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KODAI MATH. J.
17 (1994), 311—319

REMARKS ON FUNDAMENTAL GROUPS OF COMPLEMENTS OF DIVISORS ON ALGEBRAIC VARIETIES

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Introduction

We work over the complex number field $\boldsymbol{C}$, and consider the topological fundamental group of the complement of a divisor on a nonsingular projective variety.

Let V be a nonsingular connected projective variety of dimension $\geqq 2$ and let $D \subset V$ be a reduced divisor. We denote by $p: L \to V$ the line bundle corresponding to the invertible sheaf $\mathcal{O}_V(D)$, and we put

$$L^{\times} := L \backslash \{\text{the zero section}\}.$$

We fix base points $b \in V \backslash D$ and $b' \in L^{\times}$ such that $p(b')=b$. There exists a unique section $s: V \to L$ which defines D and passes through b'. By restricting s to $V \backslash D$, we get a morphism $V \backslash D \to L^{\times}$, which we denote by the same letter s. We consider the homomorphism

$$s_*: \pi_1(V \backslash D, b) \longrightarrow \pi_1(L^{\times}, b').$$

In various "good" situations (for example, when D is very ample and nonsingular), this homomorphism is an isomorphism. The following is a special case of Nori's result [3, Corollary 2.10], which is one of the corollaries of his Weak Lefschetz Theorem.

PROPOSITION (NORI). *Suppose that D is irreducible and not composed of a pencil. If the singular locus of D is of codimension $\geqq 3$ in V, then $\pi_1(V \backslash D)$ is isomorphic to $\pi_1(L^{\times})$.*

In this paper, we give another condition under which s_* is an isomorphism, using some ideas originated in [4]. As an application, we compute the fundamental groups of complements of certain singular plane curves.

Let V be as above and let $\mathfrak{d}$ be a linear system on V. We put

$$\mathrm{Bs}\,\mathfrak{d} := \{x \in V \,;\, x \in D \text{ for all } D \in \mathfrak{d}\}.$$

We also put $V^0 := V \backslash \mathrm{Bs}\,\mathfrak{d}$. Then there is a morphism $\Phi: V^0 \to \mathfrak{d}^{\vee}$ induced by $\mathfrak{d}$

Received September 13, 1993; revised April 19, 1994.

where $\mathfrak{d}^{\vee}$ is the dual projective space of $\mathfrak{d}$. As above, we denote by $p: L \to V$ the line bundle corresponding to $\mathcal{O}_V(D)$ where D is an arbitrary member of $\mathfrak{d}$, and by $L^{\times}$ the complement of the zero section of L. We also put

$$\mathfrak{d}_{nr} := \{D \in \mathfrak{d}\,;\ D \text{ is not reduced}\}.$$

The main result is as follows:

PROPOSITION 1. *Suppose that $\mathfrak{d}$ has no fixed components and the image of Φ is of dimension $\geqq 2$. Suppose also one of the following holds*; (i) $\mathfrak{d}_{nr} \subset \mathfrak{d}$ *is of codimension $\geqq 2$, or* (ii) *every fiber of Φ is of codimension $\geqq 2$ in V^0. Then, for a general member $D \in \mathfrak{d}$, s_* is an isomorphism.*

Note that if s_* is an isomorphism, we have a commutative diagram

$$\begin{array}{ccc} \pi_1(V \backslash D) & \cong & \pi_1(L^{\times}) \\ & {\scriptstyle i_*}\searrow \quad \swarrow{\scriptstyle p_*} & \\ & \pi_1(V), & \end{array} \tag{0.1}$$

where $i: V \backslash D \hookrightarrow V$ is the inclusion. By this commutative diagram, we have an exact sequence

$$\longrightarrow \pi_2(V) \xrightarrow{\ \partial\ } \boldsymbol{Z} \longrightarrow \pi_1(V \backslash D) \longrightarrow \pi_1(V) \longrightarrow 1, \tag{0.2}$$

derived from the homotopy exact sequence of $L^{\times} \to V$. It is easy to see that the image of $\boldsymbol{Z} \to \pi_1(V \backslash D)$ is contained in the center of $\pi_1(V \backslash D)$. Thus $\pi_1(V \backslash D)$ is a central extension of $\pi_1(V)$ by a cyclic group.

We shall study the boundary homomorphism ∂ in the sequence (0.2). The homology class $[D] \in H_{2n-2}(V, \boldsymbol{Z})$ of the divisor D, where $n = \dim V$, defines a linear form

$$\delta: H_2(V, \boldsymbol{Z}) \longrightarrow \boldsymbol{Z}$$

by the intersection paring $H_2(V, \boldsymbol{Z}) \times H_{2n-2}(V, \boldsymbol{Z}) \to \boldsymbol{Z}$.

PROPOSITION 2. *Suppose that s_* is an isomorphism. Then the boundary map ∂ in* (0.2) *is given by*

$$\pi_2(V) \xrightarrow{\ \eta\ } H_2(V, \boldsymbol{Z}) \xrightarrow{\ \delta\ } \boldsymbol{Z},$$

where η is the Hurewicz map and δ is the linear form defined above.

Let $\boldsymbol{Z}_{\geqq 0}$ be the set of non-negative integers. We put

$$\mathcal{S}_d := \{(i, j, k) \in (\boldsymbol{Z}_{\geqq 0})^3\,;\ i+j+k=d\}.$$

For a subset $S \subset \mathcal{S}_d$ of $\mathcal{S}_d$, we denote by $\mathfrak{d}(S)$ the linear system of all curves on $\boldsymbol{P}^2$ whose defining equations are of the form

$$\sum_{(i,j,k)\in S} a_{ijk} X_0^i X_1^j X_2^k = 0,$$

where $(X_0 : X_1 : X_2)$ are homogeneous coordinates of $\boldsymbol{P}^2$. As a corollary of Propositions 1 and 2, we have the following:

PROPOSITION 3. *Suppose that* $\mathrm{Card}(S\cap\{i=0\})\geqq 2$, $\mathrm{Card}(S\cap\{j=0\})\geqq 2$, *and* $\mathrm{Card}(S\cap\{k=0\})\geqq 2$. *Let D be a general member of $\mathfrak{d}(S)$. Then the fundamental group $\pi_1(\boldsymbol{P}^2\backslash D)$ is isomorphic to the cyclic group of order d.*

Example 1. We fix three points $P_1=(1:0:0)$, $P_2=(0:1:0)$ and $P_3=(0:0:1)$ on $\boldsymbol{P}^2$. Let m_1, m_2, m_3 and d be non-negative integers. Let $\mathfrak{d}$ be the linear system of all curves of degree d in $\boldsymbol{P}^2$ which have singularity of multiplicity m_i at each point P_i for $i=1, 2, 3$. Suppose that $m_1+m_2<d$, $m_2+m_3<d$, and $m_3+m_1<d$. Then the fundamental group of the complement of a general member of $\mathfrak{d}$ is isomorphic to $\boldsymbol{Z}/(d)$.

Example 2 (cf. [1, Chapter 4 (3.11)]). We fix affine coordinates (x, y) on $\boldsymbol{P}^2$. Let $d_1>d_2>\cdots>d_\mu$ be a decreasing sequence of positive integers with $\mu\geqq 2$. Consider the projective plane curve C defined by an inhomogeneous equation

$$f_{d_\mu}(x, y)+\cdots+f_{d_2}(x, y)+f_{d_1}(x, y)=0,$$

where $f_{d_i}(x, y)(i=1, \cdots, \mu)$ are general homogeneous polynomials of degree d_i. Then $\pi_1(\boldsymbol{P}^2\backslash C)$ is isomorphic to the cyclic group of order d_1.

In the last section, we give some other elementary examples.

1. Proof of Proposition 1

First, we shall show, by contradiction, that the assumption (ii) implies the assumption (i). Suppose that there exists an irreducible component $\mathfrak{d}'$ of $\mathfrak{d}_{nr}$ of dimension $\dim\mathfrak{d}-1$. Let $\Lambda(\mathfrak{d})\subset H^0(V, L)$ be the linear subspace corresponding to $\mathfrak{d}$, and let $C(\mathfrak{d}')\subset\Lambda(\mathfrak{d})$ be the cone over $\mathfrak{d}'$. Let $s_0\in C(\mathfrak{d}')$ be a general point. We may assume that $C(\mathfrak{d}')$ is nonsingular at s_0. Then the tangent space $T_{s_0, C(\mathfrak{d}')}$ to $C(\mathfrak{d}')$ at s_0 is canonically isomorphic to a linear subspace Λ' of codimension 1 in $\Lambda(\mathfrak{d})$. Let M be a small coordinate neighborhood of s_0 in $C(\mathfrak{d}')$, and let

$$\phi:\ \Delta:=\{z\in\boldsymbol{C}^{\dim C(\mathfrak{d}')};\ |z|<1\}\xrightarrow{\sim} M$$

be the coordinates. We denote by s_z the global section of L corresponding to $\phi(z)\in C(\mathfrak{d}')$, and by D_z the divisor defined by $s_z=0$. Since $s_0\in C(\mathfrak{d}')$ is general and M is small, there exist analytic families of divisors $\{E_z\}_{z\in\Delta}$ and $\{F_z\}_{z\in\Delta}$ over Δ such that

$$D_z=l\cdot E_z+F_z \qquad (l\geqq 2),$$

and E_z are reduced irreducible divisors for all z. Let $U \subset V$ be a classically open neighborhood of V around a general point of E_0, over which there exists a trivialization

$$L|_U \cong \boldsymbol{C} \times U$$

of the line bundle L. Then there exist families of defining functions $\{t_z\}$ and $\{u_z\}$ of E_z and F_z, respectively, on U such that

$$s_z = t_z^l \cdot u_z \tag{1.0}$$

holds on U, where we consider $s_z|_U$ as a function on U by the above trivialization. Let $s' \in T_{s_0, C(\mathfrak{d}')}$ be an arbitrary tangent vector to the cone $C(\mathfrak{d}')$ at s_0. Then we can deform (1.0) to the direction s' in the first order. Let ε be a dual number; $\varepsilon^2 = 0$. We write the first two terms of expansions of s_z, t_z and u_z of this deformation as follows;

$$s_\varepsilon = s_0 + \varepsilon s', \qquad t_\varepsilon = t_0 + \varepsilon t', \quad \text{and} \quad u_\varepsilon = u_0 + \varepsilon u'.$$

Then, considering s' as an element of $\Lambda(\mathfrak{d})$ by the canonical isomorphism $T_{s_0, C(\mathfrak{d}')} \cong \Lambda' \subset \Lambda(\mathfrak{d})$ and regarding $s'|_U$ as a function as above, we see that

$$s' = t_0^{l-1}(l t' u_0 + t_0 u')$$

holds on U. Thus, locally on U, the divisor $\{s'=0\}$ contains E_0 with multiplicity $\geqq l-1>0$. Since E_0 is irreducible, this implies that the divisor $\{s'=0\}$ contains E_0 globally. Since $T_{s_0, C(\mathfrak{d}')} \cong \Lambda' \subset \Lambda(\mathfrak{d})$ is a linear subspace of codimension 1 and $s' \in T_{s_0, C(\mathfrak{d}')}$ is arbitrary, this means that $E_0 \cap V^0$ is mapped to a point by the morphism Φ. (Note that since $\mathfrak{d}$ has no fixed components by the assumption, $\mathrm{Bs}\,\mathfrak{d}$ is of codimension $\geqq 2$ in V. Thus $E_0 \cap V^0$ is non-empty.) This contradicts the assumption (ii). Therefore we may and will assume the assumption (i) from the outset.

Let q_0 be a general point of $\mathfrak{d}^\vee$. Since $\dim \Phi(V^0) \geqq 2$, the inverse image $\Phi^{-1}(q_0) \subset V^0$ is either empty or of codimension $\geqq 2$. We put $V^1 := V^0 \setminus \Phi^{-1}(q_0)$. Let A be the space of all hyperplanes H in the projective space $\mathfrak{d}^\vee$ such that $H \not\ni q_0$. Then A is isomorphic to an affine space. We put

$$W := \{(y, H) \in V^1 \times A;\ \Phi(y) \in H\}$$

and $\mathcal{U} := (V^1 \times A) \setminus W$. Then W is a Zariski closed subset of codimension 1 in $V^1 \times A$. We give W the reduced structure. For $H \in A$, we denote by W_H the scheme theoretic intersection $W \cap (V^1 \times \{H\})$, which is regarded as a divisor of V^1. Then we have $W_H = D_H \cap V^1$, where D_H is the divisor of V corresponding to $H \in (\mathfrak{d}^\vee)^\vee = \mathfrak{d}$. Since $\mathrm{Bs}\,\mathfrak{d}$ is of codimension $\geqq 2$ in V, V^1 admits a non-singular projective compactification V such that $V \setminus V^1 = \Phi^{-1}(q_0) \cup \mathrm{Bs}\,\mathfrak{d}$ is of codimension $\geqq 2$. Combining this with the assumption (i), we can use [4, Theorem 1] and get isomorphisms

(1.1) $$\pi_1(\mathcal{V})\cong\pi_1(V^1\backslash W_H)\cong\pi_1(V\backslash D_H)$$

for a general $H\in A$. These isomorphisms are induced by the inclusions $V^1\backslash W_H \hookrightarrow \mathcal{V}$ and $V^1\backslash W_H\hookrightarrow V\backslash D_H$.

For $y\in V^1$, let l_y be the line in $\mathfrak{d}^\vee$ connecting $\Phi(y)$ and q_0. Note that since $\Phi(y)\neq q_0$, the line l_y is uniquely determined. Since $\Phi^*\mathcal{O}_{\mathfrak{d}^\vee}(1)=\mathcal{O}_{V^0}(D_H\cap V^0)$, we have isomorphisms which fit into the commutative diagram;

$$\begin{array}{ccc} L|_{V^1} & \cong & \{(y, q)\in V^1\times\mathfrak{d}^\vee \,;\, q\in l_y,\ q\neq q_0\}, \\ \cup\!\uparrow & & \cup\!\uparrow \\ L^\times|_{V^1} & \cong & \{(y, q)\in V^1\times\mathfrak{d}^\vee \,;\, q\in l_y,\ q\neq q_0,\ q\neq\Phi(y)\}, \end{array}$$

and are compatible with the projections to V^1. Under these isomorphisms, the section $s(D_H)$ of $L\to V$ defining D_H is given by

$$y \longmapsto (y,\ H\cap l_y)$$

over V_1. (Note that the above isomorphisms are unique up to the automorphisms of the fiber bundles $p: L\to V$ and $p: L^\times\to V$ by the group $\boldsymbol{C}^\times$ acting on fibers by the scalar multiplication. For the section $y\mapsto(y, H\cap l_y)$ to pass through a given point $b'\in L^\times$ as in Introduction, we have to choose the isomorphisms in a suitable way.) If $(y, H)\in\mathcal{V}$, then H intersects the line l_y at a point on $l_y\backslash\{\Phi(y), q_0\}$, because $y\notin D_H$. Therefore, using the above isomorphisms, we have a morphism

$$\begin{aligned} \mathcal{V} &\longrightarrow L^\times|_{V^1} \\ (y, H) &\longmapsto (y,\ H\cap l_y) \qquad (=s(D_H)(y)). \end{aligned}$$

This makes $\mathcal{V}$ a locally trivial fiber space over $L^\times|_{V^1}$, whose fiber is the space of all hyperplanes of $\mathfrak{d}^\vee$ passing through a given point $(\neq q_0)$ and not containing q_0, which is isomorphic to an affine space. Hence we get isomorphisms

$$\pi_1(\mathcal{V}) \cong \pi_1(L^\times|_{V^1}) \cong \pi_1(L^\times).$$

(Note that $L^\times\backslash(L^\times|_{V^1})$ is of codimension $\geqq 2$.) Combining this with (1.1), we have

$$\pi_1(V\backslash D_H) \cong \pi_1(L^\times)$$

for a general $H\in A$. By the constructions, this isomorphism is induced by the section $s(D_H): V\backslash D_H\to L^\times$. □

2. Proof of Proposition 2

We fix base points b of $V\backslash D$, and $*$ of an oriented 2-sphere S^2. Let

$$g: (S^2, *) \longrightarrow (V, b)$$

be a continuous map. We shall consider the image of the homotopy equivalence class $[g]\in\pi_2(V, b_0)$ via the boundary map ∂ in (0.2). Deforming g homotopically relative to $*$, we may assume that the image of g intersects the divisor D at its nonsingular points transversely, and $g^{-1}(g(S^2)\cap D)$ is a finite set of points. We put

$$g^{-1}(g(S^2)\cap D)=\{P_1, \cdots, P_k\}.$$

Let $\mathcal{S}\to V$ be the oriented S^1-bundle associated with $L^\times\to V$, where the orientation is induced by the complex structure of the fibers of $L^\times\to V$. The section $s=s(D): V\backslash D\to L^\times$ induces a section $r(D)$ of $\mathcal{S}|_{V\backslash D}\to V\backslash D$. By pulling back, we get a section of $g^*\mathcal{S}\to S^2$ over $S^2\backslash\{P_1, \cdots, P_k\}$, which we shall denote by $r'(D)$.

For simplicity, we fix some notation. Let P be a point of S^2, and let $\Delta\subset S^2$ be a subset which contains P in its interior and is homeomorphic to the closed disk $\{z\in\boldsymbol{C};\ |z|\leqq 1\}$. We denote by S_P^1 the fiber of $g^*\mathcal{S}\to S^2$ over P, which is oriented. There exists a trivialization

$$g^*\mathcal{S}|_\Delta\cong S_P^1\times\Delta,$$

which is the identity on S_P^1. Now suppose that there is a section $\sigma: \partial\Delta\to g^*\mathcal{S}$ of $g^*\mathcal{S}\to S^2$ over $\partial\Delta$. We define an integer $m(\sigma, \Delta, P)$ to be the mapping degree of the composition

$$\partial\Delta\xrightarrow{\sigma} g^*\mathcal{S}|_\Delta\cong S_P^1\times\Delta\xrightarrow{\mathrm{pr}_1} S_P^1,$$

where the orientation of $\partial\Delta$ is induced from that of S^2. It is easy to see that this number $m(\sigma, \Delta, P)$ is independent of the choice of the trivialization of $g^*\mathcal{S}|_\Delta$. If the section σ extends over the whole Δ, then $m(\sigma, \Delta, P)=0$. By the definition of the boundary map ∂, if Δ contains the base point $*$ in its interior and the section σ is defined over the complement $S^2\backslash$(the interior of Δ), then

$$m(\sigma, \Delta, *)=\partial([g]). \tag{2.1}$$

We fix a small closed disk Δ_i on S^2 with the center P_i. By the definition of the section $r'(D)$, we have

(2.2) $m(r'(D), \Delta_i, P_i)=$the local intersection number of $g(S^2)$ and D at P_i.

(Since $g(S^2)$ and D intersect transversely, these numbers are ± 1.)

Let $\omega_i: I\to S^2$ be a path from $*$ to a point $e_i\in\partial\Delta_i$ such that each ω_i is injective, $\omega_i(I)\cap\Delta_i=\{e_i\}$, and if $i\neq j$, then $\omega_i(I)\cap\Delta_j=\emptyset$ and $\omega_i(I)\cap\omega_j(I)=\{*\}$. We put

$$T_i:=\Delta_i\cup\omega_i(I),$$

and

$$T:=\cup_{i=1}^k T_i.$$

Let d be a standard distance on S^2. For a small positive real number $\varepsilon>0$, we put

$$T_{\iota,\varepsilon} := \{P \in S^2 ;\ \min_{Q \in T_\iota} d(P, Q) \leqq \varepsilon\},$$

$$T_\varepsilon := \{P \in S^2 ;\ \min_{Q \in T} d(P, Q) \leqq \varepsilon\}.$$

Then both of T_ε and $T_{\iota,\varepsilon}$ contain $*$ in their interiors and, if ε is small enough, they are homeomorphic to the closed disk. Moreover, since the section $r'(D)$ is defined over $S^2\backslash$(the interior of T_ε), we have, from (2.1), that

$$m(r'(D), T_\varepsilon, *) = \partial([g]). \tag{2.3}$$

On the other hand, it is obvious that $m(r'(D), \Delta_i, P_\iota) = m(r'(D), T_{\iota,\varepsilon}, *)$. Considering the limit $\varepsilon \to 0$ and taking (2.2) into account, we have

$$m(r'(D), T_\varepsilon, *) = \sum_{\iota=1}^{k} m(r'(D), T_{\iota,\varepsilon}, *) = \sum_{\iota=1}^{k} m(r'(D), \Delta_\iota, P_\iota) = \eta([g]) \cdot [D].$$

Combining this with (2.3), we complete the proof. □

3. Proof of Proposition 3

We put

$$T := \{X_0 X_1 X_2 = 0\} \subset \boldsymbol{P}^2.$$

Then the group $(\boldsymbol{G}_m)^3/(\text{diagonal})$ acts on $\boldsymbol{P}^2\backslash T$ and on $\mathfrak{d}(S)$ compatibly by

$$\boldsymbol{P}^2\backslash T \ni (a : b : c) \longmapsto (\lambda a : \mu b : \nu c) \qquad \text{for} \quad (\lambda, \mu, \nu) \in (\boldsymbol{G}_m)^3.$$

This action on $\boldsymbol{P}^2\backslash T$ is transitive. Therefore we have $\mathrm{Bs}(\mathfrak{d}(S)) \subset T$. In particular, the fixed components of $\mathfrak{d}(S)$ must be contained in T. On the other hand, the assumption implies that no lines in T can be a fixed component of $\mathfrak{d}(S)$. Hence $\mathfrak{d}(S)$ does not have any fixed component.

We consider the restriction

$$\Phi' : \boldsymbol{P}^2\backslash T \longrightarrow \mathfrak{d}(S)^\vee \cong \boldsymbol{P}^s \qquad (s+1 = \mathrm{Card}\, S)$$

of the morphism $\Phi : \boldsymbol{P}^2\backslash \mathrm{Bs}(\mathfrak{d}(S)) \to \mathfrak{d}(S)^\vee$ to $\boldsymbol{P}^2\backslash T$. Let $(U_0 : U_1 : \cdots : U_s)$ be the homogeneous coordinates of $\mathfrak{d}(S)^\vee$ dual to the basis $\{X_0^i X_1^j X_2^k\}_{(i,j,k)\in S}$ of $\mathfrak{d}(S)$. We denote by $X_0^{i_\nu} X_1^{j_\nu} X_2^{k_\nu}$ the monomial corresponding to U_ν. Then the image of Φ' is contained in

$$\boldsymbol{P}^s \backslash \{U_0 U_1 \cdots U_s = 0\}.$$

By changing the numbering if necessary, we may assume that the three vectors (i_0, j_0, k_0), (i_1, j_1, k_1), and (i_2, j_2, k_2) are linearly independent over $\boldsymbol{Q}$ because of the assumption. There exist a positive integer N and integers a_μ, b_μ, c_μ $(\mu = 0, 1, 2)$ such that

$$\sum_{\mu=0}^{2} a_\mu (i_\mu, j_\mu, k_\mu) = (N, 0, 0),$$

$$\sum_{\mu=0}^{2} b_\mu(i_\mu, j_\mu, k_\mu)=(0, N, 0),$$

$$\sum_{\mu=0}^{2} c_\mu(i_\mu, j_\mu, k_\mu)=(0, 0, N).$$

Let Ψ be the composition of the morphisms

$$\begin{aligned} \Psi: \boldsymbol{P}^s\setminus\{U_0\cdots U_s=0\} &\longrightarrow \boldsymbol{P}^2\setminus\{U_0U_1U_2=0\} \longrightarrow \boldsymbol{P}^2\setminus\{V_0V_1V_2=0\} \\ (U_0:\cdots:U_s) &\longmapsto (U_0:U_1:U_2) \longmapsto \left(\prod_{\mu=0}^{2} U_\mu^{a_\mu}: \prod_{\mu=0}^{2} U_\mu^{b_\mu}: \prod_{\mu=0}^{2} U_\mu^{c_\mu}\right). \end{aligned}$$

The composition of Φ' and Ψ is given by

$$\begin{aligned} \boldsymbol{P}^2\setminus\{X_0X_1X_2=0\} &\longrightarrow \boldsymbol{P}^2\setminus\{V_0V_1V_2=0\} \\ (X_0: X_1: X_2) &\longmapsto (X_0^N: X_1^N: X_2^N). \end{aligned}$$

Therefore it is finite and étale. Hence the image of Φ' is of dimension 2 and no curves in $\boldsymbol{P}^2\setminus T$ are mapped to a point by Φ'. On the other hand, the assumption implies that no lines in T are mapped to a point by Φ. Thus the assumption (ii) in Proposition 1 holds. Using Propositions 1 and 2, we complete the proof. □

4. Other examples

Example 3. Let $V\subset\boldsymbol{P}^N$ be a nonsingular projective variety of dimension $\geqq 2$. Suppose that V is simply connected. Let $S\subset\boldsymbol{P}^N$ be a general hypersurface of degree d. Since $\pi_2(V)\cong H_2(V, \boldsymbol{Z})\neq 0$, and the linear map $H_2(V, \boldsymbol{Z})\to\boldsymbol{Z}$ induced from the intersection with $[V\cap S]\in H_{2n-2}(V, \boldsymbol{Z})$ is non-trivial, Proposition 2 implies that $\pi_1(V\setminus S)$ is a finite cyclic group, and its order is in proportion to d.

Example 4. Let V and S be as in Example 3. If V is a complete intersection, then $\pi_1(V\setminus S)\cong\boldsymbol{Z}/(d)$. Indeed, by the generalized Lefschetz-Zariski Theorem due to Goresky-MacPherson ([2]), we have $\pi_1(H\cap(V\setminus S))\cong\pi_1(V\setminus S)$ for a general hyperplane $H\subset\boldsymbol{P}^N$ if $\dim V\geqq 3$. Therefore, we may assume that $\dim V$ is large enough compared with its multi-degree. Then V contains a line and its class generates $H_2(V, \boldsymbol{Z})\cong\boldsymbol{Z}$. Since S is general, this line intersects S at distinct d points transversely. Hence the cokernel of the boundary map $\pi_2(V)\cong H_2(V, \boldsymbol{Z})\to\boldsymbol{Z}$ is $\boldsymbol{Z}/(d)$.

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