X.-Y. LI
KODAI MATH. J.
17 (1994), 299—310

THE COMPLEX OSCILLATION THEORY OF $f''+Af'+Bf=F$, WHERE $A, B, F\not\equiv 0$ ARE TRANSCENDENTAL MEROMORPHIC FUNCTIONS

BY XIAN-YU LI

Abstract

In this paper, we investigate the complex oscillation of the differential equation

$$f''+Af'+Bf=F$$

where A, B, $F\not\equiv 0$ are finite order transcendental meromorphic functions. In some cases we obtain estimates of the order of growth and the exponent of convergence of the zero-sequence of solutions for above equation. Theorem 3 and Theorem 4 are the main results among the Theorems in this paper.

§1. Introduction and results

In this paper, we will use the standard notations of the Nevanlinna theory (e.g. see [9]). In addition, we will also use the same notations as in [1], i.e. we will use, $\lambda(f)$ and $\bar{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of $f(z)$, $\sigma(f)$ to denote the order of growth of $f(z)$. The individual notations will be shown when they appear.

G. Gundersen proved in [8]:

THEOREM A. *If $f\not\equiv 0$ is a solution of*

$$f''+Af'+Bf=0, \tag{1.1}$$

where A, B are entire such that

(i) $\sigma(B)<\sigma(A)<1/2$

or

Key words: Non-homogeneous linear differential equation, Transcendental meromorphic function, Zero-sequence, Exponent of convergence, Order of growth.

AMS No. 34A20, 30D35

Received July 19, 1993.

(ii) *A is transcendental with* $\sigma(A)=0$ *and* B *is a polynomial, then* $\sigma(f)=\infty$.

Gao Shi-an proved in [6]

THEOREM B. *For the equation*

$$f''+a_0 f=p_1 e^{p_0} \tag{1.2}$$

where a_0, p_0, p_1 *are polynomials,* $\deg a_0=n$, $\deg p_0<1+(n/2)$

(a) *If* $n>1$ *and* $\deg p_1<n$, *then every solution* f *of* (1.2) *satisfies*

$$\bar{\lambda}(f)=\lambda(f)=\sigma(f)=1+(n/2)>\deg p_0.$$

(b) *If* $\deg p_1 \geqq n \geqq 0$, *then the solution* f *of* (1.2) *either satisfies* $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=1+(n/2)>\deg p_0$, *or is of the form* $f=Qe^{p_0}$, *where* Q *is a polynomial. And if* (1.2) *has a solution of the form* Qe^{p_0} *with* Q *polynomial, then* (1.2) *must have solutions which satisfy* $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=1+(n/2)>\deg p_0$.

Chen Zong-xuan and Gao Shi-an investigated the complex oscillation of non-homogeneous linear differential equations with rational coefficients in [4].

In this paper, we will investigate the complex oscillation of the second order non-homogeneous linear differential equation

$$f''+Af'+Bf=F \tag{1.3}$$

where A, B, $F\not\equiv 0$ are transcendental meromorphic functions. We will prove the following four theorems:

THEOREM 1. *Suppose that* A, B, $F\not\equiv 0$ *are finite order meromorphic functions, that either* (i) *or* (ii) *below holds*:

(i) $\varlimsup_{r\to\infty} \log m(r, A)/\log r<\varlimsup_{r\to\infty} \log m(r, B)/\log r$

(ii) $\varliminf_{r\to\infty} m(r, B)/\log r=\infty$, *and* A *is rational.*

If non-homogeneous linear differential equation (1.3) *has meromorphic solution* $f(z)$, *then*

(a) *All meromorphic solutions of* (1.3) *satisfy*

$$\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\infty \tag{1.4}$$

with at most one possible finite order meromorphic solution f_0. *If all solutions of* (1.3) *are meromorphic, then* (1.3) *must have solutions which satisfy* (1.4).

(b) *If there exists a finite order meromorphic solution of in case* (a), *then* f_0 *satisfies*

$$\sigma(f_0)\leqq\max\{\bar{\lambda}(f_0), \sigma(F), \sigma(A), \sigma(B)\}.$$

If $\bar{\lambda}(f_0)<\sigma(f_0)$, *and* $\sigma(F)$, $\sigma(A)$, $\sigma(B)$, *are unequal each other, then*

$$\sigma(f_0)=\max\{\sigma(F), \sigma(A), \sigma(B)\}.$$

THEOREM 2. *Suppose that* A, B, $F\not\equiv 0$ *are finite order meromorphic functions having only finitely many poles, that either* (i) *or* (ii) *below holds*:

(i) $\sigma(A)<\sigma(B)$,

(ii) B *is transcendental, and* A *is rational.*

If the equation (1.3) *has meromorphic solutions* $f(z)$, *then*

(a) *All meromorphic solutions of* (1.3) *satisfy* (1.4) *with at most one possible finite order meromorphic solution* f_0. *If all solutions of* (1.3) *are meromorphic, then* (1.3) *must have solutions which satisfy* (1.4).

(b) *If there exists a finite order meromorphic solution* f_0 *in case* (a), *then* f_0 *satisfies*

$$\sigma(f_0)\leqq\max\{\bar{\lambda}(f_0),\ \sigma(B),\ \sigma(F)\}.$$

If $\bar{\lambda}(f_0)<\sigma(f_0)$, $\sigma(F)\neq\sigma(B)$, *then* $\sigma(f_0)=\max\{\sigma(B),\ \sigma(F)\}$.

THEOREM 3. *Suppose that* A, B, $F\not\equiv 0$ *are meromorphic functions having only finitely many poles,* $F\not\equiv cB$(c *is a constant*), *that either* (i) *or* (ii) *below holds*:

(i) $\sigma(B)<\sigma(A)<1/2$, *and* $\sigma(F)<\sigma(A)$

(ii) A *is transcendental and* $\sigma(A)=0$, B *and* F *are rational.*

If $f(z)$ *is a meromorphic solution of* (1.3) *then* f *satisfies* (1.4).

THEOREM 4. *Suppose that* A, B, $F\not\equiv 0$ *are finite order meromorphic functions having only finitely many poles, that either* (i) *or* (ii) *below holds*:

(i) $\sigma(B)<\sigma(A)<1/2$ *and* $\sigma(A)\leqq\sigma(F)$.

(ii) A, F *are transcendental and* $\sigma(A)=0$, B *is rational.*

If the equation (1.3) *has meromorphic solution* $f(z)$, *then*:

(a) *If* $B\equiv 0$, *then all meromorphic solutions of* (1.3) *satisfy* (1.4) *with some possible finite order solutions* $f_c=f_0+c$ (f_0 *is some finite order meromorphic solution,* C *is an arbitrary constant*).

(b) *If* $B\not\equiv 0$, *then all meromorphic solutions of* (1.3) *satisfy* (1.4) *with at most one finite order meromorphic solution* f_0.

(c) *The finite order meromorphic solution* f_c *of* (1.3) *satisfies*

$$\sigma(f_c)\leqq\max\{\sigma(F),\ \bar{\lambda}(f_c)\}.$$

If $\sigma(A)<\sigma(F)$, $\bar{\lambda}(f_c)<\sigma(f_c)$, *then* $\sigma(f_c)=\sigma(F)$

(d) *If all solutions of* (1.3) *are meromorphic, then* (1.3) *must have solutions which satisfy* (1.4).

§ 2. Lemmas

LEMMA 1. *Suppose that* $f(z)=g(z)/h(z)$ *is transcendental meromorphic function having only finitely many poles, where* $g(z)$ *is a transcendental entire function,* h *is a polynomial. Let* z *be a point with* $|z|=r$ *at which* $|g(z)|=M(r, g)$, $h(z)\neq 0$, $\nu(r)$ *denote the centralindex of the entire function* $g(z)$, *then*

$$f'(z)/f(z)=(\nu(r)/z)\ (1+o(1)) \tag{2.1}$$

holds for all $|z|=r$ *outside a subset* E *of* r *of finite logarithmic measure.*

Proof. By $f=g/h$, we have

$$f'(z)=(g'(z)/h(z))-g(z)\cdot(h'(z)/h^2(z)). \tag{2.2}$$

On the other hand, from the Wiman-Valiron theory (see [10, 11, 12]), let z be a point with $|z|=r$, at which $|g(z)|=M(r, g)$, $h(z)\neq 0$, then we have

$$g'(z)=(\nu(r)/z)g(z)(1+o(1)) \qquad r\notin E \tag{2.3}$$

where $E\subset(0, \infty)$ has finite logarithmic measure.

Substituting (2.3) into (2.2), we have

$$\begin{aligned} f'(z)&=(\nu(r)/z)(1+o(1))(g(z)/h(z))-g(z)(h'(z)/h^2(z))\\ &=(\nu(r)/z)\cdot(g(z)/h(z))[(1+o(1))-(\nu(r)/z)^{-1}h'/h]\ (r\notin E). \end{aligned} \tag{2.4}$$

Since $g(z)$ is transcendental, we have $(\nu(r))^{-1}\to o(r\to\infty)$. And $h(z)$ is a polynomial, $|z\cdot h'(z)/h(z)|=O(1)$ $(r\to\infty)$, so

$$(\nu(r)/z)^{-1}(h'/h)=o(1)\ (r\to\infty)\ (r\notin E). \tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain

$$f'(z)=(\nu(r)/z)\cdot f(z)\cdot(1+o(1))\ r\notin E.$$

This proves Lemma 1.

LEMMA 2. *Suppose that* A, B *satisfy the hypotheses of Theorem* 1. *If* $g(z)\not\equiv 0$ *is a meromorphic solution of the homogeneous linear differential equation*

$$g''+Ag'+Bg=0 \tag{2.6}$$

then $\sigma(g)=\infty$.

Proof. If $\sigma(g)<\infty$, then we have from (2.6)

$$m(r, B)\leqq m(r, A)+m(r, g''/g)+m(r, g'/g)=m(r, A)+O(\log r)$$

If A is transcendental, then

$$\overline{\lim_{r\to\infty}} \log m(r, B)/\log r\leqq\overline{\lim_{r\to\infty}} \log m(r, A)/\log r;$$

if A is rational, then $\underline{\lim}_{r\to\infty} m(r, B)/\log r\leqq\underline{\lim}_{r\to\infty} m(r, A)/\log r<M$ ($M>0$ is some constant), this contradict on the hypotheses A, B.

LEMMA 3. *Suppose that* A, B *satisfy hypotheses of Theorem* 3 *or Theorem* 4. *If* $g(z)\not\equiv 0$ *is a meromorphic solution of* (2.6), *then*: *if* $B\not\equiv 0$, *then* $\sigma(g)=\infty$;

if $B\equiv 0$, *then either* $g(z)$ *is a constant, or* $\sigma(g)=\infty$.

Proof. Assume that $g(z)$ is a transcendental meromorphic solution and $\sigma(g)=\sigma<\infty$. By (2.6) and fact that A, B have only finitely many poles, it is easy to see that $g(z)$ has only finitely many poles.

Now set

$$g(z)=u(z)/p(z), \qquad A(z)=u_A/p_A(z), \qquad B(z)=u_B(z)/p_B(z) \tag{2.7}$$

where p, p_A, p_B are polynomials, u, u_A, u_B are entire functions, u, u_A are transcedental, and $\sigma(u_A)=\sigma(A)<1/2$, $\sigma(u_B)=\sigma(B)$, $\sigma(u)=\sigma(g)=\sigma$

From Lemma 1, let z be a point with $|z|=r$ at which $|u(z)|=M(r, u)$, then

$$g'(z)/g(z)=(\nu(r)/z)\ (1+o(1)) \tag{2.8}$$

holds for all $|z|=r$ outside a set E_1 of r of finite logarithmic measure, where, $\nu(r)$ denotes the centralindex of the entire function $u(z)$.

On the other hand, by $\sigma(g)=\sigma<\infty$, and Corollary 2 of [7], we have

$$|g''(z)/g(z)|\leqq|z|^{2\sigma+1} \tag{2.9}$$

for all $|z|=r\notin E_2\cup[0, 1]$, $E_2\subset(1, \infty)$ has finite logarithmic measure.

From (2.6) and (2.7), we have

$$|u_A g'/g|\leqq|p_A\cdot g''/g|+|p_A u_B/p_B|. \tag{2.10}$$

Now divide the discussion into two cases.

CASE I. Suppose that $\sigma(u_A)=\sigma(A)>0$. Then we take ρ, τ such that

$$\sigma(u_B)=\sigma(B)<\rho<\tau<\sigma(u_A)<1/2.$$

From Theorem of $\cos(\pi\sigma)$ type in [2, 3], it is easy to know that there exists a subset $H\subset(1, +\infty)$ with infinite logarithmic measure, such that if $|z|=r\in H$, then

$$\log|u_A(z)|>r^\tau, \qquad \log|u_B(z)|<r^\rho \tag{2.11}$$

By (2.9)-(2.11), for $|z|=r\in H-(E_1\cup E_2\cup[0, 1])$, ($H-(E_1\cup E_2\cup[0, 1])$ has infinite logarithmic measure) we have as $r\to\infty$,

$$|z^2, g'(z)/g(z)|\leqq|z|^2[|p_A\cdot p_B\cdot g''(z)/g(z)|+|p_A u_B|]/|p_B u_A \tag{2.12}$$

$$|z^2\cdot g'/g|\leqq O(r^{M_1})\cdot\exp(r^\rho)/\exp(r^\tau)\longrightarrow 0, \tag{2.13}$$

where $M_1>0$ is a constant.

CASE II. Suppose that $\sigma(u_A)=\sigma(A)=0$, u_A is transcendental, then also from Theorem of $\cos(\pi\sigma)$ type, there exists a subset $H_1\subset(1, \infty)$ with infinite logarithmic measure such that if $|z|=r\in H_1$, then.

$$\min\{\log|u_A(z)| : |z|=r\}/\log r\longrightarrow\infty \qquad (r\to\infty). \tag{2.14}$$

By (2.9), (2.12) and (2.14), for $|z|=r\in H_1-(E_1\cup E_2\cup[0, 1])$ ($H_1-(E_1\cup E_2\cup [0, 1])$ has infinite logarithmic measure), we have as $r\to\infty$

$$|z^2\cdot g'(z)/g(z)|\leqq 0(r^{M_1})/\min|u_A(z)| \longrightarrow 0 . \tag{2.15}$$

Therefore, for both cases above, by (2.13) or (2.15),

$$|z^2\cdot g'(z)/g(z)| \longrightarrow 0 \qquad (r\to\infty) \tag{2.16}$$

holds for $r\in H-(E_1\cup E_2\cup[0, 1])$, or $r\in H_1-(E_1\cup E_2\cup[0, 1])$.

But by (2.8), for such z satisfying $|z|=r\in H-(E_1\cup E_2\cup[0, 1])$ or $r\in H_1-(E_1\cup E_2\cup[0, 1])$ and $|u(z)|=M(r, u)$, $r\to\infty$, we have

$$z^2\cdot g'(z)/g(z)=z\cdot\nu(r)\ (1+o(1)). \tag{2.17}$$

By (2.16) and (2.17), we have $\nu(r)\to 0$ $(r\to\infty)$. This contradicts the fact that u is a transcendental entire function if and only if $\nu(r)\to\infty$ (as $r\to\infty$). Therefore, $u(z)$ either is a polynomial, or satisfies $\sigma(u)=\infty$, i.e. $g(z)$ either is a rational function, or satisfies $\sigma(g)=\infty$.

By (2.6), it is easy to know that if $g(z)\not\equiv 0$ is a nonconstant rational function, then $g''+Ag'+Bg$ is a transcendental function with $\sigma(g''+Ag'+Bg)=\sigma(A)$, this is a contradiction; if $B\not\equiv 0$ and $g(z)$ is a constant $C\neq 0$, then $g''+Ag'+Bg=CB\not\equiv 0$, this contradicts (2.6).

LEMMA 4. *Suppose that A, B, $F\not\equiv 0$ are finite order meromorphic functions. If $f(z)$ is a meromorphic solution of equation* (1.3) *with $\sigma(f)=\infty$, then $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\infty$.*

Proof. We can write from (1.3)

$$1/f=(1/F)(f''/f)+A(f'/f)+B), \tag{2.18}$$

hence

$$N(r, 1/f)\leqq 2\bar{N}(r, 1/f)+N(r, 1/F)+N(r, A)+N(r, B). \tag{2.19}$$

Applying the Lemma of the logarithmic derivative, from (2.18), we have

$$m(r, 1/f)\leqq m(r, 1/F)+m(r, A)+m(r, B)+0\{\log T(r, f)+\log r\}\ (r\notin E) \tag{2.20}$$

where a subset $E\subset(0, \infty)$ has finite linear measure, (2.19) and (2.20) give

$$\begin{aligned} T(r, f)&=T(r, 1/f)+O(1)\\ &\leqq 2\bar{N}(r, 1/f)+T(r, 1/F)+T(r, A)+T(r, B)+O\{\log T(r, f)+\log r\}\ (r\notin E). \end{aligned} \tag{2.21}$$

Since $\sigma(f)=\infty$, there exists $\{r'_n\}(r'_n\to\infty)$ such that

$$\lim_{r'_n\to\infty} \log T(r'_n, f)/\log r'_n=\infty .$$

Setting the linear measure of E, $mE=\delta<\infty$, then there exists a point $r_n \in [r'_n, r'_n+\delta+1]-E$. From

$$\log T(r_n, f)/\log r_n \geqq \log T(r'_n, f)/\log(r'_n+\delta+1)$$
$$=\log T(r'_n f)/[\log r'_n+\log[1+(\delta+1)/r'_n)],$$

we have

$$\varliminf_{r_n\to\infty} \log T(r_n, f)/\log r_n \tag{2.22}$$
$$\geqq \lim_{r'_n\to\infty} \log T(r'_n, f)/[\log r'_n+\log(1+(\delta+1)/r'_n]=\infty.$$

For a given arbitrary large $\beta(\beta>c=\max\{\sigma(A), \sigma(B), \sigma(F)\})$, by (2.22),

$$T(r_n, f)\geqq r_n^{\beta} \tag{2.23}$$

hold for sufficiently large r_n.

On the other hand, for a given ε $(0<\varepsilon<\beta-c)$, for sufficiently large r_n, we have

$$T(r_n, A)<r_n^{c+\varepsilon}, \qquad T(r_n, B)<r_n^{c+\varepsilon}, \qquad T(r_n, F)<r_n^{c+\varepsilon}.$$

By (2.23) as $r_n\to\infty$, we have

$$T(r_n, A)/T(r_n, f)<r_n^{c+\varepsilon-\beta} \longrightarrow 0$$
$$T(r_n, B)/T(r_n, f)<r_n^{c+\varepsilon-\beta} \longrightarrow 0$$
$$T(r_n, F)/T(r_n, f)<r_n^{c+\varepsilon-\beta} \longrightarrow 0$$

Therefore,

$$T(r_n, A)<(1/5)T(r_n, f) \tag{2.24}$$
$$T(r_n, B)<(1/5)T(r_n, f) \tag{2.25}$$
$$T(r_n, F)<(1/5)T(r_n, f) \tag{2.26}$$

hold for sufficiently large r_n. From

$$O\{\log T(r_n, f)+\log r_n\}=o\{T(r_n, f)\},$$

we obtain that

$$O\{\log T(r_n, f)+\log r_n\}\leqq(1/5)T(r_n, f) \tag{2.27}$$

also holds for sufficiently large r_n. Substituting (2.24)-(2.27) into (2.21), we obtain

$$T(r_n, f)<10\overline{N}(r, 1/f). \tag{2.28}$$

By (2.22) and (2.28), we have

$$\infty=\lim_{r_n\to\infty} \log T(r_n, f)/\log r_n \leqq \varlimsup_{r_n\to\infty} \log \overline{N}(r_n, 1/f)/\log r_n \leqq \bar{\lambda}(f)$$

therefore, $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\infty$.

§ 3. Proofs of theorems

Proof of Theorem 1. (a) Assume that f_0 is a meromorphic solution of (1.3) with $\sigma(f_0)=\sigma<\infty$. If $f_1(\not\equiv f_0)$ is second finite order meromorphic solution of (1.3), then $\sigma(f_1-f_0)<\infty$, and f_1-f_0 is a meromorphic solution of the corresponding homogeneous equation (2.6) of (1.3). But $\sigma(f_1-f_0)=\infty$ from Lemma 2, this is a centradiction.

Now assume that $f(z)$ is an infinite order meromorphic solution of (1.3), then $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\infty$ from Lemma 4.

If all solutions of (1.3) are meromorphic functions, then all solutions of the corresponding homogeneous equation (2.6) of (1.3) are meromorphic functions. Assume $\{f_1, f_2\}$ is fundamental solution set of (2.6). By [5, p. 412], we have

$$m(r, B)=O\{\log[\max(T(r, f_S), s=1, 2)]+O(\log r)\}.$$

Since B is transcendental, there exists at least f_1 or f_2 with infinite order of growth. If f_0 is a solution of (1.3), then every solution f of (1.3) can be written in the form

$$f=c_1f_1+c_2f_2+f_0$$

where c_1, c_2 are arbitrary constants. Hence (1.3) must have infinite order solutions, and all infinite order solutions satisfy (1.4) from Lemma 4.

(b) For the finite order meromorphic solution f_0 of (1.3), using the analogous proof as in Lemma 4, and remarking $m(r, f^{(j)}/f)=O\{\log r\}(j=1, 2)$ from $\sigma(f_0)=\sigma<\infty$, we easily know that

$$T(r, f_0)\leqq 2\bar{N}(r, 1/f_0)+T(r, F)+T(r, A)+T(r, B)+O\{\log r\} \tag{3.1}$$

holds for all r. Hence

$$\sigma(f_0)\leqq\max\{\bar{\lambda}(f_0), \sigma(F), \sigma(A), \sigma(B)\} \tag{3.2}$$

If $\bar{\lambda}(f_0)<\sigma(f_0)$, and $\sigma(F)$, $\sigma(A)$, $\sigma(B)$ are different from each other, then from (1.3), we have

$$\sigma(f_0)\geqq\max\{\sigma(F), \sigma(A), \sigma(B)\}. \tag{3.3}$$

Therefore, (3.2) and (3.3) give

$$\sigma(f_0)=\max\{\sigma(F), \sigma(A), \sigma(B)\}. \tag{3.4}$$

Proof of Theorem 2. Theorem 2 immediately follows from Theorem 1.

Proof of Theorem 3. From $F\not\equiv cB$, we know that (1.3) has no constant solutions. If f is a nonconstant rational function, then for case (i), we have $\sigma(f''+Af'+Bf)=\sigma(A)>\sigma(F)$; for case (ii), we have $f''+Af'+Bf$ is transcendental, but F is a rational function. Hence (1.3) has no rational solutions, i.e. f must be a transcendental meromorphic solution.

Now assume that f is a transcendental meromorphic solution with $\sigma(f)=\sigma<\infty$. From (1.3) and fact that A, B, F have only finitely many poles, we know that f has only finitely many poles.

Set

$$f(z)=u(z)/p(z),\ A(z)=u_A/p_A,\ B=u_B/p_B,\ F=u_F/p_F \tag{3.5}$$

where u, u_A, u_B, u_F are entire and u, u_A are transcendental p, p_A, p_B, p_F are polynomials, $\sigma(u)=\sigma(f)=\sigma$, $\sigma(u_A)=\sigma(u)$, $\sigma(u_B)=\sigma(B)$, $\sigma(u_F)=\sigma(F)$.

For f, using the same reasoning as in Lemma 3, by Lemma 1, we have

$$f'(z)/f(z)=(\nu(r)/z)(1+o(1)) \qquad r\notin E_1, \tag{3.6}$$

where $|z|=r$, $|u(z)|=M(r, u)$, $E_1\subset(1, \infty)$ has finite logarithmic measure, $\nu(r)$ denotes the centralindex of $u(z)$. From Corollary 2 of [7], we have

$$|f''(z)/f(z)|\leqq|z|^{2\sigma+1} \qquad r\notin E_2\cup[0, 1] \tag{3.7}$$

where $E_2\subset(1, \infty)$ has finite logarithmic measure. By (3.5) and (1.3), we obtain

$$|u_Af'/f|\leqq[|p_A\cdot p_B\cdot f''/f|+|p_Au_B|]/|p_B|+|u_Fpp_A|/|p_Fu|. \tag{3.8}$$

From $u(z)$ is a transcendental entire function, we take z satisfying $|z|=r$, $|u(z)|=M(r, u)$, then for sufficiently large $|z|$, we have $|u(z)|>1$ and $|u_Fpp_A|/|p_Fu|<|u_Fpp_A|/|p_F|$. By (3.8), we have

$$|u_Af'/f|\leqq[|p_Ap_Bf''/f|+|p_Au_B|]/|p_B|+|u_Fpp_A|/|p_F| \tag{3.9}$$

for sufficiently large $|z|$, and z satisfying $|z|=r$, $|u(z)|=M(r, u)$.

Divide the discussion into two cases.

CASE I. Suppose that $\sigma(u_A)=\sigma(A)>0$, then we take ρ, τ, such that

$$\max\{\sigma(u_B),\ \sigma(u_F)\}<\rho<\tau<\sigma(u_A)<1/2.$$

From theorem of $\cos(\pi\sigma)$ type [2, 3], it is easy to know that there exists a subset $H\subset(1, \infty)$ with infinite logarithmic measure such that if $|z|=r\in H$, then

$$\log|u_A(z)|>r^\tau,\ \log|u_B(z)|<r^\rho,\ \log|u_F(z)|<r^\rho. \tag{3.10}$$

By (3.6)-(3.10), for $|z|=r\in H-(E_1\cup E_2\cup[0, 1])$ and z satifying $|u(z)|=M(r, u)$, $r\to\infty$, we have

$$\begin{aligned}|z^2\cdot f'(z)/f(z)|&\leqq[|z^2p_Ap_Bp_Ff''/f|+|z^2p_Ap_Fu_B|\\&+|z^2u_Fpp_Ap_B|]/|p_Fp_Bu_A|<O(r^{M_1})\exp(r^\rho)/\exp(r^\tau)\longrightarrow 0\end{aligned} \tag{3.11}$$

where $M_1>0$ is a constant.

CASE II. Suppose that $\sigma(u_A)=\sigma(A)=0$, u_A is transcendental, then also from

Theorem of $\cos(\pi\sigma)$ type, there exists a subset $H_1 \subset (1, \infty)$ with infinite logarithmic, measure such that if $|z| = r \in H_1$, then

$$\min\{\log|u_A(z)| : |z| = r\}/\log r \longrightarrow \infty \qquad (r \to \infty). \tag{3.12}$$

By (3.6)-(3.9), (3.12), and the fact that B, F are rational function, for $|z| = r \in H_1 - (E_1 U E_2 U [0.1])$, and z satisfying $|u(z)| = M(r, u)$, $r \to \infty$, we have

$$|z^2 \cdot f'(z)/f(z)| \leqq O(r^{M_1})/\min|u_A(z)| \longrightarrow 0. \tag{3.13}$$

Therefore, for both cases above, by (3.11) or (3.13), for $r \in H - (E_1 \cup E_2 \cup [0.1])$ (or $r \in H_1 - (E_1 \cup E_2 \cup [0, 1])$ and z satisfying $|u(z)| = M(r, u)$, $r \to \infty$, we have

$$|z^2 f'(z)/f(z)| \longrightarrow 0. \tag{3.14}$$

On the other hand, for $r \in H - (E_1 \cup E_2 \cup [0, 1])$ (or $r \in H_1 - (E_1 \cup E_2 \cup [0, 1])$ and z satisfying $|z| = r$, $|u(z)| = M(r, u)$, by (3.6) as $r \to \infty$, we heve

$$z^2 f'(z)/f(z) \sim z \cdot \nu(r). \tag{3.15}$$

(3.15) and (3.14) give $\nu(r) \to 0 \, (r \to \infty)$, this contradicts the fact that u is a transcendental entire function if and only if $\nu(r) \to \infty \, (r \to \infty)$. Therefore, we have $\sigma(f) = \infty$. From Lemma 4, we know that f satisfies (1.4).

Proof of Theorem 4. (a) If $B \equiv 0$, then arbitrary constant c is a solution of the corresponding homogeneous equation (2.6) of (1.3). Assume f_0 is a finite order meromorphic solution of (1.3), then $f_c = f_0 + c$ are also solutions of (1.3). If $f_1 (\not\equiv f_0)$ is second finite order meromorphic solution of (1.3), then $f_1 - f_0$ is a constant solution of the corresponding homogeneous equation (2.6) of (1.3). From Lemma 3 and $\sigma(f_1 - f_0) < \infty$, all finite order meromorphic solutions of (1.3) are of the form $f_c = f_0 + c$.

If f is a meromorphic solution of (1.3) with $\sigma(f) = \infty$, then $\bar{\lambda}(f) = \lambda(f) = \sigma(f) = \infty$ from Lemma 4.

(b) If $B \not\equiv 0$, using the same reasoning as in Theorem 1 by Lemma 3, we know that (1.3) has at most one finite order meromorphic solution f_0. If f is a meromorphic solution of (1.3) with $\sigma(f) = \infty$, then $\bar{\lambda}(f) = \lambda(f) = \sigma(f) = \infty$ from Lemma 4.

(c) For the finite order meromorphic solution f_c of (1.3), using the same reasoning as in Theorem 1, and remarking $\sigma(A) \leqq \sigma(F)$, we can obtain

$$\sigma(f_c) \leqq \max\{\bar{\lambda}(f_c), \sigma(F)\} \tag{3.16}$$

If $\bar{\lambda}(f_c) < \sigma(f_c)$ and $\sigma(A) < \sigma(F)$, then $\sigma(f_c) \geqq \sigma(F)$ from (1.3), combining (3.4), we have $\sigma(f_c) = \sigma(F)$.

(d) We can use the same proof as in Theorem 1 (a).

§4. Examples for having finite order solutions

Example 1. The equation

$$f''-2zf'+(\sin z-2)f=\exp(z^2)\cdot\sin z$$

satisfies hypothses of Theorem 1 or Theorem 2, it has a finite order solution $f=\exp(z^2)$.

Example 2. Suppose A is a transcendental meromorphic function satisfying the additional hypothesis of A in Theorem 4, then the equation

$$f''+Af'+zf=(A+z+1)e^z$$

has finite order solution $f_0=e^z$.

Acknowledgment. The author thank the referees for valuable suggestions that improved our paper.

REFERENCES

[1] S. BANK AND I. LAINE, On the oscillation theory of $f''+Af=0$, where A is entine, Trars. Amer. Math. Soc., **273** (1982), 351-363.
[2] P.D. BARRY, On a theorem of Besicoivitch, Quart. J. Math, Oxford (2), **14** (1963), 293-302.
[3] P.D. BARRY, Some theorems related to the $\cos(\pi\rho)$ theorem, Proc. London Math. Soc. (3), **21** (1970), 334-360.
[4] ZONG-XUAN CHEN AND SHI-AN GAO, On the complex oscillation of non-homogeneous linear differential equations with meromorphic coefficients, Kodai Math. J., **15** (1992), 65-78.
[5] G. FRANK AND S. HELLERSTEIN, On the meromorphic solutions of non-homogeneous linear differential equations with polynomial coefficients, Proc. London Math. Soc. (3), **53** (1986), 407-428.
[6] SHI-AN GAO, On the complex oscillation of solution of non-homogeneous linear differential equations with polynomial coefficients, Comment. Math. Univ. St. Pauli, **38** (1989), 11-20.
[7] G. GUNDERSEN, Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates, J. London Math. Soc. (2), **37** (1988), 88-104.
[8] G. GUNDERSEN, Finite order solutions of second order linear differential equations, Trans. Amer. Math. Soc., **305** (1988), 415-429.
[9] W. HAYMAN, Meromorphic Functions, Clarendon Press, Oxford, 1964.
[10] W. HAYMAN, The local growth of power series: a survey of the Wiman-Valiron method, Canad. Math. Bull., **17** (1974), 317-358.
[11] YU-ZAN HE AND XIU-ZHI XIAO, Algebroid Functions and Ordinary Differential Equations, Science Press, 1988 (in Chinese).
[12] G. VALIRON, Functions Analytiques, Presses Universitaires de France, Paris, 1954.

Department of Mathematics
Jiangxi Normal University
Nanchang
P.R. China