with some global assumptions, Naka (= Miyaoka) obtained partial positive answers to this question.

3. Our results

First we solve the problem in the case where $N = 4$.

THEOREM 2 ([10]). *Let $f : M \to X^4(c)$ be a minimal immersion of a 2-dimensional manifold M into $X^4(c)$. Suppose that the induced metric ds^2 satisfies the Ricci condition with respect to c except at isolated points where the Gaussian curvature $= c$. Then $f(M)$ lies in a totally geodesic $X^3(c)$.*

Remark 2. (i) Theorem 2 says that $S(3, c)$ and $S(4, c)$ are disjoint.

(ii) When $c = 0$, Theorem 2 is included in [6].

(iii) In the case where $c > 0$, Theorem 2 is not true if we replace $X^4(c)$ by $X^5(c)$ (cf. Example 3).

(iv) In [10], with an additional assumption, we give a result also in higher codimensional cases.

In [3] Johnson studied a class of minimal surfaces in $X^N(c)$, which are called exceptional minimal surfaces and are related to the theory of harmonic sequences (cf. [2] and [11]). Next we discuss exceptional minimal surfaces in $X^N(c)$ whose induced metrics satisfy the Ricci condition with respect to c.

THEOREM 3 ([9]). *Let $f : M \to X^N(c)$ be an exceptional minimal immersion of a 2-dimensional manifold M into $X^N(c)$ where $c > 0$. Suppose that the induced metric ds^2 satisfies the Ricci condition with respect to c except at isolated points where the Gaussian curvature $= c$. Then either (i) $f(M)$ lies fully in a totally geodesics $X^{4m+1}(c)$ and ds^2 is flat, or (ii) $f(M)$ lies fully in a totally geodesic $X^{4m+3}(c)$.*

THEOREM 4 ([9]). *Let $f : M \to X^N(c)$ be an exceptional minimal immersion of a 2-dimensional manifold M into $X^N(c)$ where $c < 0$. Suppose that the induced metric ds^2 satisfies the Ricci condition with respect to c except at isolated points where the Gaussian curvature $= c$. Then $f(M)$ lies in a totally geodesic $X^3(c)$.*

Remark 3. (i) There are flat exceptional minimal surfaces lying fully in $X^{2n+1}(c)$, where $c > 0$ (see [9]).

(ii) There are non-flat exceptional minimal surfaces lying fully in $X^{4m+3}(c)$ whose induced metrics satisfy the Ricci condition with respect to c, where $c > 0$ (see [9]).

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