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ON THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF ARRANGEMENTS

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Let V be a vector space of finite dimension. An arrangement of hyperplanes in V is a finite collection $\mathcal{A}$ of hyperplanes of V. An arrangement $\mathcal{A}$ will be said to be real (resp. complex) if V is a real (resp. complex) vector space. The complexification of a hyperplanes H of $\mathbf{R}^n$ is the hyperplane $H_{\mathbf{C}}$ of $\mathbf{C}^n$ having the same equation as H. Given an arrangement $\mathcal{A}$ in $\mathbf{R}^n$, we have its complexification $\mathcal{A}_{\mathbf{C}}$ to be the complex arrangement $\{H_{\mathbf{C}}; H \in \mathcal{A}\}$ in $\mathbf{C}^n$.

Given an arrangement $\mathcal{A}$, we are interested in finding a presentation for the fundamental group $\pi_1(M)$ of the complement

$$M = V - \bigcup_{H \in \mathcal{A}} H$$

in case $\mathcal{A}$ is a complex arrangement, and $\pi_1(M_{\mathbf{C}})$ of the complement of $\mathcal{A}_{\mathbf{C}}$ in case $\mathcal{A}$ is a real arrangement. In [2] we have suggested a geometrical method to compute the fundamental group of a manifold equipped with a suitable cellular decomposition. Also, given a real arrangement $\mathcal{A}$ in $\mathbf{R}^n$, we have introduced a certain cellular decomposition $\mathcal{C}(\mathbf{C}^n, \mathcal{A})$ of $\mathbf{C}^n$, induced from the arrangement $\mathcal{A}$. In this note, we will apply our method to this decomposition to find a presentation for $\pi_1(M_{\mathbf{C}})$ of any real arrangement $\mathcal{A}$. Such a presentation has been given by M. Salvetti in [4] using his complex. After reducing the problem to the case of dimension 2, W. Arvola has suggested an algorithm to find a presentation for the complement of a complex arrangement. In a sequent paper [3] we will also treat the case when $\mathcal{A}$ is a complex arrangement.

We first recall of our method suggested in [2]. Let $\mathcal{M}$ be a connected topological-manifold of dimension n with a locally finite CW-semicomplex structure $\mathcal{C}_{\mathcal{M}}$ such that $\mathcal{M}$ is 1-codimensionally regular (see [2] for the notion of CW-semicomplex and 1-codimensional regularity). Each $(n-1)$-cell σ of $\mathcal{M}$ is a face of exactly two n-cells, say c and c'. Then we have two n-intervals $[c, \sigma, c']$ and $[c', \sigma, c]$. We specify one of them by $[\sigma]$ and the other by $[\sigma]^{-1}$. A n-path γ on $\mathcal{M}$ is a join of a finite number of n-intervals

$$\gamma = [\sigma_1]^{\epsilon_1} \vee [\sigma_2]^{\epsilon_2} \vee \cdots \vee [\sigma_k]^{\epsilon_k},$$

where $\epsilon_i = \pm 1$, σ_i are $(n-1)$-cells of $\mathcal{M}$, $1 \leq i \leq k$. If $[\sigma_1]^{\epsilon_1} = [c, \sigma_1, c_1]$ and $[\sigma_k]^{\epsilon_k} = [c_k, \sigma_k, c']$, for some n-cells c, c_1, c_k and c' we say that γ is a n-path from c to c'. Among n-paths on the manifold $\mathcal{M}$ we have defined in [2] a certain equivalence relation.

Let $\mathcal{M}$ be given a base point $*$ belonging to a certain n-cell c_0. Then, the equivalence classes of closed n-paths at $*$ form a group denoted by $\pi_1(\mathcal{C}_{\mathcal{M}}, *)$. In [2] we have proved the isomorphism $\pi_1(\mathcal{C}_{\mathcal{M}}, *) \cong \pi_1(\mathcal{M}, *)$. So, in order to compute $\pi_1(\mathcal{M}, *)$, it suffices to compute $\pi_1(\mathcal{C}_{\mathcal{M}}, *)$. And the latter can be determined by means of the decomposition $\mathcal{C}_{\mathcal{M}}$ as below

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We call a n-tree T of $\mathcal{M}$ a family of n-cells and $(n-1)$-cells of $\mathcal{C}_{\mathcal{M}}$ such that each $(n-1)$-cell of T is a face of exactly two n-cells of T and the union of all cells in T is simply connected. A n-tree T is maximal if it is not contained in any othe n-tree. It is easily seen that such a maximal n-tree exists and it contains all n-cells of $\mathcal{C}_{\mathcal{M}}$. Let K be a subspace of $\mathcal{M}$ which is the underlying space of a CW-subsemicomplex of codimension 2 of $\mathcal{M}$. Then we have

THEOREM 1 (see [2]). *Let $\mathcal{M}$ be a connected manifold with a 1-codimensionally regular CW-semicomplex structure $\mathcal{C}_{\mathcal{M}}$ and T a maximal n-tree of $\mathcal{C}_{\mathcal{M}}$. Let K as above. Then $\pi_1(\mathcal{M}-K,*)$ has a presentation with generators g_s indexed by the set $\mathcal{M}^{(n-1)}-\mathcal{M}^{(n-2)}$ of all $(n-1)$-cells of $\mathcal{M}$, with following defining relations*

(i) $g_\sigma = 1$ *if* $\sigma \in T$

(ii) $g_{\sigma_1}^{\epsilon_1}\cdot g_{\sigma_2}^{\epsilon_2}\cdots g_{\sigma_q}^{\epsilon_q} = 1$ *if* $[\sigma_1]^{\epsilon_1}\vee[\sigma_2]^{\epsilon_2}\vee\cdots\vee[\sigma_q]^{\epsilon_q}$ *is well defined and is a closed n-path around some $(n-2)$-cell of $\mathcal{M}-K$, i.e. $\sigma_1,\ldots,\sigma_q$ having a common $(n-2)$-face in $\mathcal{M}-K$.*

Next we recall of the decomposition $\mathcal{C}(\mathbf{C}^n,\mathcal{A})$ induced from the arrangement $\mathcal{A}=\{H_s; s\in I\}$ of hyperplanes in a n-dimensional real vector space $V=\mathbf{R}^n$. For each $s\in I$ we have a linear function $u_s: V\to\mathbf{R}$ with $H_s=\ker u_s$. Clearly, $V-H_s$ consists of two components $V_s^+=\{x\in V: u_s(x)>0\}$ and $V_s^-=\{x\in V; u_s(x)<0\}$. In a natural way, the arrangement $\mathcal{A}$ induces a cellular decomposition $\mathcal{C}(V,\mathcal{A})$ of V. An arbitrary cell of $\mathcal{C}(V,\mathcal{A})$ is an equivalence class with respect to the following equivalence relation

"*x and y are equivalent if $u_s(x)\cdot u_s(y)>0$ or $u_s(x)=u_s(y)=0$ for all $s\in I$.*"

It is easily seen that a n-cell of $\mathcal{C}(V,\mathcal{A})$ is a connected component of $V-\cup_{s\in I}H_s$ and is usually called a chamber of $\mathcal{A}$. A cell of dimension lower than n is contained in the closure $\bar{c}$ of a certain chamber c. The closure of such a cell is the intersection $\bar{c}\cap\{\bigcap_{j=1}^{l}H_{s_j}\}$ of $\bar{c}$ and some hyperplanes of $\mathcal{A}$. We denote this cell by $c_{s_1,\ldots,s_l}$. For each cell e of $\mathcal{C}(V,\mathcal{A})$ we denote by $|e|$ the intersection of all hyperplanes of $\mathcal{A}$ containing e and call it the support of e. Then it is easily seen that $|c_{s_1,\ldots,s_l}|=\cap_{j=1}^{l}H_{s_j}$.

Let us consider the complexification $\mathbf{C}^n=V+\iota V$ of V, where ι denotes $\sqrt{-1}$, the imaginary unit. Then we have the complexification of the arrangement $\mathcal{A}$, $\mathcal{A}_{\mathbf{C}}=\{H_s+\iota H_s; s\in I\}$, in $\mathbf{C}^n$ with the complement $M_{\mathbf{C}}=V_{\mathbf{C}}-\cup_{s\in I}(H_s+\iota H_s)$. The arrangement $\mathcal{A}$ induces also a cellular decomposition $\mathcal{C}(\mathbf{C}^n,\mathcal{A})$ of $\mathbf{C}^n$ as follows.

(a) An arbitrary $2n$-cell in $\mathcal{C}(\mathbf{C}^n,\mathcal{A})$ is of the form $c+\iota V$, where c is a chamber of $\mathcal{C}(V,\mathcal{A})$.

(b) An arbitrary $(2n-1)$-cell in $\mathcal{C}(\mathbf{C}^n,\mathcal{A})$ is of the form $c_s+\iota V_s^\epsilon, \epsilon=+$ or $-$, with c_s and V_s^ϵ as in the above notations.

(c) A cell of dimension $\leq 2n-1$ is of the form

$$c_{s_1,\ldots,s_k}+\iota D,$$

where D is any cell in $\mathcal{C}(V,\{H_{s_1},\ldots,H_{s_k}\})$.

Remark 2. Denoting $V_{s_1,\ldots,s_k}^{\epsilon_1,\ldots,\epsilon_k}$ to be the intersection $V_{s_1}^{\epsilon_1}\cap\ldots\cap V_{s_k}^{\epsilon_k}$, with $s_j\in I,\epsilon_j=+$ or $-$, we observe that any $(2n-2)$-cell of $\mathcal{C}(\mathbf{C}^n,\mathcal{A})$ lying in the complement M will be of the form

$$c_{s_1,\ldots,s_k}+\iota V_{s_1,\ldots,s_k}^{\epsilon_1,\ldots,\epsilon_k},$$

where $c_{s_1,\ldots,s_k}$ is a cell of codimension 2 of $\mathcal{C}(V,\mathcal{A})$.

Now we will use theorem 1 and this decomposition $\mathcal{C}(\mathbf{C}^n,\mathcal{A})$ to give a presentation for the group $\pi_1(M_{\mathbf{C}},*)$. According to Theorem 1, we have a presentation of the fundamental group of the complement $\pi_1(M)$ with generators corresponding to $(2n-1)$-cells of $\mathcal{C}(\mathbf{C}^n,\mathcal{A})$ and with defining relations given by $(2n-2)$-cells of $\mathcal{C}(\mathbf{C}^n,\mathcal{A})$ lying in M modulo a chosen maximal $2n$-tree T. We will denote by g_σ^+ and g_σ^- the generators corresponding to $\sigma+\iota V_s^+$ and $\sigma+V_s^-$ respectively, where σ is any $(n-1)$-cell of $\mathcal{C}(V,\mathcal{A})$ with the support $|\sigma|=H_s\in\mathcal{A}$. Now we try to simplify this presentation.

LEMMA 3. *We can choose a maximal $2n$-tree T_0 of $\mathcal{C}(\mathbf{C}^n,\mathcal{A})$ so that in the group $\pi_1(M)$ we have $g_\sigma^-=1$ for any $(n-1)$-cell σ of $\mathcal{C}(V,\mathcal{A})$.*

Let T be the maximal $2n$-tree as in Lemma 3. According to Theorem 1 (i), the fundamental group $\pi_1(M)$ has the set of generators $\{g_\sigma^+;\sigma \text{ is a } (n-1)\text{-cell of } \mathcal{C}(V,\mathcal{A})\}$. For the sake of simplicity, from now on, we will denote g_σ^+ simply by g_σ. We can also prove that two generators corresponding to $(n-1)$-cells of $\mathcal{C}(V,\mathcal{A})$ having the same support are conjugate. So, there are as many generators as there are hyperplanes in $\mathcal{A}$. Denote the generator corresponding to the hyperplane H_s by $g_s, s\in I$.

Next we consider the defining relations of the group $\pi_1(M)$. According to Theorem 1 (ii), these relations are given by $(2n-2)$-cells in M. By Remark 2, any $(2n-2)$-cell in M will be of the form

$$c_{s_1,\ldots,s_k}^{\epsilon_1,\ldots,\epsilon_k}=c_{s_1,\ldots,s_k}+\iota V_{s_1,\ldots,s_k}^{\epsilon_1,\ldots,\epsilon_k},$$

where $\epsilon_i=+$ or $-$, $s_i\in I$ and $c_{s_1,\ldots,s_k}$ is a codimension 2 cell of $\mathcal{C}(V,\mathcal{A})$. We first observe that the $(2n-1)$-cells of $\mathcal{C}(\mathbf{C}^n,\mathcal{A})$ having $c_{s_1,\ldots,s_k}^{\epsilon_1,\ldots,\epsilon_k}$ as a common face are $\sigma_1+\iota V_{s_1}^{\epsilon_1},\ldots,\sigma_k+\iota V_{s_k}^{\epsilon_k}$, and $\sigma_1'+V_{s_1}^{\epsilon_1},\ldots,\sigma_k'+\iota V_{s_k}^{\epsilon_k}$, where $\sigma_i,\sigma_i', 1\le i\le k$, are codimension 1 cells of $\mathcal{C}(V,\mathcal{A})$ having $c_{s_1,\ldots,s_k}$ as a common face and $|\sigma_i|=|\sigma_i'|=H_{s_i}, 1\le i\le k$. Looking at a 2-plane perpendicular to $c_{s_1,\ldots,s_k}^{\epsilon_1,\ldots,\epsilon_k}$ at an inner point of this cell, we obtain

LEMMA 4. *The relation given by $c_{s_1,\ldots,s_k}^{\epsilon_1,\ldots,\epsilon_k}$ is*

$$g_{\sigma_1}^{\epsilon_1}\cdots g_{\sigma_k}^{\epsilon_k}\cdot(g_{\sigma_1'}^{\epsilon_1})^{-1}\cdots(g_{\sigma_k'}^{\epsilon_k})^{-1}=1.$$

Let us consider the codimension 2 cell $c_{s_1,\ldots,s_l}$ of $\mathcal{C}(V,\mathcal{A})$. Changing the sign of linear functions $u_{s_i}, 1\le i\le l$, if necessary, we can assume that $V_{s_1,\ldots,s_l}^{+,\ldots,+}=V_{s_1}^+\cap\ldots\cap V_{s_l}^+\neq\emptyset$. Then in $\mathcal{C}(V_{\mathbf{C}},\mathcal{A})$ we will have the following $(2n-2)$-cells : $c_{s_1,\ldots,s_l}^{\epsilon_1,\ldots,\epsilon_l}$, where $(\epsilon_1,\ldots,\epsilon_l)$ runs over the set

$$\{(+,+,\ldots,+,+),(-,+,\ldots,+,+),\ldots,(-,-,\ldots,-,+),(-,-,\ldots,-,-)\}.$$

By a direct computation, from the defining relations of the group $\pi_1(M_{\mathbf{C}})$ corresponding to these $(2n-2)$-cells as determined in Lemma 4 we obtain the set of relations

$$\{g_{\sigma_1}g_{\sigma_2}\cdots g_{\sigma_l}=g_{\sigma_{s(1)}}g_{\sigma_{s(2)}}\cdots g_{\sigma_{s(l)}};s\in C\},$$

where C is the set of cyclic permutations of $\{1,\ldots,l\}$. Denote this set of relations by $R(c_{s_1,\ldots,s_l})$.

Next we consider two cells of codimension 2 of $\mathcal{C}(V,\mathcal{A})$ having the same support. Suppose they are $c_{s_1,\ldots,s_l}$ and $\tilde{c}_{s_1,\ldots,s_l}$. Then we have $(n-1)$-cells $\sigma_1,\ldots,\sigma_l$ and $\tilde{\sigma}_1,\ldots,\tilde{\sigma}_l$

around $c_{s_1,\ldots,s_l}$ and $\tilde{c}_{s_1,\ldots,s_l}$respectively. Moreover $|\sigma_i| = |\tilde{\sigma}_i| = H_{s_i}, 1 \leq i \leq l$. We can prove that there is a class of n-paths λ such that $\lambda g_{\tilde{\sigma}_i}\lambda^{-1} \simeq g_{\sigma_i}, 1 \leq i \leq l$. Combining this and relations $R(c_{s_1,\ldots,s_l})$ and $R(\tilde{c}_{s_1,\ldots,s_l})$ we see that $c_{s_1,\ldots,s_l}$ and $\tilde{c}_{s_1,\ldots,s_l}$ determine the same relation of the group $\pi_1(M)$.

Now substituting to g_{σ_i} the right conjugate of g_{s_i}, the relations $R(c_{s_1,\ldots,s_l})$ translate into the relations amongs g_{s_i}. We denote these relations by $R(s_1,\ldots,s_l)$.

So from above investigations we obtain

THEOREM 5. *The fundamental group $\pi_1(M_{\mathbf{C}})$ accepts a presentation with the set of generators $\{g_s; s \in I\}$ and with defining relation $R(s_1,\ldots,s_l)$ whenever $H_{s_1}\cap\ldots\cap H_{s_l}$ is of codimension* 2.

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