S. SEGAWA
KODAI MATH. J.
17 (1994), 256—261

HARMONIC DIMENSION AND EXTREMAL LENGTH

Dedicated to Professor Nobuyuki Suita on his sixtieth birthday

BY SHIGEO SEGAWA

Consider an open Riemann surface R with a single ideal boundary component. A subregion $V(\neq R)$ of R is said to be an *end* of R if V is relatively noncompact in R and the relative boundary ∂V consists of finitely many analytic Jordan curves. Denote by $\mathcal{P}(V)$ the class of nonnegative harmonic functions on V with vanishing boundary values on ∂V:

$$\mathcal{P}(V)=\{h\in HP(V): h|\partial V=0\},$$

where $HP(V)$ is the class of nonnegative harmonic functions on V. The dimension of the linear space $\mathcal{P}(V)\ominus\mathcal{P}(V)=\{h_1-h_2: h_1, h_2\in\mathcal{P}(V)\}$ is referred to as the *harmonic dimension* of V (cf. Heins [4]), $\dim\mathcal{P}(V)$ in notation. It is known that $\dim\mathcal{P}(V)$ does not depend on a choice of an end V of R (cf. [4]): $\dim\mathcal{P}(V_1)=\dim\mathcal{P}(V_2)$ for any pair (V_1, V_2) of ends of R.

Denote by O_G the class of open Riemann surfaces of null boundary and by M the class of open Riemann surfaces $R\in O_G$ such that there exists an end V of R with $\dim\mathcal{P}(V)=1$. In terms of Martin compactification an R belongs to M if and only if R is of null boundary and the Martin boundary of R consists of a single point (cf. e.g. Constantinescu and Cornea [3]). We are particularly interested in the following result by Heins [4] (see also [7]):

THEOREM A. *Let V be an end and $\{A_n\}$ be a sequence of mutually disjoint annuli in V satisfying that A_{n+1} separates A_n from the ideal boundary of V for every n. If the sum of moduli of A_n diverges, then* $\dim\mathcal{P}(V)=1$.

We also denote by O_G'' the class of open Riemann surfaces having a regular exhaustion $\{R_n\}_{n=0}^{\infty}$ such that each $A_n=R_{2n}-\overline{R_{2n-1}}$ $(n=1, 2, \cdots)$ is a doubly connected region and $\sum_{n=1}^{\infty}\operatorname{mod}A_n=\infty$, where $\operatorname{mod}A_n$ is the modulus of A_n. Then the above Heins' result is restated as

$$O_G''\subset M.$$

* This work was partially supported by Grant-in Aid for Scientific Research, No. 04302006, Japanese Ministry of Education, Science and Culture.
Received June 5, 1993.

The main purpose of this paper is to show that *the inclusion* $O_G^{\prime\prime}\subset M$ *is strict.*

In § 1, we study harmonic dimension of an end which is a two-sheeted covering surface of the punctured unit disc $\{0<|z|<1\}$. In § 2, applying a fact obtained in § 1, we give an example which shows that the inclusion $O_G^{\prime\prime}\subset M$ is strict.

Harmonic dimension of two-sheeted covering surfaces

1.1. Consider two sequences $\{a_n\}$ and $\{b_n\}$ satisfying $0<b_{n+1}<a_n<b_n<1$ and $\lim_{n\to\infty} a_n=0$. Denote by $D=D(\{a_n\}, \{b_n\})$ the region $\Delta-\bigcup_{n=1}^{\infty} I_n$, where $I_n=[a_n, b_n]$ and $\Delta=\{0<|z|<1\}$. Take N copies $D_1, D_2, \cdots, D_N$ of D. Joining the upper edge of I_n in D_m and the lower edge of I_n in D_{m+1} (mod. N) for every n, we obtain an N-sheeted covering surface $W=W(\{a_n\}, \{b_n\})$ of $\{0<|z|<1\}$. We can view W as an end of an N-sheeted covering surface of $\{0<|z|\leqq\infty\}$. From Theorem A it follows that if $\sum_{n=1}^{\infty}\log(b_n/a_n)=\infty$ then $\dim\mathcal{P}(W)=1$. Heins also showed the following (cf. [4]):

THEOREM B. *Let D and W be the same as above. Then* (i) $\dim\mathcal{P}(W)$ *is at most N and* (ii) $\dim\mathcal{P}(W)=N$ *if the set $I=\bigcup_{n=1}^{\infty} I_n$ is sufficiently 'thin' at the point $z=0$ such as* $\limsup_{R\ni x\to 0}\hat{R}^I_{\log(1/|z|)}(x)<\infty$, *where $\hat{R}^I_{\log(1/|z|)}$ is the balayage of* $\log(1/|z|)$ *on* $\{|z|<1\}$ *with respect to I.*

Here and hereafter we restrict our attention to the case $N=2$. Thus V is an end of a two-sheeted covering surface R of $\{0<|z|\leqq\infty\}$ which is ramified over $\bigcup_{n=1}^{\infty}\{a_n, b_n\}$. Denote by π the projection of R onto $\{0<|z|\leqq\infty\}$. From (i) of Theorem B it follows that $\dim\mathcal{P}(W)=1$ or 2, since $\mathcal{P}(W)\neq\emptyset$. We first prove the following which sharpens the above result in the case $N=2$:

THEOREM 1. *Suppose that $N=2$. Then* $\dim\mathcal{P}(W)=2$ *if and only if the point $z=0$ is an irregular boundary point of the domain D with respect to Dirichlet problem.*

The proof is given in 1.2 and 1.3.

1.2. To begin with we state Heins' duality relation between harmonic dimensions and bounded harmonic functions. Heins [4] proved the following which is applied to the proof of Theorem 1:

THEOREM C. *Let V be an end. Then* $\dim\mathcal{P}(V)=1$ *if and only if every bounded harmonic function on V has a limit at the ideal boundary of V.*

We are in the stage of proving 'if part' of Theorem 1. Denote by u_1 (resp. u_2) be the bounded harmonic function on D_1 (resp. D_2) with boundary values 1 (resp. -1) on $\{|z|=1\}$ and 0 on $I=\bigcup_{n=1}^{\infty} I_n$. Let w be a function on W such that $w=u_i$ ($i=1, 2$) on D_i. By the Schwarz reflection principle we see that w can be considered as a bounded harmonic function on W. Since u_1 has a positive upper limit at $z=0$ by assumption (cf. e.g. Helms [5]), w does not have a limit at the ideal boundary of W. Therefore Theorems B and C imply that $\dim \mathcal{P}(W)=2$.

1.3. Suppose that $z=0$ is a regular boundary point of D. We show that $\dim \mathcal{P}(W)=1$. We may assume that

$$(1) \qquad \sum_{n=1}^{\infty} \log \frac{a_n}{b_{n+1}} = \infty .$$

In fact, otherwise, we have $\sum_{n=1}^{\infty} \log (a_n/b_n)=\infty$, which implies that W satisfies the condition of Theorem A. Hence Theorem A yields that $\dim \mathcal{P}(W)=1$.

Let p be a point in W. If p belongs to the sheet D_i ($i=1, 2$), then we denote by $\bar{p}$ the point which belongs to D_i and lies over $\overline{\pi(p)}$. Take an arbitrary $u \in HB(W)$, the space of bounded harmonic functions on W, and set $u^*(p) =(1/2)(u(p)+u(\bar{p}))$. Observe that u^* is a bounded harmonic function on W satisfying $u^*(p)=u^*(\bar{p})$. We also set

$$v(z)=\frac{1}{2}(u(p_1)+u(p_2))$$

for $z \in \Delta$, where $\pi^{-1}(z)=\{p_1, p_2\}$. Then v is bounded and harmonic on Δ, and hence on $\{|z|<1\}$. In particular we see that v is continuous on $I \cup \{0\}$. Consider two functions $v_i=u^* | D_i$ ($i=1, 2$). Note that $v(z)=u^*(p_1)=u^*(p_2)$ for every $z \in I$. Hence we have that v_i can be viewed as a bounded harmonic function on D with continuous boundary values $v | I \cup \{0\}$ on $I \cup \{0\}$. Therefore v_i has a limit at $z=0$ by assumption.

We show that u has a limit at the ideal boundary of W, which completes the proof by virtue of Theorem C. Put $J_n=[b_{n+1}, a_n]$ ($n=1, 2, \cdots$) and $J=\bigcup_{n=1}^{\infty} J_n$. Consider two functions $u_i=u | D_i$ ($i=1, 2$), which are viewed as bounded harmonic functions on D. Note that $u_i=v_i$ ($i=1, 2$) on J. Hence, as proved in the preceding paragraph, we have

$$(2) \qquad \lim_{J \ni z \to 0} (u_1(z)-u_2(z))=0 .$$

For $r \in J$, denote by $\delta_i(r)$ the oscillation of u_i on $\{|z|=r\}$ ($i=1, 2$):

$$\delta_i(r)=\max_{|z|=r} u_i(z)-\min_{|z|=r} u_i(z) .$$

We also denote by $\delta(r)$ the oscillation of u on $\{|\pi(p)|=r\}$:

$$\delta(r)=\max_{|\pi(p)|=r} u(p)-\min_{|\pi(p)|=r} u(p) .$$

It is easily seen that

(3) $$\delta(r)\leqq\delta_1(r)+\delta_2(r)+|u_1(r)-u_2(r)|.$$

Set $\delta_n=\min_{r\in J_n}(\delta_1(r)+\delta_2(r))$. Then we have

$$\delta_n\leqq\delta_1(r)+\delta_2(r)\leqq\sum_{j=1}^{2}\int_0^{2\pi}\left|\frac{\partial u_j(re^{i\theta})}{\partial\theta}\right|d\theta$$

for $r\in J_n$. Hence, by the Schwarz inequality and integration on J_n, we obtain

$$\delta_n^2\mu_n\leqq 4\pi\sum_{j=1}^{2}\int_{b_{n+1}}^{a_n}\int_0^{2\pi}\frac{1}{r^2}\left|\frac{\partial u_j}{\partial\theta}\right|^2 rdrd\theta\leqq 4\pi\sum_{j=1}^{2}D_n(u_j),$$

where $\mu_n=\log(a_n/b_{n+1})$ and $D_n(u_i)$ is Dirichlet integral of u_i on $\{b_{n+1}<|z|<a_n\}$. Since u has finite Dirichlet integral on $\{0<|\pi(p)|<a_1\}$, this implies that $\sum_{n=1}^{\infty}\delta_n^2\mu_n$ converges. Therefore it follows from (1) that $\liminf_{n\to\infty}\delta_n=0$. Hence, by (2) and (3), there exists a sequence $\{r_n\}$ such that $r_n\in J_n$ and $\lim_{n\to\infty}\delta(r_n)=0$. By means of maximum principle, this implies that u has a limit at the ideal boundary. The proof is herewith complete.

Extremal length of dividing curves

2.1. Consider an open Riemann surface R and its relatively compact subregion $F(\neq\emptyset)$. Let $\Gamma=\Gamma(R-\bar{F})$ be the family of closed curves in $R-\bar{F}$ which separate the ideal boundary of R from F and $\lambda(\Gamma)=\lambda(\Gamma(R-\bar{F}))$ be the extremal length of Γ. For the detail of extremal length, we refer to e.g. Ahlfors and Sario [2]. Denote by O'_S the class of open Riemann surfaces R such that $\lambda(\Gamma(R-\bar{F}))=0$ for an F. It is well-known that the property $\lambda(\Gamma(R-\bar{F}))=0$ does not depend on a choice of F. Kusunoki [6] showed the following (see also Shiga [8]):

THEOREM D. $O''_S\subset O'_S\subset M$.

Set $a_n=e^{-n}(1-e^{-n^2})$ and $b_n=e^{-n}$. For these sequences $\{a_n\}$ and $\{b_n\}$ and for $N=2$, let D_0 and W_0 be the region $D(\{a_n\}, \{b_n\})$ and the end $W(\{a_n\}, \{b_n\})$, respectively, which are considered in no. 1.1. We claim the following:

THEOREM 2. *For the end W_0 given above,* $\dim\mathcal{P}(W_0)=1$ *and* $\lambda(\Gamma)>0$, *where Γ is the family of closed curves in W_0 which separate the ideal boundary of W_0 from ∂W_0.*

The proof is given in 2.2 and 2.3.

2.2. Let R be the two-sheeted covering surface of $\{0<|z|\leqq\infty\}$ which branches over $\bigcup_{n=1}^{\infty}\{a_n, b_n\}$ and φ be the projection of R onto $\{0<|z|\leqq\infty\}$. Theorem 2 implies that $R\in M-O'_S$. Combining this with Theorem D it is immediately seen that $O'_S<M$, and hence $O''_S<M$, where $<$ means strict inclusion.

It is easily proved that $\dim \mathcal{P}(W_0)=1$. In fact, since

$$\sum_{n=1}^{\infty} \frac{n}{\log(4/(b_n-a_n))} \geqq \sum_{n=1}^{\infty} \frac{1}{n+3}=\infty,$$

the Wiener criterion yields that the point $z=0$ is a regular boundary point of D_0 (cf. e.g. Tsuji [9]). Hence, by virtue of Theorem 1, we have the conclusion.

In order to prove that $\lambda(\Gamma)>0$, we provide the following lemma:

LEMMA. *Let A_n be the annulus $\{|z|<1\}-([0, a_n]\cup[b_n, 1])$. Then* $\mathrm{mod}\, A_n<\pi^2/n^2$.

In fact A_n is conformally equivalent to the region $C-([-1, 0]\cup[r_n, +\infty])$ where $r_n=(1-a_nb_n)(b_n-a_n)a_n^{-1}(1+b_n)^{-2}$. Hence we have $\mathrm{mod}\, A_n \leqq \pi^2/\log(16/r_n)$ (cf. e.g. Ahlfors [1]). It is easy to see that $\log(16/r_n)>n^2$.

2.3. We start on the proof of $\lambda(\Gamma)>0$. Let Γ_n be the family of $\gamma\in\Gamma$ satisfying that $\varphi(\gamma)\cap I_n\neq\emptyset$. Observe that

$$\Gamma=\bigcup_{n=1}^{\infty}\Gamma_n. \tag{4}$$

Set $\Gamma_n^*=\{\varphi(\gamma):\gamma\in\Gamma_n\}$. Then Γ_n^* is a family of curves in $\Delta=\{0<|z|<1\}$. It is easily seen that

$$2\lambda(\Gamma_n)\geqq\lambda(\Gamma_n^*) \tag{5}$$

for each n. Denote by u the harmonic measure of the annulus A_n with respect to the outer boundary and set $\rho=|\nabla u|$. We may assume that ρ is defined on Δ. Let C_n be the family of closed curves in the annulus A_n which separate the inner boundary from the outer boundary. Set $\bar{\gamma}=\{\bar{z}:z\in\gamma\}$ for $\gamma\in\Gamma_n^*$. Observe that $\gamma\cup\bar{\gamma}$ for each $\gamma\in\Gamma_n^*$ contains a closed curve which is approximated by a sequence in C_n. Hence we have

$$4\lambda(\Gamma_n^*)\geqq \inf_{\gamma\in\Gamma_n^*}\frac{(2L(\gamma, \rho))^2}{A(\rho)}\geqq \inf_{\gamma\in\Gamma_n^*}\frac{L(\gamma\cup\bar{\gamma}, \rho)^2}{A(\rho)}\geqq \inf_{c\in C_n}\frac{L(c, \rho)^2}{A(\rho)}=\frac{2\pi}{\mathrm{mod}\, A_n},$$

where $L(\gamma, \rho)=\int_\gamma \rho|dz|$ and $A(\rho)=\iint_\Delta \rho^2 dxdy$ $(z=x+iy)$. By means of (4), (5) and Lemma, this yields that

$$\lambda(\Gamma)^{-1}\leqq\sum_{n=1}^{\infty}\lambda(\Gamma_n)^{-1}\leqq\frac{4}{\pi}\sum_{n=1}^{\infty}\mathrm{mod}\, A_n\leqq 4\pi\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty,$$

which implies $\lambda(\Gamma)>0$.

REFERENCES

[1] L. AHLFORS, Conformal Invariants, McGraw-Hill, 1973.
[2] L. AHLFORS AND L. SARIO, Riemann Surfaces, Princeton, 1960.
[3] C. CONSTANTINESCU AND A. CORNEA. Ideale Ränder Riemannscher Flächen, Springer, 1963.
[4] M. HEINS, Riemann surfaces of infinite genus, Ann. of Math., **55** (1952), 296-317.
[5] L. HELMS, Introduction to Potential Theory, Wiley-Interscience, 1969.
[6] Y. KUSUNOKI, On Riemann's period relations on open Riemann surfaces, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math., **30** (1956), 1-22.
[7] S. SEGAWA, A duality relation for harmonic dimensions and its applications, Kodai Math. J., **4** (1981), 508-514.
[8] H. SHIGA, On harmonic dimensions and bilinear relations on open Riemann surfaces, J. Math. Kyoto Univ., **21** (1981), 861-879.
[9] M. TSUJI, Potential Theory in Modern Function Theory, Chelsea, 1975.

DEPARTMENT OF MATHEMATICS
DAIDO INSTITUTE OF TECHNOLOGY
DAIDO, MINAMI, NAGOYA 457
JAPAN