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KODAI MATH J.
17 (1994), 246—255

SOME RESULTS ON RIGIDITY OF HOLOMORPHIC MAPPINGS

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1. In this paper we study rigidity properties of holomorphic mappings. Let X and Y be complex normed spaces. Let D_1 be a balanced domain in X and D_2 be a bounded convex balanced domain in Y. We consider holomorphic mappings f from D_1 into D_2. We prove two theorems. One of them is a generalization of the Schwarz lemma, which gives an upper bound for $\mu_{D_2}(f(x))$, $x \in D_1$. Here μ_{D_2} denotes the Minkowski functional of D_2. We also discuss the extremal mappings related to the Schwarz lemma. We deduce as a corollary the following fact: if $f: X \to Y$ is a holomorphic mapping which satisfies $\|f(x)\| = \|x\|$ for all $x \in X$, then f is linear. Another theorem gives a lower bound for $\mu_{D_2}(f(x))$, $x \in D_1$. Finally we are concerned with the limits of sequences of automorphisms of bounded domains. It is known that if D is a bounded domain in $\boldsymbol{C}^n$ and if a mapping $f: D \to D$ is a pointwise limit of a sequence of automorphisms of D, then f is also an automorphism of D. However, in the case that D is a bounded domain in a complex normed space X the limit $f: D \to D$ need not be an automorphism of D. We give a simple counterexample. Using the above two theorems we show that the limit f is one-to-one.

2. We summarize the main notation and terminology used in this paper. Let X be a complex normed space and let D be a domain in X. The Minkowski functional μ_D of D is defined by

$$\mu_D(x) = \inf\{t > 0 : t^{-1}x \in D\} \qquad (x \in X).$$

We denote the open ball with center at a and radius r in X by $B(a, r)$. Then we have that $\mu_{B(0,r)}(x) = r^{-1}\|x\|$.

Let X and Y be complex normed spaces and let D be a domain in X. A mapping $f: D \to Y$ is said to be holomorphic in D if, corresponding to every $a \in D$, there exist a power series $\sum_{k=0}^{\infty} P_k$ and a positive number ρ such that f is expressed by

$$f(x) = \sum_{k=0}^{\infty} P_k(x-a) \qquad (x \in B(a, \rho)).$$

Received May 7, 1993; revised October 29, 1993.

Here P_k is a continuous k-homogeneous polynomial from X to Y, and the convergence is uniform on $B(a, r)$ for every r with $0<r<\rho$. We use the notation

$$\hat{d}^k f(a)=k\,!\,P_k \qquad (k=0, 1, 2, \cdots),$$

and we call the power series

$$\sum_{k=0}^{\infty} \frac{1}{k\,!}\hat{d}^k f(a)(x-a)$$

the Taylor series of f at a. We refer to [1], [2] and [6] for further details.

3. There are some generalizations of the Schwarz lemma in complex normed spaces (see, for example, [2], [3], [4] and [5]). We also give a generalization of the Schwarz lemma. We adapt Rudin's proof [8, Theorem 8.1.2] in which only the finite dimensional case is considered.

Let X and Y be complex normed spaces and let D_1 and D_2 be balanced domains in X and Y, respectively. Suppose that $D_1 \neq X$. Then there exists a point $x \in X$ with $\mu_{D_1}(x)>0$. For a holomorphic mapping $f: D_1 \to D_2$ we define

$$\tilde{\lambda}_f=\inf\left\{\frac{\mu_{D_2}(\hat{d} f(0)(x))}{\mu_{D_1}(x)} : x \in X,\ \mu_{D_1}(x)>0\right\}$$

and

$$\tilde{\Lambda}_f=\sup\left\{\frac{\mu_{D_2}(\hat{d} f(0)(x))}{\mu_{D_1}(x)} : x \in X,\ \mu_{D_1}(x)>0\right\}.$$

Here $\hat{d} f(0)=\hat{d}^1 f(0)$ is the linear part of the Taylor series of f at the origin 0 of X.

THEOREM 1. *Let X and Y be complex normed spaces. Suppose that*

(i) *D_1 is a balanced domain in X,*

(ii) *D_2 is a bounded convex balanced domain in Y,*

(iii) *$f: D_1 \to D_2$ is a holomorphic mapping with $f(0)=0$.*

Then

(a) $\mu_{D_2}(f(x)) \leqq \mu_{D_1}(x) \quad (x \in D_1)$,

(b) $\mu_{D_2}(\hat{d} f(0)(x)) \leqq \mu_{D_1}(x) \quad (x \in D_1)$,

(c) *if $D_1 \neq X$, then $0 \leqq \tilde{\lambda}_f \leqq \tilde{\Lambda}_f \leqq 1$.*

The equality $\tilde{\lambda}_f=1$ holds if and only if the equality $\mu_{D_2}(f(x))=\mu_{D_1}(x)$ holds for all $x \in D_1$.

Proof. We first note that since D_2 is bounded, convex and balanced, μ_{D_2} is a norm on Y and that since D_1 is balanced, the power series

$$\sum_{k=1}^{\infty} \frac{1}{k\,!}\hat{d}^k f(0)(x)$$

converges to f uniformly on every compact subset of D_1. (See [1], Corollary 5.2).

Take a point $x_0 \in D_1$ with $\mu_{D_1}(x_0)>0$. Put $y=\mu_{D_1}(x_0)^{-1}x_0$. Then $y \in \partial D_1$. Let φ be a continuous linear functional on Y of norm 1, i. e.,

$$\|\varphi\|=\sup\{|\varphi(x)| : x \in Y,\ \mu_{D_2}(x)=1\}=1\,.$$

We define the function g by

$$g(\zeta)=\varphi(f(\zeta y)) \qquad (\zeta \in \boldsymbol{C},\ |\zeta|<1)\,.$$

Then g is holomorphic and $|g(\zeta)| \leqq 1$ in $\Delta=\{\zeta \in \boldsymbol{C} : |\zeta|<1\}$, and $g(0)=0$. Now applying the classical Schwarz lemma we have

$$|g(\zeta)| \leqq |\zeta| \qquad (\zeta \in \Delta)\,,$$

$$|g'(0)| \leqq 1\,.$$

Since $g'(0)=\varphi(\hat{d}f(0)(y))=\mu_{D_1}(x_0)^{-1}\varphi(\hat{d}f(0)(x_0))$, these inequalities imply

$$|\varphi(f(x_0))| \leqq \mu_{D_1}(x_0)\,,$$

$$|\varphi(\hat{d}f(0)(x_0))| \leqq \mu_{D_1}(x_0)\,.$$

The Hahn-Banach theorem assures the existence of continuous linear functionals φ_0 and φ_1 on Y of norm 1 such that $\varphi_0(f(x_0))=\mu_{D_2}(f(x_0))$ and $\varphi_1(\hat{d}f(0)(x_0))=\mu_{D_2}(\hat{d}f(0)(x_0))$. Hence we obtain the inequalites

$$\mu_{D_2}(f(x_0)) \leqq \mu_{D_1}(x_0)\,,$$

$$\mu_{D_2}(\hat{d}f(0)(x_0)) \leqq \mu_{D_1}(x_0)\,.$$

Suppose that $x_0 \in D_1$ and $\mu_{D_1}(x_0)=0$. For every $\varepsilon>0$ there is a positive number t such that $0<t<\varepsilon$ and $t^{-1}x_0 \in D_1$. Considering the function

$$g(\zeta)=\varphi(f(\zeta t^{-1}x_0)) \qquad (\zeta \in \Delta)\,,$$

we have

$$\mu_{D_2}(f(x_0)) \leqq t\,, \qquad \mu_{D_2}(\hat{d}f(0)(x_0)) \leqq t\,.$$

Hence $\mu_{D_2}(f(x_0))=\mu_{D_2}(\hat{d}f(0)(x_0))=0$. Thus (a) and (b) are proved.

Now the inequality

$$0 \leqq \tilde{\lambda}_f \leqq \tilde{\Lambda}_f \leqq 1$$

is an immediate consequence of (b).

Suppose that $\tilde{\lambda}_f=1$. Then $\mu_{D_2}(\hat{d}f(0)(x))=\mu_{D_1}(x)$ for all $x \in X$. Let $x_0 \in D_1$. If $\mu_{D_1}(x_0)=0$, then $\mu_{D_2}(f(x_0))=0$ as we have shown. If $\mu_{D_1}(x_0)>0$, we consider the function

$$g_1(\zeta)=\varphi_1(f(\zeta y))\,, \qquad y=\mu_{D_1}(x_0)^{-1}x_0\,.$$

Then

$$g_1'(0)=\mu_{D_1}(x_0)^{-1}\varphi_1(\hat{d}f(0)(x_0))=\mu_{D_1}(x_0)^{-1}\mu_{D_2}(\hat{d}f(0)(x_0))=1$$

and so $g_1(\zeta)=\zeta$ for all $\zeta\in\Delta$. Hence

$$\mu_{D_1}(x_0)=g_1(\mu_{D_1}(x_0))=\varphi_1(f(x_0))\leqq\|\varphi_1\|\mu_{D_2}(f(x_0))=\mu_{D_2}(f(x_0)).$$

Consequently we obtain that

$$\mu_{D_2}(f(x))=\mu_{D_1}(x) \qquad (x\in D_1).$$

Conversely, suppose that $\mu_{D_2}(f(x_0))=\mu_{D_1}(x_0)$ for some $x_0\in D_1$ with $\mu_{D_1}(x_0)>0$. We consider the function

$$g_0(\zeta)=\varphi_0(f(\zeta y)), \qquad y=\mu_{D_1}(x_0)^{-1}x_0.$$

Since $g_0(\mu_{D_1}(x_0))=\mu_{D_1}(x_0)$, we have $g_0(\zeta)=\zeta$ for all $\zeta\in\Delta$ and so $g_0'(0)=1$. Hence we obtain

$$\mu_{D_1}(x_0)=\varphi_0(\hat{d}f(0)(x_0))\leqq\mu_{D_2}(\hat{d}f(0)(x_0)).$$

Thus if $\mu_{D_2}(f(x))=\mu_{D_1}(x)$ for all $x\in D_1$, then $\tilde{\lambda}_f\geqq 1$ and hence $\tilde{\lambda}_f=1$.

Remarks. If $\mu_{D_2}(f(x_0))=\mu_{D_1}(x_0)$ for some $x_0\in D_1$ with $\mu_{D_1}(x_0)>0$, then $\tilde{\Lambda}_f=1$. However, $\tilde{\Lambda}_f=1$ does not imply that there exists a point $x_0\in D_1$ with $\mu_{D_1}(x_0)>0$ such that $\mu_{D_2}(f(x_0))=\mu_{D_1}(x_0)$. Indeed, let $X=Y=c_{00}$ and $D_1=D_2=\{x\in c_{00}:\|x\|<1\}$. Here c_{00} is the vector space of all sequences $x=(x_1, x_2, \cdots, x_n, \cdots)$ of complex numbers having only a finite number of non-vanishing terms, with norm

$$\|x\|=\max_n|x_n|.$$

The mapping $f:X\to Y$ defined by

$$f:(x_1, x_2, \cdots, x_n, \cdots)\longmapsto\left(\frac{1}{2}x_1, \frac{2}{3}x_2, \cdots, \frac{n}{n+1}x_n, \cdots\right)$$

maps D_1 into D_2 and satisfies $f(0)=0$ and $\tilde{\Lambda}_f=1$. But $\mu_{D_2}(f(x))=\|f(x)\|\neq\|x\|=\mu_{D_1}(x)$ for every $x\in D_1$ with $\mu_{D_1}(x)=\|x\|>0$.

Moreover, we consider the mapping $g:X\to Y$ defined by

$$g:(x_1, x_2, x_3, \cdots, x_n, \cdots)\longmapsto(x_1^2, x_1, x_2, \cdots, x_{n-1}, \cdots).$$

Then g maps D_1 into D_2 and satisfies $g(0)=0$ and $\tilde{\lambda}_g=1$. But $g\neq\hat{d}g(0)$. Thus $\tilde{\lambda}_g=1$ does not imply that $g=\hat{d}g(0)$.

COROLLARY. *Let X and Y be complex normed spaces. If $f:X\to Y$ is a holomorphic mapping which satisfies $\|f(x)\|=\|x\|$ for all $x\in X$, then f is linear.*

Proof. Let M be a positive number. By the assumptions it follows that f

maps $B(0, M)$ into $B(0, M)$ and satisfies $f(0)=0$ and

$$\mu_{B(0,M)}(f(x))=\mu_{B(0,M)}(x) \qquad (x\in B(0, M)).$$

Hence by Theorem 1 we have $\tilde{\lambda}_f=1$, and so

$$\mu_{B(0,M)}(\hat{d}f(0)(x))=\mu_{B(0,M)}(x) \qquad (x\in X).$$

Therefore it follows that the equality

$$\|f(x)\|=\|\hat{d}f(0)(x)\|$$

holds for all $x\in X$. Replacing x by ζx we have

$$\left\|\sum_{k=1}^{\infty}\frac{1}{k!}\zeta^{k-1}\hat{d}^k f(0)(x)\right\|=\|\hat{d}f(0)(x)\| \qquad (x\in X,\ \zeta\in \boldsymbol{C}).$$

Let φ be a continuous linear functional on Y. Then the function

$$h(\zeta)=\varphi\Big(\sum_{k=1}^{\infty}\frac{1}{k!}\zeta^{k-1}\hat{d}^k f(0)(x)\Big)$$

is holomorphic and bounded in $\boldsymbol{C}$. Hence the Liouville theorem says that h is constant. Since the dual space Y^* of Y separates points on Y, it now follows that

$$\sum_{k=2}^{\infty}\frac{1}{k!}\zeta^{k-1}\hat{d}^k f(0)(x)=0 \qquad (\zeta\in \boldsymbol{C}).$$

Therefore we conclude that $f=\hat{d}f(0)$.

Next we prove a theorem which gives a lower bound for $\mu_{D_2}(f(x))$, $x\in D_1$. In our proof the following fact plays an important role.

PROPOSITION. *Let X be a complex normed space and D be a convex balanced domain in X. Let k_D denote the Kobayashi pseudodistance of D. Then*

$$k_D(0, x)=\frac{1}{2}\log\frac{1+\mu_D(x)}{1-\mu_D(x)}$$

and

$$\{x\in D : k_D(0, x)<\alpha\}=rD, \qquad r=\frac{e^{2\alpha}-1}{e^{2\alpha}+1}.$$

(See [2], Theorem IV.1.8).

THEOREM 2. *Let X and Y be complex normed spaces. Suppose that*

(i) *D_1 is a balanced domain in X,*

(ii) *D_2 is a bounded convex balanced domain in Y,*

(iii) *$f : D_1\to D_2$ is a holomorphic mapping with $f(0)=0$.*

Then

$$\mu_{D_2}(f(x)) \geqq \frac{\mu_{D_1}(x)(\tilde{\lambda}_f - \mu_{D_1}(x))}{1 - \tilde{\lambda}_f \mu_{D_1}(x)} \qquad (x \in D_1).$$

Proof. Take a point $x_0 \in D_1$ with $\mu_{D_1}(x_0) > 0$. Put $y = \mu_{D_1}(x_0)^{-1} x_0$ and define the mapping g from Δ into D_2 by

$$g(\zeta) = f(\zeta y) \qquad (\zeta \in \Delta).$$

Since D_1 is balanced, the Taylor series of f at 0

$$\sum_{k=1}^{\infty} \frac{1}{k!} \hat{d}^k f(0)(x)$$

converges to f uniformly on every compact subset of D_1. Hence we have

$$g(\zeta) = \sum_{k=1}^{\infty} \frac{1}{k!} \zeta^k \hat{d}^k f(0)(y) \qquad (\zeta \in \Delta),$$

and hence we can write

$$g(\zeta) = \zeta h(\zeta) \qquad (\zeta \in \Delta),$$

where h is a holomorphic mapping from Δ into Y with the Taylor series

$$h(\zeta) = \sum_{k=1}^{\infty} \frac{1}{k!} \zeta^{k-1} \hat{d}^k f(0)(y)$$

at 0. Let $0 < t < 1$. By Proposition and the definition of k_{D_2} we have that if $\zeta h(\zeta) \neq 0$, then

$$k_{D_2}(0, t\zeta h(\zeta)) < k_{D_2}(0, \zeta h(\zeta)) = k_{D_2}(g(0), g(\zeta)) \leqq \frac{1}{2} \log \frac{1+|\zeta|}{1-|\zeta|},$$

and so

$$t\zeta h(\zeta) \in |\zeta| D_2.$$

Since D_2 is balanced, this implies that

$$th(\zeta) \in D_2 \qquad (\zeta \in \Delta, \zeta \neq 0).$$

By Theorem 1, $h(0) = \hat{d} f(0)(y) \in \bar{D}_2$ and so $th(0) \in D_2$. Thus th is also a holomorphic mapping from Δ into D_2. Therefore by Proposition and the definition of k_{D_2} we have the following two inequalities:

$$\begin{aligned} k_{D_2}(0, th(0)) = k_{D_2}(0, t\hat{d} f(0)(y)) &= \frac{1}{2} \log \frac{1 + t\mu_{D_2}(\hat{d} f(0)(y))}{1 - t\mu_{D_2}(\hat{d} f(0)(y))} \\ &\geqq \frac{1}{2} \log \frac{1 + t\tilde{\lambda}_f}{1 - t\tilde{\lambda}_f}, \end{aligned}$$

and

$$k_{D_2}(th(0), th(\zeta)) \leqq \frac{1}{2} \log \frac{1+|\zeta|}{1-|\zeta|}.$$

Moreover, by the triangle inequality we have

$$k_{D_2}(0,\ th(\zeta)) \geqq k_{D_2}(0,\ th(0)) - k_{D_2}(th(0),\ th(\zeta)).$$

Combining these inequalities we obtain

$$k_{D_2}(0,\ th(\zeta)) \geqq \frac{1}{2} \log \frac{(1+t\tilde{\lambda}_f)(1-|\zeta|)}{(1-t\tilde{\lambda}_f)(1+|\zeta|)}.$$

Now this inequality and Proposition show that

$$\begin{aligned}\mu_{D_2}(tg(\zeta)) &= |\zeta|\mu_{D_2}(th(\zeta)) = |\zeta|\Phi(k_{D_2}(0,\ th(\zeta))) \\ &\geqq |\zeta|\Phi\Big(\frac{1}{2} \log \frac{(1+t\tilde{\lambda}_f)(1-|\zeta|)}{(1-t\tilde{\lambda}_f)(1+|\zeta|)}\Big) = \frac{|\zeta|(t\tilde{\lambda}_f - |\zeta|)}{1-t\tilde{\lambda}_f|\zeta|},\end{aligned}$$

where $\Phi(s)=(e^{2s}-1)/(e^{2s}+1)$. Letting $t\to 1$ and putting $\zeta=\mu_{D_1}(x_0)$ we obtain the desired inequality

$$\mu_{D_2}(f(x_0)) \geqq \frac{\mu_{D_1}(x_0)(\tilde{\lambda}_f - \mu_{D_1}(x_0))}{1-\tilde{\lambda}_f\mu_{D_1}(x_0)}.$$

4. Finally we study the limits of sequences of automorphisms of bounded domains. If D is a bounded domain in $\boldsymbol{C}^n$ and if $f: D\to D$ is a pointwise limit of a sequence $\{F_n\}$ of automorphisms of D, then f is also an automorphism of D. This follows from the fact that $\{F_n\}$ has a subsequence $\{F_{n_k}\}$ which converges to f uniformly on every compact subset of D. (See [7], pp. 78-82). However, in the case that D is a bounded domain in a complex normed space X the limit $f: D\to D$ of a sequence of automorphisms of D need not be an automorphism of D. In this section using Theorems 1 and 2 we prove that f is one-to-one.

Let X be a complex normed space and D be a domain in X. The automorphisms of D are the biholomorphic mappings from D onto D. We denote by $\mathrm{Aut}(D)$ the group of all automorphisms of D. We begin with a simple example.

Example. Let $X=c_{00}$ and $D=\{x\in X: \|x\|<1\}$. Define the mappings F_n, $n=1, 2, \cdots$, and f by

$$F_n: (x_1, x_2, x_3, \cdots, x_n, x_{n+1}, \cdots) \longmapsto (x_n, x_1, x_2, \cdots, x_{n-1}, x_{n+1}, \cdots)$$

and

$$f: (x_1, x_2, x_3, \cdots, x_n, \cdots) \longmapsto (0, x_1, x_2, \cdots, x_{n-1}, \cdots).$$

Then D is a bounded domain in X and $F_n\in\mathrm{Aut}(D)$, $n=1, 2, \cdots$. Moreover,

$$f(x)=\lim_{n\to\infty} F_n(x) \qquad (x\in D).$$

However, $f\notin\mathrm{Aut}(D)$.

We next prove two lemmas. For a holomorphic mapping $f: D \to X$ we define

$$\lambda_f(a) = \inf\left\{\frac{\|\hat{d}f(a)(x)\|}{\|x\|} : x \in X,\ x \neq 0\right\} \qquad (a \in D),$$

and

$$\Lambda_f(a) = \sup\left\{\frac{\|\hat{d}f(a)(x)\|}{\|x\|} : x \in X,\ x \neq 0\right\} \qquad (a \in D).$$

Here we note that if f maps $B(0, r_1)$ into $B(0, r_2)$, then

$$\lambda_f(0) = \frac{r_2}{r_1}\tilde{\lambda}_f, \qquad \Lambda_f(0) = \frac{r_2}{r_1}\tilde{\Lambda}_f.$$

LEMMA 1. *Let D be a bounded domain in X. Let $F \in \mathrm{Aut}(D)$ and $a \in D$. If M, r and ρ are positive numbers such that $D \subset B(0, M)$, $B(a, r) \subset D$ and $B(F(a), \rho) \subset D$, then*

$$\frac{\rho}{2M} \leqq \lambda_F(a) \leqq \Lambda_F(a) \leqq \frac{2M}{r}.$$

Proof. Since $F^{-1} \circ F = F \circ F^{-1} = \mathrm{id.}$, we have $\hat{d}F^{-1}(F(a)) \circ \hat{d}F(a) = \hat{d}F(a) \circ \hat{d}F^{-1}(F(a)) = \mathrm{id.}$ Hence it follows that $\hat{d}F(a) \in \mathrm{Aut}(X)$ and $\lambda_F(a)\Lambda_{F^{-1}}(F(a)) = 1$.

Put $G(x) = F(x+a) - F(a)$. Then G maps $B(0, r)$ into $B(0, 2M)$ and $G(0) = 0$. Hence, by Theorem 1, we obtain

$$\Lambda_F(a) = \Lambda_G(0) = \frac{2M}{r}\tilde{\Lambda}_G \leqq \frac{2M}{r}.$$

On the other hand, since F^{-1} maps $B(F(a), \rho)$ into $B(0, M)$, we have

$$\lambda_F(a) = \frac{1}{\Lambda_{F^{-1}}(F(a))} \geqq \frac{\rho}{2M}.$$

LEMMA 2. *Let D be a bounded domain in X. Let $F \in \mathrm{Aut}(D)$, $a \in D$ and $\lambda_F(a) \geqq \lambda_0 > 0$. Let M and r be positive numbers such that $B(a, r) \subset D \subset B(0, M)$. If $0 < t < (r^2\lambda_0/4M)$, then*

$$F(B(a, t)) \supset B\left(F(a), \frac{\lambda_0}{2}t\right).$$

Proof. Put $G(x) = F(x+a) - F(a)$. Then G maps $B(0, r)$ into $B(0, 2M)$ and $G(0) = 0$. Hence applying Theorem 2 to G we have

$$\|G(x)\| \geqq \frac{2M}{r^2}\frac{\|x\|(r^2\lambda_G(0) - 2M\|x\|)}{2M - \lambda_G(0)\|x\|} \qquad (x \in B(0, r)).$$

Since $\lambda_G(0) = \lambda_F(a) \geqq \lambda_0$, we have

$$\|F(x) - F(a)\| \geqq \frac{2M}{r^2}\frac{\|x-a\|(r^2\lambda_0 - 2M\|x-a\|)}{2M - \lambda_0\|x-a\|} \qquad (x \in B(a, r)).$$

Hence using the inequality $\lambda_0 \leqq \lambda_F(a) \leqq \Lambda_F(a) \leqq (2M/r)$, we obtain that if $\|x-a\| \leqq (r^2\lambda_0/4M)$, then

$$\|F(x)-F(a)\| \geqq \frac{4M^2\lambda_0}{8M^2-r^2\lambda_0^2}\|x-a\| \geqq \frac{\lambda_0}{2}\|x-a\|.$$

Since $F \in \mathrm{Aut}(D)$, this inequality shows that if $0<t<(r^2\lambda_0/4M)$, then

$$F(B(a, t)) \supset B\left(F(a), \frac{\lambda_0}{2}t\right).$$

Now we can prove the following theorem.

THEOREM 3. *Let D be a bounded domain in X. Suppose that $f: D \to D$ is a pointwise limit of a sequence $\{F_n\}$ of automorphisms of D. Then f is one-to-one in D.*

Proof. Assume that there exist two distinct points a_1 and a_2 in D such that $f(a_1)=f(a_2)=b$. Since $b \in D$, there is a positive number ρ with $B(b, 2\rho) \subset D$. Hence we can choose a positive integer n_0 such that if $n>n_0$, then $B(F_n(a_i), \rho) \subset D$, $i=1, 2$. Take positive numbers M, r_1 and r_2 such that $B(0, M) \supset D$, $B(a_1, r_1) \subset D$ and $B(a_2, r_2) \subset D$. Then, by Lemma 1, we have, if $n>n_0$, then

$$\lambda_{F_n}(a_i) \geqq \frac{\rho}{2M} \qquad (i=1, 2),$$

and hence, by Lemma 2, if $n>n_0$ and $0<t<(r_i^2\rho/8M^2)$, then

$$F_n(B(a_i, t)) \supset B\left(F_n(a_i), \frac{\rho}{4M}t\right) \qquad (i=1, 2).$$

On the other hand, we can choose a positive number t and a positive integer n satisfying conditions:

(i) $0<t<\min\left\{\frac{r_1^2\rho}{8M^2}, \frac{r_2^2\rho}{8M^2}\right\}$, and $B(a_1, t) \cap B(a_2, t)=\emptyset$,

(ii) $n>n_0$ and $B\left(F_n(a_1), \frac{\rho}{4M}t\right) \cap B\left(F_n(a_2), \frac{\rho}{4M}t\right) \neq \emptyset$.

These facts contradict that $F_n \in \mathrm{Aut}(D)$. Therefore it follows that f is one-to-one in D.

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