H. MASAOKA AND S. SEGAWA
KODAI MATH. J.
17 (1994), 351—359

HARMONIC DIMENSION OF COVERING SURFACES

Dedicated to Professor Mitsuru Nakai on his sixtieth birthday

BY HIROAKI MASAOKA AND SHIGEO SEGAWA

Introduction

Let R be an open Riemann surfaces of null boundary which has a single ideal boundary component in the sense of Kerékjártó-Stoïlow. A relatively noncompact subregion Ω of R is said to be an *end* of R if the relative boundary $\partial\Omega$ consists of finitely many analytic Jordan curves (cf. Heins [4]). We denote by $\mathcal{P}(\Omega)$ the class of nonnegative harmonic functions on Ω with vanishing boundary values on $\partial\Omega$. The *harmonic dimension* of Ω, $\dim\mathcal{P}(\Omega)$ in notation, is defined as the minimum number of elements of $\mathcal{P}(\Omega)$ generating $\mathcal{P}(\Omega)$ provided that such a finite set exists, otherwise as ∞. It is known that $\dim\mathcal{P}(\Omega)$ does not depend on a choice of end of R: $\dim\mathcal{P}(\Omega)=\dim\mathcal{P}(\Omega')$ for any pair (Ω, Ω') of ends of R(cf. [4]). In terms of the Martin compactification $\dim\mathcal{P}(\Omega)$ coincides with the number of minimal points over the ideal boundary (cf. Constantinescu and Cornea [3]).

In this paper we especially concern with ends W which are subregion of p-sheeted unlimited covering surfaces of $\{0<|z|\leqq\infty\}$. For these W it is known that $1\leqq\dim\mathcal{P}(W)\leqq p$ (cf. [4]). Consider two positive sequences $\{a_n\}$ and $\{b_n\}$ satisfying $b_{n+1}<a_n<b_n<1$ and $\lim_{n\to\infty}a_n=0$. Set $G=\{0<|z|<1\}-I$ where $I=\bigcup_{n=1}^{\infty}I_n$ and $I_n=[a_n, b_n]$. We take $p(>1)$ copies $G_1, \cdots, G_p$ of G. Joining the upper edge of I_n on G_j and the lower edge of I_n on G_{j+1} (j mod p) for every n, we obtain a p-sheeted covering surface $W=W_p^I$ of $\{0<|z|<1\}$ which is naturally considered as an end of a p-sheeted covering surface of $\{0<|z|\leqq\infty\}$. In [4] Heins proved the followings:

(A) *If I is sufficiently 'thin' at $z=0$ such as*

$$\limsup_{R\ni x\to 0}\hat{R}^I_{G_0}(x)<+\infty,$$

then $\dim\mathcal{P}(W)=p$, *where $\hat{R}^I_{G_0}$ is the balayage of $G_0(z)=\log(1/|z|)$ relative to I on D;*

(B) *if I is sufficiently 'thick' at $z=0$ such as*

Received September 6, 1993.

$$\sum_{n=1}^{\infty} \log \frac{b_n}{a_n} = \infty ,$$

then $\dim \mathcal{P}(W)=1$.

The purpose of this paper is to extend these Heins' results. For example our Theorem 1 (cf. § 1) in more general setting for I implies that if I is *thin* at $z=0$, in the sense that $z=0$ is an irregular boundary point of G with respect to Dirichlet problem, then $\dim \mathcal{P}(W)=p$, which sharpens the above (A). Restricted to the case $p=2^m (m \in \boldsymbol{N})$ our Theorem 2 (cf. § 1) in a bit more general setting for I implies that if I is not thin at $z=0$, then $\dim \mathcal{P}(W)=1$, which partially sharpens the above (B). Consequently we have the following which completely determines the harmonic dimension of $W=W_p^I$ in the case $p=2^m$ (cf. [6]):

THEOREM. *Suppose that* $p=2^m$. *Then*
(i) $\dim \mathcal{P}(W)=p$ *if and only if* I *is thin at* $z=0$;
(ii) $\dim \mathcal{P}(W)=1$ *if and anly if* I *is not thin at* $z=0$.

In § 1 we give preliminaries and state main results Theorems 1 and 2. The proof of Theorem 1 (resp. Theorem 2) is given in § 2 (resp. § 3).

§ 1. Preliminaries from potential theory and statement of main results

1.1. We begin with recalling the definition of balayage. Consider an open Riemann surface F possessing the Green's function. Denote by $\mathcal{S}$ the class of nonnegative superharmonic functions on F. Let E be a subset of F and s belong to $\mathcal{S}$. Then the *balayage* $\hat{R}_s^E = {}^F\hat{R}_s^E$ of s relative to E on F is defined by

$$\hat{R}_s^E(z) = \liminf_{x \to z} \inf \{u(x) : u \in \mathcal{S},\ u \geqq s \text{ on } E\}$$

(cf. e.g. [1]). Let $G_\xi^F(\cdot)$ be the Green's function on F with pole at ξ. We here review fundamental properties of balayage (cf. [1], [2], [5], etc.).

PROPOSITION 1.1. (i) *If* $E_1 \subset E_2$, *then* $\hat{R}_s^{E_1} \leqq \hat{R}_s^{E_2}$;
(ii) $\hat{R}_s^{E_1 \cup E_2} \leqq \hat{R}_s^{E_1} + \hat{R}_s^{E_2}$;
(iii) $\hat{R}_{u+v}^E = \hat{R}_u^E + \hat{R}_v^E$;
(iv) *if* N *is a polar set, then* $\hat{R}_s^{E \cup N} = \hat{R}_s^E$;
(v) *if* E *is a closed subset of* F, *then* $\hat{R}_s^E(z)=s(z)$ *on* E *except possibly for those* $z \in \partial E$ *which are irregular boundary points of* $F-E$;
(vi) $\hat{R}_{G_z^F}^E(x) = \hat{R}_{G_x^F}^E(z)$ *for every* z *and* x *in* F.

Next we state the definition of thinness (cf. [2]).

DEFINITION 1.1. Let z be a point of F and E a subset of F. We say that E is *thin* at z if $\hat{R}_{G_z^F}^E \neq G_z^F$.

Assuming that E is closed and z belongs to E in the above definition, it is well-known that E is thin at z if and only if z is an irregular boundary point of $F-E$ with respect to Dirichlet problem (cf. e.g. [1, p. 348]).

1.2. In the complex plane $\boldsymbol{C}$, we introduce the weakest topology which makes all positive superharmonic functions in $\boldsymbol{C}$ continuous. This topology is called *fine topology* (cf. e.g. [2]). It is well-known that a subset U of $\boldsymbol{C}$ is a fine neighborhood of a point z in $\boldsymbol{C}$ if and only if $\boldsymbol{C}-U$ is thin at z. Here and hereafter, for simplicity, we denote by $G_{\xi}(\cdot)$ the Green's function on $\{|z|<1\}$ with pole at ξ. In § 2 we will be in need of the following proposition (cf. [2]):

PROPOSITION 1.2. *Let E be a domain in $\boldsymbol{C}$ such that the point $z=0$ belongs to ∂E. Suppose that $\boldsymbol{C}-E$ is thin at $z=0$ and h is a positive superharmonic function on E. Then h/G_0 has a fine limit $f\text{-}\lim_{E\ni z\to 0} h(z)/G_0(z)$ at $z=0$, where the fine limit of h/G_0 at $z=0$ is the limit of h/G_0 at $z=0$ with respect to the fine topology.*

1.3. In order to state main results, we begin with fixing the notations. Denote by D the open unit disc $\{|z|<1\}$. Let $\{J_n\}_{n=1}^{\infty}$ be a family of closed segments J_n in $D-\{0\}$ such that $J_m\cap J_n=\emptyset$ for every m and n with $m\neq n$ and $\{J_n\}_{n=1}^{\infty}$ accumulates only at $z=0$ in $D\cup\partial D$. Set $J=\bigcup_{n=1}^{\infty}J_n$ and $S=D-\{0\}-J$. By definition of S, S has two edges on each J_n. Then we denote by J_n^+ one of them and by J_n^- the other. Take $p(>1)$ copies $S_1, \cdots, S_p$ of S and identify along each J_n, the edge J_n^+ on S_j being joined to the edge J_n^- on S_{j+1} (j mod p). We thereby obtain a p-sheeted covering surface $W=W_p$ of $\{0<|z|<1\}$ which is naturally considered as an end of a p-sheeted covering surface of $\{0<|z|\leqq\infty\}$. The followings are our main results.

THEOREM 1. *If J is thin at the origin, then* $\dim\mathcal{P}(W)=p$.

THEOREM 2. *Suppose that $p=2^m (m\in\boldsymbol{N})$ and that J is symmetric with respect to the real axis. If neither of $J\cap\boldsymbol{R}$ and $\boldsymbol{R}-J$ is thin at the origin, then* $\dim\mathcal{P}(W)=1$.

It is easily checked that Theorem in Introduction follows from Theorems 1 and 2.

§ 2. Proof of Theorem 1

2.1. First we give the following proposition:

PROPOSITION 2.1. *Suppose that J and S are the same as in Theorem 1. Then,*

$$f\text{-}\lim_{S\ni z\to 0}\frac{\hat{R}^{J}_{G_0}(z)}{G_0(z)}=0$$

if and only if J is thin at $z=0$, where $\hat{R}^{J}_{G_0}={}^{D}\hat{R}^{J}_{G_0}$.

Proof. The 'only if' part of the assertion follows from the definition of thinness. Suppose that J is thin at $z=0$. Set $D_0=\{z\in D: \mathrm{Re}\,z\geqq 0\}$ and $D_1=\{z\in D: \mathrm{Re}\,z\leqq 0\}$. By (ii) of Proposition 1.1, we only have to prove that

$$f\text{-}\lim_{S\ni z\to 0}\frac{\hat{R}^{J\cap D_k}_{G_0}(z)}{G_0(z)}=0 \tag{1}$$

for $k=0$ and 1. We prove (1) only for $k=0$, since the proof works similarly for $k=1$. By the fact that the open segment $(-1, 0)$ is not thin at $z=0$ and by Proposition 1.2, we have only to prove that

$$\lim_{R\ni z\to -0}\frac{\hat{R}^{J\cap D_0}_{G_0}(z)}{G_0(z)}=0. \tag{2}$$

We take points x in $D_0\cap J$ and z in $(-1, 0)$. From simple calculation we obtain the inequality

$$G_z(x)=\log\left|\frac{1-zx}{x-z}\right|\leqq\log\frac{1}{|z|}=G_z(0).$$

Hence we have

$$\hat{R}^{E}_{G_z}\leqq\hat{R}^{E}_{G_z(0)}=G_z(0)\hat{R}^{E}_{1} \tag{3}$$

on D for $z\in(-1, 0)$ and a subset E of $J\cap D_0$. Let ρ be a real number with $\rho>1$ and set $D(N)=\{|z|<e^{-\rho^N}\}(N\in\boldsymbol{N})$. Then Wiener's criterion implies that

$$\lim_{N\to\infty}\hat{R}^{J\cap D_0\cap D(N)}_{1}(0)=0 \tag{4}$$

(cf. [2, p. 80]). By (ii) and (vi) of Proposition 1.1 and by (3), we have

$$\begin{aligned}\limsup_{R\ni z\to -0}\frac{\hat{R}^{J\cap D_0}_{G_0}(z)}{G_0(z)}&\leqq\limsup_{R\ni z\to -0}\left(\frac{\hat{R}^{J\cap D_0\cap D(N)}_{G_0}(z)}{G_0(z)}+\frac{\hat{R}^{J\cap D_0-D(N)}_{G_0}(z)}{G_0(z)}\right)\\&\leqq\limsup_{R\ni z\to -0}\frac{\hat{R}^{J\cap D_0\cap D(N)}_{G_z}(0)}{G_z(0)}\\&\leqq\hat{R}^{J\cap D\cap D_0(N)}_{1}(0).\end{aligned}$$

Therefore, by letting $N\to\infty$ and by (4), we have the equality (2).

2.2. *Proof of Theorem* 1. Suppose that J is thin at $z=0$. Let π be the projection from W onto $D-\{0\}$. For every $\xi\in S$, we denote by ξ_j the point in W such that $\pi(\xi_j)=\xi$ and $\xi_j\in S_j(j=1, \cdots, p)$. Since the origin is a finely interior point of $S\cup\{0\}$, there exists the fine limit $f\text{-}\lim_{S_j\ni\xi_j\to 0}G^{W}_{\xi_j}(\eta)$ for every $\eta\in W$ (cf. [2]), and hence the fine limit $f\text{-}\lim_{S_j\ni\xi_j\to 0}G^{W}_{\xi_j}(\eta)$ determines an element, denoted by $h_j(\eta)$, belonging to $\mathcal{P}(W)$ for each $j=1, \cdots, p$. Thus, by the fact that $\dim \mathcal{P}(W) \leqq p$, we have only to prove that the family $\{h_1, \cdots, h_p\}$ in $\mathcal{P}(W)$ is linearly independent. To see this, we define positive harmonic functions h_{jk} on S as follows: $h_{jk}(z)=h_j(z_k)$, $j, k=1, \cdots, p$, where $\pi^{-1}(z)=\{z_1, \cdots, z_p\}$ and $z_k \in S_k$. Then, we have only to prove the equality

$$(5) \qquad f\text{-}\lim_{S \ni z \to 0} \frac{h_{jk}(z)}{G_0(z)}=\delta_{jk}$$

where δ_{jk} is the Kronecker delta, since it instantly follows from (5) that the family $\{h_1, \cdots, h_p\}$ is linearly independent.

It is easily seen that

$$(6) \qquad G_\xi(z)=G_z(\xi)=\sum_{j=1}^{p} G^W_{z_k}(\xi_j)=\sum_{j=1}^{p} G^W_{\xi_j}(z_k)$$

for each $z \in S$ and for each $k=1, \cdots, p$ (cf. [4]). Hence, by definitions of h_j and h_{jk}, we obtain the equality

$$(7) \qquad G_0(z)=\sum_{j=1}^{p} h_{jk}(z)$$

on S for each $k=1, \cdots, p$. On the other hand, by (6), we have

$$(8) \qquad G_\xi(z) \geqq \sum_{j \neq k} G^W_{\xi_j}(z_k)$$

for every $z \in S$ and for every $k=1, \cdots, p$. Hence, by (iv) and (v) of Proposition 1.1 and by maximum principle, we find that

$$(9) \qquad \hat{R}^J_{G_\xi}(z)=\hat{R}^{J \cup \{0\}}_{G_\xi}(z) \geqq \sum_{j \neq k} G^W_{\xi_j}(z_k) \qquad (z \in S,\ k=1, \cdots, p),$$

since $\sum_{j \neq k} G^W_{\xi_j}(z_k)$ is considered as a bounded harmonic function on S. Thus, by letting $\xi \to 0$ with respect to the fine topology and by (vi) of Proposition 1.1, we have

$$\hat{R}^J_{G_0}(z) \geqq \sum_{j \neq k} h_{jk}(z).$$

Therefore, by virtue of Proposition 2.1, we obtain

$$(10) \qquad f\text{-}\lim_{S \ni z \to 0} \frac{\sum_{j \neq k} h_{jk}(z)}{G_0(z)}=0.$$

It is easily seen that the equality (5) follows from (7) and (10). The proof is herewith complete.

§3. Proof of Theorem 2

3.1. We first give the following lemma which is useful in the sequel:

LEMMA **3.1.** *Let F be an open Riemann surface, $\tilde{F}$ an unlimited covering* *surface of F, E a subset of F, s a positive superharmonic functin on F and* π *the canonical projection from* $\tilde{F}$ *onto F. Then, it holds that*

$$^{F}\hat{R}_{s}^{E}\circ\pi={}^{\tilde{F}}\hat{R}_{s\circ\pi}^{\pi^{-1}(E)}$$

on $\tilde{F}$.

Proof. Let $\tilde{u}$ be a positive superharmonic function on $\tilde{F}$ satisfying that $\tilde{u}\geqq s\circ\pi$ on $\pi^{-1}(E)$. Setting

$$u(w)=\liminf_{x\to w}\inf\{\tilde{u}(z):\ z\in\pi^{-1}(x)\}$$

on F, we find that u is a positive superharmonic function on F and $u\geqq s$ on E. Hence we have $\tilde{u}\geqq u\circ\pi\geqq{}^{F}\hat{R}_{s}^{E}\circ\pi$ on $\tilde{F}$, which implies that

$$^{\tilde{F}}\hat{R}_{s\circ\pi}^{\pi^{-1}(E)}\geqq{}^{F}\hat{R}_{s}^{E}\circ\pi$$

on $\tilde{F}$. Therefore, by a trivial relation $^{F}\hat{R}_{s}^{E}\circ\pi\geqq{}^{\tilde{F}}\hat{R}_{s\circ\pi}^{\pi^{-1}(E)}$, we have the desired assertion.

3.2. Essential part of the proof of Theorem 2 is to prove the following proposition:

PROPOSITION 3.1. *Suppose that* $p=2$ *and that* J *is symmetric with respect to the real axis. If neither of* $J\cap\boldsymbol{R}$ *and* $\boldsymbol{R}-J$ *is thin at the origin, then* $\dim\mathcal{P}(W)=1$.

Proof. Let h be an element of $\mathcal{P}(W)$ and π the projection from W onto $D-\{0\}$. For a point $z\in W$ which belongs to $S_i(i=1, 2)$, we denote by $\bar{z}$ the point in S_i whose projection coincides with $\overline{\pi(z)}$. Defining $\bar{h}$ by $\bar{h}(z)=h(\bar{z})$ on W, we find that $\bar{h}\in\mathcal{P}(W)$.

First we show that $h(\in\mathcal{P}(W))$ is a constant multiple of $G_0(\pi(z))$ if $h=\bar{h}$. Let τ be the sheet interchange of W. Then, we find a positive constant c such that

$$cG_0(\pi(z))=h(z)+h\circ\tau(z) \tag{11}$$

on W. Since $J\cap\boldsymbol{R}$ is not thin at the origin, by Lemma 3.1, (11) and (iii) of Proposition 1.1, we have

$$\begin{aligned}cG_0(\pi(z))&={}^{D}\hat{R}_{cG_0}^{J\cap\boldsymbol{R}}(\pi(z))\\&={}^{W}\hat{R}_{cG_0\circ\pi}^{\pi^{-1}(J\cap\boldsymbol{R})}(z)\\&\leqq{}^{W}\hat{R}_{h}^{\pi^{-1}(J\cap\boldsymbol{R})}(z)+{}^{W}\hat{R}_{h\circ\tau}^{\pi^{-1}(J\cap\boldsymbol{R})}(z)\\&\leqq h(z)+h\circ\tau(z)\\&=cG_0(\pi(z))\end{aligned}$$

on W and, in particular,

$$h(z)={}^{W}\hat{R}_{h}^{\pi^{-1}(J\cap\boldsymbol{R})}(z) \tag{12}$$

on W. On the other hand, by (11), we also have

$$h(z)=\frac{c}{2}G_{0}(\pi(z)) \tag{13}$$

for every $z\in\pi^{-1}(J\cap\boldsymbol{R})$, because $h=\bar{h}=h\circ\tau$ on $\pi^{-1}(J\cap\boldsymbol{R})$ except possibly a polar subset of $\pi^{-1}(J\cap\boldsymbol{R})$. By means of (12), (13), Lemma 3.1 and the assumption, we conclude that

$$\begin{aligned} h(z)&={}^{W}\hat{R}_{h}^{\pi^{-1}(J\cap\boldsymbol{R})}(z)\\ &={}^{W}\hat{R}_{(c/2)G_{0}\circ\pi}^{\pi^{-1}(J\cap\boldsymbol{R})}(z)\\ &={}^{D}\hat{R}_{(c/2)G_{0}}^{J\cap\boldsymbol{R}}(\pi(z))\\ &=\frac{c}{2}G_{0}(\pi(z)) \end{aligned} \tag{14}$$

on W.

Next we consider the general case. Let $h\in\mathcal{P}(W)$ be a minimal function (cf. e.g. [2]). By the fact that $h+\bar{h}=\overline{h+\bar{h}}$ on W and by the above observation, we find a positive constant a such that

$$h(z)+\bar{h}(z)=aG_{0}(\pi(z))$$

on W, and hence

$$h(z)=\frac{a}{2}G_{0}(\pi(z)) \tag{15}$$

on $\pi^{-1}(\boldsymbol{R}-J)$, because $\bar{z}=z$ on $\pi^{-1}(\boldsymbol{R}-J)$. Since $\boldsymbol{R}-J$ is not thin at the origin, by (15) and Lemma 3.1, we have

$$\begin{aligned} h(z)&\geqq{}^{W}\hat{R}_{h}^{\pi^{-1}(\boldsymbol{R}-J)}(z)\\ &={}^{W}\hat{R}_{(a/2)G_{0}\circ\pi}^{\pi^{-1}(\boldsymbol{R}-J)}(z)\\ &={}^{D}\hat{R}_{(a/2)G_{0}}^{(\boldsymbol{R}-J)\cap D}(\pi(z))\\ &=\frac{a}{2}G_{0}(\pi(z)) \end{aligned}$$

on W. Therefore, by the minimality of h, we find a positive constant k such that

$$h(z)=kG_{0}(\pi(z))$$

on W, which implies that $\dim\mathcal{P}(W)=1$.

3.3. *Proof of Theorem* 2. Take a minimal function h in $\mathcal{P}(W_{p})$, where $p=2^{m}(m\in\boldsymbol{N})$. Let θ be the covering transformation of W_{p}:

$$\theta(w_i)=w_{i+1}, \qquad (i \bmod p,\ i=1, \cdots, p)$$

where $\pi^{-1}(w)=\{w_1, \cdots, w_p\}$ and $w_i \in S_i$ for $w \in D-\{0\}$. Set

$$f_j=\sum_{k=0}^{2m-1-1} h \circ \theta^{2k+j} \qquad (j=0, 1),$$

where $\theta^0=$id.. Then we can consider f_0 as a function in $\mathcal{P}(W_2)$. Hence, by Proposition 3.1, we find a positive constant b such that

$$f_0(z)=bG_0(\pi(z))$$

on W, and hence, by the fact that $f_1=f_0 \circ \theta$, we have

$$f_0(z)=f_1(z)$$

on W. Therefore, by the uniqueness of Martin's integral representation (cf. e.g. [2], [3], [5] etc), we can find an integer l such that

$$h=h \circ \theta^{2l+1}, \tag{16}$$

since $h \circ \theta^i$ is a minimal function for each $i=1, \cdots, p$. On the other hand, we can find two integers α and β such that $\alpha(2l+1)+\beta 2^m=1$. Therefore, by the fact that $\theta^{2^m}=$id. and by (16), we have

$$h=h \circ \theta .$$

From this it follows that $\dim \mathcal{P}(W_p)=1$.

3.4. By applying the same argument as in 3.3 and by the fact that $\dim \mathcal{P}(W_n) \leqq n$, we obtain the following:

THEOREM 3. *Suppose that $p=2^m n$, where $m \in \boldsymbol{N}$ and n is an odd integer. Under the same condition for J as in Theorem 2, it holds that* $\dim \mathcal{P}(W_p) \leqq n$.

Remark. In Theorem 2, we can not omit the condition that $\boldsymbol{R}-J$ is not thin at $z=0$. For example, we assume that $p=2$, $J \subset \boldsymbol{R}$ and $\boldsymbol{R}-J$ is thin at $z=0$. Denote by $\{J'_n\}_{n=1}^{\infty}$ the family of the connected components of $(\boldsymbol{R}-J) \cap D-\{0\}$ and by $\bar{J}'_n$ the closure of J'_n for each n. By replacing $\{J_n\}_{n=1}^{\infty}$ in 1.3 with $\{\bar{J}'_n\}_{n=1}^{\infty}$, we construct a 2-sheeted covering surface W' of $\{0<|z|<1\}$ in the same way as in 1.3. Then $\bigcup_{n=1}^{\infty} \bar{J}'_n$ is thin at $z=0$, and hence Theorem 1 yields that $\dim \mathcal{P}(W')=2$. Therefore we find that $\dim \mathcal{P}(W)=2$ since W is conformally equivalent to W'.

REFERENCES

[1] J. BLIEDTNER AND W. HANSEN, Potential Theory, Springer-Verlag, 1986.
[2] M. BRELOT, On Topologies and Boundaries in Potential Theory, Lecture Notes in Math., 175, Springer-Verlag 1971.

[3] C. Constantinescu and A. Cornea, Ideale Ränder Riemannscher Flächen, Springer-Verlag, 1963.
[4] M. Heins, Riemann surfaces of infinite genus, Ann. of Math., **55** (1952), 296-317.
[5] L. Helms, Introduction to Potential Theory, Wiley-Interscience, 1969.
[6] S. Segawa, Harmonic dimension and extremal length, Kodai Math. J., **17** (1994), 256-261.

Department of Mathematics
Faculty of Science
Kyoto Sangyo University
Kamigamo-motoyama, Kitaku, Kyoto 603
Japan

Department of Mathematics
Daido Institute of Technology
Daido, Minami, Nagoya 457
Japan