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KODAI MATH. J.
17 (1994), 402—408

FACTORED ARRANGEMENTS OF HYPERPLANES

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Let $\mathbf{K}$ be a field and let V be a vector space over $\mathbf{K}$. An *arrangement of hyperplanes* in V is a finite family $\mathcal{A}$ of hyperplanes of V through the origin. An arrangement $\mathcal{A}$ of hyperplanes is said to be *real* (resp. *complex*) if $\mathbf{K} = \mathbf{R}$ is the field of real numbers (resp. if $\mathbf{K} = \mathbf{C}$ is the field of complex numbers).

With an arrangement $\mathcal{A}$ of hyperplanes, one can associate a graded torsion-free $\mathbf{Z}$-algebra $A(\mathcal{A})$, called the *Orlik-Solomon algebra* of $\mathcal{A}$. If $\mathcal{A}$ is a complex arrangement, then $A(\mathcal{A})$ is isomorphic to the cohomology algebra of the *complement*

$$M(\mathcal{A}) = V - (\cup_{H\in\mathcal{A}} H)$$

of $\mathcal{A}$ (see [OS1]). The *Poincaré polynomial* $\mathrm{Poin}(\mathcal{A},t)$ of $\mathcal{A}$ is the Poincaré polynomial of $A(\mathcal{A})$, namely,

$$\mathrm{Poin}(\mathcal{A},t) = \sum_{n=0}^{\infty} \dim(A^n(\mathcal{A}))t^n \ .$$

We refer to [Or] and [OT] for good expositions on the theory of arrangements of hyperplanes and, more precisely, on Orlik-Solomon algebras.

Let $\mathcal{A}$ be a real arrangement of hyperplanes. A *chamber* of $\mathcal{A}$ is a connected component of $V - (\cup_{H\in\mathcal{A}} H)$. We denote by $\mathcal{C}(\mathcal{A})$ the set of chambers of $\mathcal{A}$. For $C, D \in \mathcal{C}(\mathcal{A})$, we denote by $\mathcal{S}(C,D)$ the set of hyperplanes of $\mathcal{A}$ which separate C and D. For a fixed chamber $C_0 \in \mathcal{C}(\mathcal{A})$, we partially order $\mathcal{C}(\mathcal{A})$ by

$$C \leq D \quad \text{if} \quad \mathcal{S}(C_0,C) \subseteq \mathcal{S}(C_0,D) \ .$$

$\mathcal{C}(\mathcal{A})$ provided with this order is denoted by $P(\mathcal{A},C_0)$. It is a ranked bounded poset of finite rank, where $\mathrm{rank}(C) = |\mathcal{S}(C_0,C)|$ for $C \in \mathcal{C}(\mathcal{A})$. Its smallest element is C_0 and its greatest one is the chamber $-C_0$ opposite to C_0. The *rank-generating function* of $P(\mathcal{A},C_0)$ is

$$\zeta(P(\mathcal{A},C_0),t) = \sum_{C\in\mathcal{C}(\mathcal{A})} t^{\mathrm{rank}(C)} \ .$$

The poset $P(\mathcal{A},C_0)$ has been introduced and investigated by Björner, Edelman and Ziegler [Ed] [BEZ].

Let $\mathcal{A}$ be a real arrangement of hyperplanes. If $\mathcal{A}$ is either a supersolvable arrangement or a Coxeter arrangement, then there exist some integers $b_1,\ldots,b_l$ and a chamber $C_0 \in \mathcal{C}(\mathcal{A})$ such that the Poincaré polynomial of $\mathcal{A}$ factors as

$$\mathrm{Poin}(\mathcal{A},t) = \prod_{i=1}^{l}(1+b_it) \tag{1}$$

Received June 19, 1993.

(see [St] for supersolvable arrangements and [OS2] for Coxeter arrangements), the poset $P(\mathcal{A}, C_0)$ is a lattice (see [BEZ]), and the rank-generating function of $P(\mathcal{A}, C_0)$ factors as

$$\zeta(P(\mathcal{A}, C_0), t) = \prod_{i=1}^{l}(1 + t + \ldots + t^{b_i}) \tag{2}$$

(see [BEZ] and [Pa] for supersolvable arrangements and [So] for Coxeter arrangements).

So, it is natural to ask whether there exists some relation between the Poincaré polynomial of a real arrangement $\mathcal{A}$ and the poset $P(\mathcal{A}, C_0)$ for some suitable chamber $C_0 \in \mathcal{C}(\mathcal{A})$, and whether there exist other arrangements with such properties.

There is a class of arrangements of hyperplanes called *free arrangements*, introduced by Terao [Te1], and which contains supersolvable arrangements (see [JT]) and Coxeter arrangements (see [Te2]). If $\mathcal{A}$ is a free arrangement, then there exist some integers $b_1, \ldots, b_l$ such that the Poincaré polynomial factors as (1) (see [Te1]). However, if $\mathcal{A}$ is a free real arrangement of hyperplanes, then there does not necessarily exist a chamber $C_0 \in \mathcal{C}(\mathcal{A})$ such that the rank-generating function of $P(\mathcal{A}, C_0)$ factors as (2); Terao [Te3] has found that the arrangement $A_4(17)$ from Grünbaum's list [Gr] is a counterexample.

Let $\mathcal{A}$ be an arrangement of hyperplanes. The *intersection lattice* of $\mathcal{A}$ is the geometric lattice

$$\mathcal{L}(\mathcal{A}) = \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \right\} ,$$

ordered by reverse inclusion. $V = \cap_{H \in \emptyset} H$ is the smallest element of $\mathcal{L}(\mathcal{A})$ and $\cap_{H \in \mathcal{A}} H$ is its greatest one. For $X \in \mathcal{L}(\mathcal{A})$, we set

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supseteq X\} ,$$
$$\mathcal{A}^X = \{H \cap X \mid H \in \mathcal{A} - \mathcal{A}_X\} .$$

A partition $\Pi = (\Pi_1, \ldots, \Pi_l)$ of $\mathcal{A}$ into l disjoint subsets is called *independent* if, for any choice of hyperplanes $H_i \in \Pi_i$ $(i = 1, \ldots, l)$, the rank of $H_1 \cap \ldots \cap H_l$ is exactly l. If $X \in \mathcal{L}(\mathcal{A})$, then Π *induces* a partition Π_X of $\mathcal{A}_X$ which blocks are the nonempty subsets $\Pi_i \cap \mathcal{A}_X$. The partition Π is a *factorization* (or a *nice partition*) if

1) Π is independent,
2) if $X \in \mathcal{L}(\mathcal{A}) - \{V\}$, then Π_X has at least a block which is a singleton.

In particular, one of the blocks Π_i is a singleton. We say that $\mathcal{A}$ is *factored* if $\mathcal{A}$ has a factorization.

Factored arrangements have been introduced and investigated by Jambu, Falk and Terao [JF] [Te4]. Supersolvable arrangements are factored (see [Ja]).

The homogeneous component $A^1(\mathcal{A})$ of degree 1 of the Orlik-Solomon algebra $A(\mathcal{A})$ of an arrangement $\mathcal{A}$ can be viewed as a free $\mathbf{Z}$-module spanned by the hyperplanes of $\mathcal{A}$. For $\mathcal{B} \subseteq \mathcal{A}$, we denote by $B(\mathcal{B})$ the submodule of $A^1(\mathcal{A})$ spanned by the hyperplanes included in $\mathcal{B}$. Following theorem is due to Terao [Te4].

THEOREM 1 (Terao [Te4]). *Let $\mathcal{A}$ be an arrangement of hyperplanes and let $\Pi = (\Pi_1, \ldots, \Pi_l)$ be a partition of $\mathcal{A}$. The Orlik-Solomon algebra of $\mathcal{A}$, viewed as a graded $\mathbf{Z}$-module, factors as*

$$A(\mathcal{A}) = (\mathbf{Z} \oplus B(\Pi_1)) \otimes \ldots \otimes (\mathbf{Z} \oplus B(\Pi_l))$$

if and only if Π *is a factorization.*

COROLLARY 2 (Terao [Te4]). *Let* $\mathcal{A}$ *be a factored arrangement of hyperplanes. Let* $\Pi = (\Pi_1, \dots, \Pi_l)$ *be a factorization of* $\mathcal{A}$.

1) *The Poincaré polynomial of* $\mathcal{A}$ *factors as*

$$\mathrm{Poin}(\mathcal{A}, t) = \prod_{i=1}^{l} (1 + |\Pi_i| t) \ .$$

2) *The multiset* $\{|\Pi_1|, \dots, |\Pi_l|\}$ *depends only on* $\mathcal{A}$.

3) *For* $X \in \mathcal{L}(\mathcal{A})$,

$$\mathrm{rank}(X) = |\{i \mid \Pi_i \cap \mathcal{A}_X \neq \emptyset\}| \ .$$

In particular, l *is the rank of* $\mathcal{A}$.

Let $\mathcal{A}$ be a factored arrangement of hyperplanes and let $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_l)$ be a factorization of $\mathcal{A}$. We say that a hyperplane $H_0 \in \mathcal{A}$ is *distinguished* if Π induces a factorization Π' of $\mathcal{A}' = \mathcal{A} - \{H_0\}$, namely, the nonempty subsets $\Pi_i \cap \mathcal{A}'$ form a factorization of $\mathcal{A}'$ (note that $\Pi_i \cap \mathcal{A}' = \Pi_i \neq \emptyset$ if $H_0 \notin \Pi_i$). Given a distinguished hyperplane $H_0 \in \Pi_1$, we write $\Pi_i'' = \{H \cap H_0 \mid H \in \Pi_i\}$ for $i = 2, \dots, l$. We prove in [JP] the following result.

$\Pi'' = (\Pi_2'', \dots, \Pi_l'')$ *is a factorization of* $\mathcal{A}'' = \mathcal{A}^{H_0}$.

A factorization $\Pi = (\Pi_1, \dots, \Pi_l)$ of an arrangement $\mathcal{A}$ of hyperplanes is said to be an *inductive factorization* if there exists a distinguished hyperplane $H_0 \in \mathcal{A}$ such that Π' is an inductive factorization of $\mathcal{A}' = \mathcal{A} - \{H_0\}$ and Π'' is an inductive factorization of $\mathcal{A}'' = \mathcal{A}^{H_0}$. We say that $\mathcal{A}$ is *inductively factored* if $\mathcal{A}$ has an inductive factorization.

A factored arrangement is not necessarily inductively factored, for example, the nine planes in $\mathbf{C}^3$ which satisfy the equation $(x^3 - y^3)(x^3 - z^3)(y^3 - z^3) = 0$ form a factored arrangement which is not inductively factored. Nevertheless, we do not know any arrangement $\mathcal{A}$ of hyperplanes in a real vector space which is factored and not inductively factored. Supersolvable arrangements are inductively factored.

In [JP] we prove the following results:

THEOREM 3. 1) *Inductively factored arrangements are free.*

2) *Let* $\mathcal{A}$ *be an inductively factored arrangement of hyperplanes in a real vector space. Let* $\Pi = (\Pi_1, \dots, \Pi_l)$ *be an inductive factorization of* $\mathcal{A}$. *There exists a chamber* $C_0 \in \mathcal{C}(\mathcal{A})$ *such that* $P(\mathcal{A}, C_0)$ *is a lattice and its rank-generating function factors as*

$$\zeta(P(\mathcal{A}, C_0), t) = \prod_{i=1}^{l} (1 + t + \ldots + t^{|\Pi_i|})$$

We do not know if 1) and 2) hold for any factored arrangement of hyperplanes.

An arrangement $\mathcal{A}$ of hyperplanes is *essential* if $\cap_{H \in \mathcal{A}} H = \{0\}$. Let $\mathcal{A}$ be an essential arrangement of hyperplanes in a l-dimensional real vector space $V = \mathbf{R}^l$. We provide V with an arbitrary scalar product. Let $\mathbf{S}^{l-1}$ denote the unit sphere. The arrangement

$\mathcal{A}$ determines a cellular decomposition (defined in Section 4) of $\mathbf{S}^{l-1}$ called the *dual decomposition* of $\mathbf{S}^{l-1}$ *induced* by $\mathcal{A}$.

THEOREM 4. *If $\mathcal{A}$ has an inductive factorization $\Pi = (\Pi_1, \ldots, \Pi_l)$, then $\mathbf{S}^{l-1}$ can be viewed as the boundary $\partial\Omega$ of the l-cube*

$$\Omega =]0, |\Pi_1|[\times \ldots \times]0, |\Pi_l|[\ ,$$

and each cell of the dual decomposition of $\mathbf{S}^{l-1} = \partial\Omega$ has the form $I_1 \times \ldots \times I_l$, where, for $i = 1, \ldots, l$, either $I_i = \{a_i\}$ ($a_i \in \mathbf{N}$) or $I_i =]a_i, b_i[$ ($a_i, b_i \in \mathbf{N}$).

In [Sa] Salvetti associated with a real arrangement $\mathcal{A}$ of hyperplanes a cellular complex Sal($\mathcal{A}$) and proved that Sal($\mathcal{A}$) has the same homotopy type as the complement $M(\mathcal{A}_{\mathbf{C}})$ of the complexification of $\mathcal{A}$. ¿From the dual decomposition of $\mathbf{S}^{l-1}$ induced by $\mathcal{A}$, we define a cellular decomposition of the (closed) unit disk $\mathbf{B}^l$ by attaching a l-cell to $\mathbf{S}^{l-1}$. For every chamber $C \in \mathcal{C}(\mathcal{A})$, we fix a copy $\mathbf{B}(C)$ of $\mathbf{B}^l$ provided with this decomposition. The complex Sal($\mathcal{A}$) can be defined by

$$\mathrm{Sal}(\mathcal{A}) = \left\{ \coprod_{C \in \mathcal{C}(\mathcal{A})} \mathbf{B}(C) \right\} / \sim \ ,$$

where $\sim$ is an equivalence relation which, for each pair (C, D) of chambers, identifies some cells of $\mathbf{B}(C)$ with their corresponding cells of $\mathbf{B}(D)$. So, Theorem 4 may be certainly used to investigate the homotopy of the complement $M(\mathcal{A}_{\mathbf{C}})$ of the complexification of an inductively factored arrangement $\mathcal{A}$.

Let $\mathcal{A}$ be a real arrangement of hyperplanes. Assume that $\mathcal{A}$ has a factorization $\Pi = (\Pi_1, \ldots, \Pi_l)$. For $C, D \in \mathcal{C}(\mathcal{A})$ and $i \in \{1, \ldots, l\}$, we set $\mathcal{S}_i(C, D) = \mathcal{S}(C, D) \cap \Pi_i$. For $b \in \mathbf{N}$, we write $[b] = \{0, 1, \ldots, b\}$. The *counting map* of $\mathcal{A}$ *with respect* to a chamber $C_0 \in \mathcal{C}(\mathcal{A})$ is the morphism $\phi(\Pi, C_0)$ of ranked posets defined by

$$\begin{array}{rccc} \phi(\Pi, C_0): & P(\mathcal{A}, C_0) & \longrightarrow & [|\Pi_1|] \times \ldots \times [|\Pi_l|] \\ & C & \longmapsto & (|\mathcal{S}_1(C_0, C)|, \ldots, |\mathcal{S}_l(C_0, C)|) \end{array}$$

where $[|\Pi_1|] \times \ldots \times [|\Pi_l|]$ is partially ordered by

$$(a_1, \ldots, a_l) \le (b_1, \ldots, b_l) \quad \text{if} \quad a_1 \le b_1, \ldots, a_l \le b_l \ ,$$

and, for $(a_1, \ldots, a_l) \in [|\Pi_1|] \times \ldots \times [|\Pi_l|]$,

$$\mathrm{rank}(a_1, \ldots, a_l) = a_1 + \ldots + a_l \ .$$

Note that, in general, the counting map $\phi(\Pi, C_0)$ is not an isomorphism of ranked posets, even if $\phi(\Pi, C_0)$ is a bijection; two chambers C and D may be not comparable meanwhile their images are comparable.

Let $\Pi = (\Pi_1, \ldots, \Pi_l)$ be a factorization of $\mathcal{A}$. We say that Π is a *hyperfactorization* if there exists a chamber $C_0 \in \mathcal{C}(\mathcal{A})$ such that $\phi(\Pi, C_0)$ is a bijection. We say that $\mathcal{A}$ is *hyperfactored* if $\mathcal{A}$ has a hyperfactorization.

THEOREM 5. *Inductively factored real arrangements are hyperfactored.*

Most of the results on inductively factored arrangements mentioned above actually are proved for hyperfactored arrangements. We do not know any real arrangement $\mathcal{A}$

which is hyperfactored and not inductively factored. In fact, we introduce this notion of "hyperfactorization" for technical reasons, indeed, in order to prove that inductively factored arrangements are hyperfactored, we need several preliminary results on hyperfactored arrangements. So, all results stated in this paper for hyperfactored arrangements hold for inductively factored arrangements.

Example. Let $\mathcal{A}$ be the arrangement in $\mathbf{R}^3$ projectively pictured in Figure 1 ($\mathcal{A}$ contains the "line" at infinity). Set $\Pi_1 = \{H_1, H_3, H_4\}$, $\Pi_2 = \{H_2, H_5, H_6\}$, and $\Pi_3 = \{H_7\}$. Then $\Pi = (\Pi_1, \Pi_2, \Pi_3)$ is a factorization of $\mathcal{A}$. One can easily verify that H_1 is distinguished in $\mathcal{A}$, H_2 is distinguished in $\mathcal{A}-\{H_1\}$, H_3 is distinguished in $\mathcal{A}-\{H_1, H_2\}$, and so on. Hence, Π is an inductive factorization.

The *adjacency graph* of $\mathcal{A}$ is defined to be the graph which vertices are the chambers of $\mathcal{A}$, and where two chambers are joined by an edge if they are adjacent. It is shown in Figure 2. From this graph, one can verify that $\phi(\Pi, C_0)$ is a bijection from $\mathcal{C}(\mathcal{A})$ to $[3] \times [3] \times [1]$, thus Π is a hyperfactorization. Moreover, $P(\mathcal{A}, C_0)$ is a lattice and

$$\zeta(P(\mathcal{A}, C_0), t) = (1 + t + t^2 + t^3)^2(1 + t) .$$

We set $\Omega =]0,3[\times]0,3[\times]0,1[$. The dual decomposition of $\mathbf{S}^2 = \partial\Omega$ induced by $\mathcal{A}$ is shown in Figure 3.

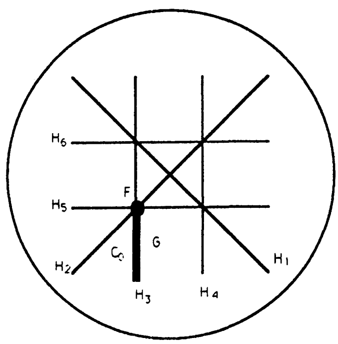

Figure 1

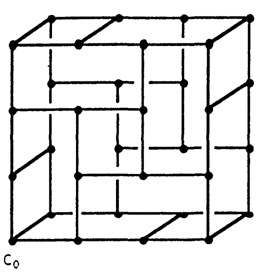

Figure 2

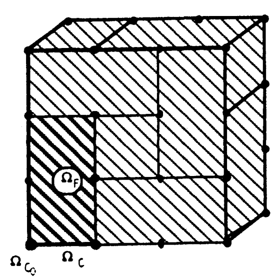

Figure 3

Our paper [JP] is organized as follows.

In Section 2 we mainly prove that, if $\mathcal{A}$ is a factored arrangement of hyperplanes and $H_0 \in \mathcal{A}$ is distinguished, then $\mathcal{A}'' = \mathcal{A}^{H_0}$ is factored. We also prove that inductively factored arrangements are free.

In Section 3 we prove several results on hyperfactored arrangements. In particular, we characterize the chambers $C_0 \in \mathcal{C}(\mathcal{A})$ such that $\phi(\Pi, C_0)$ is a bijection , we prove there are $2^{\mathrm{rank}(\mathcal{A})}$ of them, we prove that, for such a chamber C_0, the rank-generating function of $P(\mathcal{A}, C_0)$ factors as

$$\zeta(P(\mathcal{A}, C_0), t) = \prod_{i=1}^{l}(1 + t + \ldots + t^{|\Pi_i|}),$$

and, if C is a chamber of a hyperfactored arrangement $\mathcal{A}$, then C has at most $2.\mathrm{rank}(\mathcal{A}) - 1$ walls.

In Section 4 we prove that the dual decomposition of the sphere $\mathbf{S}^{l-1}$ induced by a hyperfactored arrangement can be viewed as the decomposition of the boundary $\partial\Omega$ of the l-cube Ω described above.

In Section 5 we prove that, if $\mathcal{A}$ is a hyperfactored arrangement and $C_0 \in \mathcal{C}(\mathcal{A})$ is a suitable chamber, then the poset $P(\mathcal{A}, C_0)$ is a lattice.

In Section 6 we prove that inductively factored real arrangements are hyperfactored.

ACKNOWLEDGEMENT: The first author is grateful to all the organizers of this Workshop for their hospitality and he would like to thank more particularly Professor Huynh Mui, Professor Yukio Matsumoto and Professor Mutsuo Oka.

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