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THE HADAMARD VARIATION OF THE GROUND STATE VALUE OF SOME QUASI-LINEAR ELLIPTIC EQUATIONS

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1. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^N$ $(N\geqq 2)$ with smooth boundary $\partial\Omega$. Let $\rho(x)$ be a real smooth function on $\partial\Omega$ and ν_x be the exterior unit normal vector at $x\in\partial\Omega$. For any sufficiently small $\varepsilon\geqq 0$, let Ω_ε be the domain bounded by

$$\partial\Omega_\varepsilon=\{x+\varepsilon\rho(x)\nu_x\,;\ x\in\partial\Omega\}.$$

Fix $p\in(1, \infty)$ and let q be a fixed number satisfying $0<q<p^*-1$, where $p^*=\infty$ if $p\geqq N$ and $p^*=Np/(N-p)$ if $p<N$. Then we consider the following problem.

$$\lambda(\varepsilon)=\inf_{X_\varepsilon}\int_{\Omega_\varepsilon}|\nabla u|^p dx\,, \tag{1.1$_\varepsilon$}$$

where

$$X_\varepsilon=\{u\in W_0^{1,p}(\Omega_\varepsilon)\,;\ \|u\|_{L^{q+1}(\Omega_\varepsilon)}=1,\ u\geqq 0 \text{ a. e.}\}.$$

It is easy to see that there exists at least one non-negative solution u_ε which attains $(1.1)_\varepsilon$ and which satisfies

$$\begin{aligned} -\operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon(x))&=\lambda(\varepsilon)u_\varepsilon^q(x) \qquad x\in\Omega_\varepsilon \\ u_\varepsilon(x)&=0 \qquad x\in\partial\Omega_\varepsilon \\ u_\varepsilon(x)&\geqq 0 \qquad \text{a. e. } x\in\Omega_\varepsilon. \end{aligned} \tag{1.2}$$

Furthermore $u_\varepsilon\in C^{1+\alpha}(\bar{\Omega}_\varepsilon)$ for some $\alpha\in(0, 1)$.

In this note we want to show the following.

THEOREM 1. *Assume that $p\geqq 2$ and $q\geqq p-1$. Assume that the minimizer u_0 of $(1.1)_0$ is unique. Then, the following asymptotic behaviour of $\lambda(\varepsilon)$ holds.*

$$\lambda(\varepsilon)-\lambda(0)=-\varepsilon(p-1)\int_{\partial\Omega}\left|\frac{\partial u_0}{\partial\nu_x}(x)\right|^p\rho(x)d\sigma_x+o(\varepsilon). \tag{1.3}$$

Here $\partial/\partial\nu_x$ denotes the derivative along the exterior normal direction.

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Remarks. When $p=2$ and $q=1$, the formula (1.3) can be found, for example, in Hadamard [7], Garabedian-Schiffer [3].

When $p=2$ and $q>1$, the formula (1.3) can be found in Osawa [11] with the additional assumption that $\mathrm{Ker}(\Delta+\lambda(0)qu_0^{q-1})=\{0\}$. Therefore the result of this paper is an improvement of Osawa [11, Theorem 1, pp. 258-259]. Furthermore he treated the Hadamard variation of (1.2) under the Robin boundary condition and the Neumann boundary condition. As an application of [11], the problem of asymptotic behaviour of non-linear eigenvalues under singular variation of domains is studied by Ozawa [12], Ozawa-Roppongi [13].

When $p=q-1$, the uniqueness of the minimizer of $(1.1)_0$ is shown in Lindqvist [10]. When $p=2$, $q>1$ and Ω is a ball, the uniqueness of the minimizer of $(1.1)_0$ is shown in Gidas, Ni and Nirenberg [4].

The regularity of the non-negative solution u_ε of (1.2) is discussed, for example, in Dibenedetto [1], Guedda-Veron [6], Lieberman [9], Sakaguchi [14], Tolksdorf [16], [17]. It should be noticed that the solution of (1.2) with $p\neq 2$ does not always belong to $C^2(\bar{\Omega}_\varepsilon)$, since the p-Laplacian is degenerate elliptic when $p\neq 2$.

The reader who is unfamiliar with Hadamard's variation may be referred to Hadamard [7], Garabedian-Schiffer [3], Fujiwara-Ozawa [2], Shimakura [15].

Section 2 contains preliminary material. The asymptotic formula (1.3) is established in section 3. In Appendix we give some regularity properties of the solution of (1.2) and give some inequalities. Throughout section 2 and section 3 we assume all the assumption in Theorem 1.

2. Preliminary Lemma

In this section we would like to construct a nice C^∞-diffeomorphism between $\bar{\Omega}$ and $\bar{\Omega}_\varepsilon$ for any sufficiently small $\varepsilon>0$. Let U_0 be a neighbourhood of $\partial\Omega$ in $\boldsymbol{R}^N$ such that there exists a unique $P\in C^\infty(U_0, \partial\Omega)$ satisfying $|x-P(x)|=\mathrm{dist}(x, \partial\Omega)$ for $x\in U_0$. Let O be a neighbourhood of $\partial\Omega$ in Ω as in Lemma A.2 in the Appendix. Then $u_0\in C^2(\bar{O})$. Let Ω' (Ω'', respectively) be a bounded domain with a smooth boundary $\partial\Omega'=\{x-\delta\nu_x\,;\,x\in\partial\Omega\}$ ($\partial\Omega''=\{x-2\delta\nu_x\,;\,x\in\partial\Omega\}$, respectively) for any sufficiently small $\delta>0$. We fix $\delta>0$ so that $\Omega\setminus\Omega''\Subset U_0$ and $\Omega\setminus O\Subset\Omega''\Subset\Omega'\Subset\Omega$ hold. Then $\Omega'\Subset\Omega_\varepsilon$ holds for any sufficiently small $\varepsilon>0$.

We take a $\phi\in C^\infty(\bar{\Omega}, \boldsymbol{R})$ such that $0\leqq\phi\leqq 1$, $\phi=0$ on Ω'' and $\phi=1$ on $\bar{\Omega}\setminus\Omega'$. We put

$$\Phi_\varepsilon(x)=\begin{cases} x & x\in\Omega'' \\ x+\varepsilon\phi(x)\rho(P(x))\nu_{P(x)} & x\in\bar{\Omega}\setminus\Omega'', \end{cases}$$

where $\nu_{P(x)}$ denotes the exterior unit normal vector at $P(x)\in\partial\Omega$.

Then we can see that $\Phi_\varepsilon:\bar{\Omega}\to\bar{\Omega}_\varepsilon$ is a surjective diffeomorphism for any sufficiently small $\varepsilon>0$ and that the following properties (2.1), (2.2), (2.3) and (2.4) hold.

(2.1) We put $\Phi_\varepsilon(x)=x+\varepsilon S(x)$ for $x\in\bar{\Omega}$. Then

$$S\in C^\infty(\bar{\Omega}, \boldsymbol{R}^N) \quad \text{and} \quad \|S\|_{C^m(\bar{\Omega})^N}\leqq C_m \quad (m=0, 1, 2, \cdots)$$

holds for a constant C_m independent of ε.

(2.2) There exists a $t^{(\varepsilon)}\in C^\infty(\bar{\Omega}_\varepsilon, \boldsymbol{R}^N)$ satisfying

$$\Phi_\varepsilon^{-1}(x)=x+\varepsilon t^{(\varepsilon)}(x) \quad \text{for } x\in\bar{\Omega}_\varepsilon \quad \text{and}$$

$$\|t^{(\varepsilon)}\|_{C^m(\bar{\Omega}_\varepsilon)^N}\leqq C_m \qquad (m=0, 1, 2, \cdots)$$

holds for a constant C_m independent of ε. Here Φ_ε^{-1} denotes the inverse function of Φ_ε.

$$\begin{aligned} S(x)&=\rho(x)\nu_x && x\in\partial\Omega \\ &=0 && x\in\partial\Omega''. \end{aligned} \tag{2.3}$$

$$u_0\in C^2(\overline{\Omega\setminus\Omega''}) \quad \text{and} \quad S(x)=0 \qquad \text{for } x\in\Omega''. \tag{2.4}$$

For a function f on Ω_ε, we define function $\tilde{f}$ on Ω by $\tilde{f}(x)=f(\Phi_\varepsilon(x))$ for $x\in\Omega$. For a function g on Ω, we define function $\hat{g}$ on Ω_ε by $\hat{g}(y)=g(\Phi_\varepsilon^{-1}(y))$ for $y\in\Omega_\varepsilon$.

Then we have the following.

LEMMA 2.1. (i) *Let $J\Phi_\varepsilon(x)$ be the Jacobian of $\Phi_\varepsilon(x)$. Then*

$$|J\Phi_\varepsilon(x)|=1+\varepsilon\sum_{i=1}^N\frac{\partial S_i}{\partial x_i}(x)+O(\varepsilon^2) \tag{2.5}$$

holds uniformly for $x\in\bar{\Omega}$, where $S_i(x)$ denotes the i-th element of $S(x)\in\boldsymbol{R}^N$ $(1\leqq i\leqq N)$.

(ii) $\tilde{}: W_0^{1,p}(\Omega_\varepsilon)\ni f\mapsto\tilde{f}\in W_0^{1,p}(\Omega)$ *is a bounded linear operator and its operator norm is uniformly bounded for any sufficiently small $\varepsilon>0$.*

The same is true for $\hat{}: W_0^{1,p}(\Omega)\ni g\mapsto\hat{g}\in W_0^{1,p}(\Omega_\varepsilon)$.

(iii)

$$\begin{aligned} \int_{\Omega_\varepsilon}|(\nabla f)(y)|^p dy=&\int_\Omega|(\nabla\tilde{f})(x)|^p dx \\ &+\varepsilon\int_\Omega|(\nabla\tilde{f})(x)|^p\sum_{i=1}^N\frac{\partial S_i}{\partial x_i}dx \\ &-\varepsilon p\int_\Omega|(\nabla\tilde{f})(x)|^{p-2}\sum_{j,k=1}^N\frac{\partial S_k}{\partial x_j}\frac{\partial\tilde{f}}{\partial x_j}\frac{\partial\tilde{f}}{\partial x_k}dx \\ &+O(\varepsilon^2) \end{aligned} \tag{2.6}$$

holds for any $f\in W_0^{1,p}(\Omega_\varepsilon)$.

Furthermore, if $\|f\|_{W_0^{1,p}(\Omega_\varepsilon)}\leqq C$ *holds for a constant* C *independent of* ε, *then the remainder term in the right hand side of* (2.6) *is uniform with respect to* f.

Proof. (i) and (ii) easily follow from (2.1) and (2.2). Therefore we give a proof of (iii).

We take an arbitrary $f\in W_0^{1,p}(\Omega_\varepsilon)$ and the transformation of co-ordinates; $\Phi_\varepsilon^{-1}: \Omega_\varepsilon\ni y\mapsto x=\Phi_\varepsilon^{-1}(y)\in\Omega$. Since $x=y+\varepsilon t^{(\varepsilon)}(y)$ for $y\in\Omega_\varepsilon$, we have

$$\frac{\partial x_i}{\partial y_j}=\delta_{i,j}+\varepsilon\frac{\partial t_i^{(\varepsilon)}}{\partial y_j}(y)\qquad(1\leqq i,\ j\leqq N),\tag{2.7}$$

where $\delta_{i,j}$ denotes Kronecker's delta and $t_i^{(\varepsilon)}(y)$ denotes the i-th element of $t^{(\varepsilon)}(y)\in\boldsymbol{R}^N$. On the other hand, since $y=\Phi_\varepsilon(x)=x+\varepsilon S(x)=y+\varepsilon t^{(\varepsilon)}(y)+\varepsilon S(x)$ hold for $y\in\Omega_\varepsilon$, we have

$$t^{(\varepsilon)}(y)+S(x)=0\qquad(y\in\Omega_\varepsilon,\ \varepsilon>0).$$

Thus we get

$$\frac{\partial t_k^{(\varepsilon)}}{\partial y_j}(y)+\sum_{i=1}^N\frac{\partial x_i}{\partial y_j}\frac{\partial S_k}{\partial x_i}(x)=0\qquad(1\leqq j,\ k\leqq N).\tag{2.8}$$

From (2.7) and (2.8),

$$\begin{aligned}\frac{\partial x_k}{\partial y_j}&=\delta_{j,k}-\varepsilon\sum_{i=1}^N\frac{\partial x_i}{\partial y_j}\frac{\partial S_k}{\partial x_i}(x)\\&=\delta_{j,k}-\varepsilon\sum_{i=1}^N\left(\delta_{i,j}+\varepsilon\frac{\partial t_i^{(\varepsilon)}}{\partial y_j}(y)\right)\frac{\partial S_k}{\partial x_i}(x)\\&=\delta_{j,k}-\varepsilon\frac{\partial S_k}{\partial x_j}(x)-\varepsilon^2\sum_{i=1}^N\frac{\partial t_i^{(\varepsilon)}}{\partial y_j}(y)\frac{\partial S_k}{\partial x_i}(x)\end{aligned}$$

hold for $1\leqq j,\ k\leqq N$. Hence we get

$$\begin{aligned}\frac{\partial f}{\partial y_j}(y)&=\sum_{k=1}^N\frac{\partial x_k}{\partial y_j}\frac{\partial}{\partial x_k}f(\Phi_\varepsilon(x))\\&=\sum_{k=1}^N\left(\delta_{j,k}-\varepsilon\frac{\partial S_k}{\partial x_j}(x)-\varepsilon^2\sum_{i=1}^N\frac{\partial t_i^{(\varepsilon)}}{\partial y_j}(y)\frac{\partial S_k}{\partial x_i}(x)\right)\frac{\partial\tilde f}{\partial x_k}(x)\\&=\frac{\partial\tilde f}{\partial x_j}(x)-\varepsilon\sum_{k=1}^N\frac{\partial S_k}{\partial x_j}(x)\frac{\partial\tilde f}{\partial x_k}(x)\\&\qquad-\varepsilon^2\sum_{i,k=1}^N\frac{\partial t_i^{(\varepsilon)}}{\partial y_j}(y)\frac{\partial S_k}{\partial x_i}(x)\frac{\partial\tilde f}{\partial x_k}(x)\end{aligned}\tag{2.9}$$

for $1\leqq j\leqq N$.

From (2.5) and (2.9) we can see that

$$(2.10)\qquad |(\nabla f)(\Phi_\varepsilon(x))|^p |J\Phi_\varepsilon(x)| = |(\nabla f)(\Phi_\varepsilon(x))|^p + \varepsilon|(\nabla \tilde{f})(x)|^p \sum_{i=1}^N \frac{\partial S_i}{\partial x_i}(x) + R(\varepsilon, x, \tilde{f})$$

holds for $x \in \Omega$, where

$$|R(\varepsilon, x, \tilde{f})| \leqq C\varepsilon^2 |(\nabla \tilde{f})(x)|^p .$$

Here C denotes a positive constant independent of ε, x and $\tilde{f}$.

On the other hand, by (2.9) and using Lemma A.3 in the Appendix with $w_1 = (\nabla \tilde{f})(x)$ and $w_2 = (\nabla f)(y) = (\nabla f)(\Phi_\varepsilon(x))$, we have the following.

$$(2.11)\qquad |(\nabla f)(\Phi_\varepsilon(x))|^p = |(\nabla \tilde{f})(x)|^p - \varepsilon p |(\nabla \tilde{f})(x)|^{p-2} \sum_{j,k=1}^N \frac{\partial S_k}{\partial x_j}(x) \frac{\partial \tilde{f}}{\partial x_k} \frac{\partial \tilde{f}}{\partial x_j} + R'(\varepsilon, x, \tilde{f})$$

holds for $x \in \Omega$, where

$$\begin{aligned} &|R'(\varepsilon, x, \tilde{f})| \\ &\leqq p(p-1)(|(\nabla f)(x)| + |(\nabla \tilde{f})(y) - (\nabla \tilde{f})(x)|)^{p-2} |(\nabla f)(y) - (\nabla \tilde{f})(x)|^2 \\ &\quad + \varepsilon^2 p |(\nabla \tilde{f})(x)|^{p-2} \left| \sum_{i,j,k=1}^N \frac{\partial t_i^{(\varepsilon)}}{\partial y_j}(y) \frac{\partial S_k}{\partial x_i}(x) \frac{\partial \tilde{f}}{\partial x_k} \frac{\partial \tilde{f}}{\partial x_j} \right| \\ &\leqq C'\varepsilon^2 |(\nabla \tilde{f})(x)|^p . \end{aligned}$$

Here C' denotes a positive constant independent of ε, x and $\tilde{f}$.

Since

$$\int_{\Omega_\varepsilon} |(\nabla f)(y)|^p dy = \int_\Omega |(\nabla f)(\Phi_\varepsilon(x))|^p |J\Phi_\varepsilon(x)| dx ,$$

(2.6) follows from (2.10) and (2.11). Furthermore the absolute value of the remainder term in the right hand side of (2.6) is bounded from above by

$$(C+C')\varepsilon^2 \|\tilde{f}\|^p_{W_0^{1,p}(\Omega)} \leqq C''\varepsilon^2 \|f\|^p_{W_0^{1,p}(\Omega_\varepsilon)} .$$

Thus the proof is complete. q.e.d.

3. Proof of Theorem 1

For the sake of simplicity we write $\|\cdot\|_{L^r(\Omega)}$ ($\|\cdot\|_{L^r(\Omega_\varepsilon)}$, respectively) as $\|\cdot\|_r$ ($\|\cdot\|_{r,\varepsilon}$, respectively) for $r \geqq 1$.

Since $\hat{u}_0/\|\hat{u}_0\|_{q+1,\varepsilon} \in X_\varepsilon$, we have

$$(3.1)\qquad \lambda(\varepsilon) \leqq \left(\int_{\Omega_\varepsilon} |(\nabla \hat{u}_0)(y)|^p dy\right)\left(\int_{\Omega_\varepsilon} |\hat{u}_0(y)|^{q+1} dy\right)^{-p/(q+1)} .$$

Notice that $\lambda(0) = \|\nabla u_0\|_p^p$, $\|u_0\|_{q+1} = 1$ and $\tilde{\hat{u}}_0 = u_0$ on Ω. Thus, from (2.5) and

(2.6), we see

$$\int_{\Omega_\varepsilon}|\hat{u}_0(y)|^{q+1}dy=\int_\Omega|\tilde{u}_0(x)|^{q+1}|J\Phi_\varepsilon(x)|dx \tag{3.2}$$

$$=1+\varepsilon\int_\Omega u_0^{q+1}\sum_{i=1}^N\frac{\partial S_i}{\partial x_i}dx+O(\varepsilon^2)$$

and

$$\int_{\Omega_\varepsilon}|(\nabla\hat{u}_0)(y)|^p dy=\lambda(0)+\varepsilon\int_\Omega|\nabla u_0|^p\sum_{i=1}^N\frac{\partial S_i}{\partial x_i}dx \tag{3.3}$$

$$-\varepsilon p\int_\Omega|\nabla u_0|^{p-2}\sum_{j,k=1}^N\frac{\partial S_k}{\partial x_j}\frac{\partial u_0}{\partial x_j}\frac{\partial u_0}{\partial x_k}dx$$

$$+O(\varepsilon^2).$$

By (3.1), (3.2) and (3.3) we get the following.

LEMMA 3.1. *For any sufficiently small* $\varepsilon>0$

$$\lambda(\varepsilon)\leqq\lambda(0)+\mu\varepsilon+O(\varepsilon^2) \tag{3.4}$$

holds, where

$$\mu=\int_\Omega(|\nabla u_0|^p-p\lambda(0)(q+1)^{-1}u_0^{q+1})\sum_{i=1}^N\frac{\partial S_i}{\partial x_i}dx$$

$$-p\int_\Omega|\nabla u_0|^{p-2}\sum_{j,k=1}^N\frac{\partial S_k}{\partial x_j}\frac{\partial u_0}{\partial x_j}\frac{\partial u_0}{\partial x_k}dx.$$

On the other hand, since $\tilde{u}_\varepsilon/\|\tilde{u}_\varepsilon\|_{q+1}\in X_0$, we have

$$\lambda(0)\leqq\left(\int_\Omega|(\nabla\tilde{u}_\varepsilon)(x)|^p dx\right)\left(\int_\Omega|\tilde{u}_\varepsilon(x)|^{q+1}dx\right)^{-p/(q+1)}. \tag{3.5}$$

Notice that $\lambda(\varepsilon)=\|\nabla u_\varepsilon\|_{p,\varepsilon}^p\leqq C$ (independent of ε) and $\|u_\varepsilon\|_{q+1,\varepsilon}=1$. Thus, from (2.5) and (2.6), we see

$$1=\int_\Omega|\tilde{u}_\varepsilon(x)|^{q+1}|J\Phi_\varepsilon(x)|dx \tag{3.6}$$

$$=\int_\Omega\tilde{u}_\varepsilon^{q+1}dx+\varepsilon\int_\Omega\tilde{u}_\varepsilon^{q+1}\sum_{i=1}^N\frac{\partial S_i}{\partial x_i}dx+O(\varepsilon^2)$$

and

$$\lambda(\varepsilon)=\int_\Omega|\nabla\tilde{u}_\varepsilon|^p dx+\varepsilon\int_\Omega|\nabla\tilde{u}_\varepsilon|^p\sum_{i=1}^N\frac{\partial S_i}{\partial x_i}dx \tag{3.7}$$

$$-\varepsilon p\int_\Omega|\nabla\tilde{u}_\varepsilon|^{p-2}\sum_{j,k=1}^N\frac{\partial S_k}{\partial x_j}\frac{\partial\tilde{u}_\varepsilon}{\partial x_j}\frac{\partial\tilde{u}_\varepsilon}{\partial x_k}dx+O(\varepsilon^2).$$

Since $\|\nabla u_\varepsilon\|_{p,\varepsilon}\leqq C$, we can see that $\|\tilde{u}_\varepsilon\|_{q+1}\leqq C'\|\nabla\tilde{u}_\varepsilon\|_p\leqq C''$ by (ii) of Lemma 2.1 and the Sobolev embedding: $W_0^{1,p}(\Omega)\hookrightarrow L^{q+1}(\Omega)$. Therefore, from (3.5), (3.6) and (3.7), we see

$$\int_\Omega \tilde{u}_\varepsilon^{q+1}dx=1+O(\varepsilon), \qquad \int_\Omega |\nabla\tilde{u}_\varepsilon|^p dx=\lambda(\varepsilon)+O(\varepsilon) \tag{3.8}$$

and $\lambda(0)\leqq\lambda(\varepsilon)+O(\varepsilon)$. On the other hand, by Lemma 3.1, $\lambda(\varepsilon)\leqq\lambda(0)+O(\varepsilon)$ holds. Thus we have

$$\lambda(\varepsilon)=\lambda(0)+O(\varepsilon)\,. \tag{3.9}$$

Next we want to show that

$$\tilde{u}_\varepsilon \longrightarrow u_0 \quad \text{weakly in } W_0^{1,p}(\Omega) \qquad \text{as } \varepsilon\to 0\,. \tag{3.10}$$

Assume that (3.10) does not hold. Then there exist $\eta>0$, $F\in(W_0^{1,p}(\Omega))^*$, and a sequence $\{\varepsilon_n\}_{n=0}^\infty$ satisfying $\varepsilon_n\downarrow 0$ $(n\to\infty)$ such that

$$|F(\tilde{u}_{\varepsilon_n})-F(u_0)|\geqq\eta \tag{3.11}$$

holds. Since $\{\tilde{u}_{\varepsilon_n}\}$ is bounded in $W_0^{1,p}(\Omega)$ and the Sobolev embedding: $W_0^{1,p}(\Omega)\hookrightarrow L^{q+1}(\Omega)$ is compact, there exist a subsequence $\{\tilde{u}_{\varepsilon_{n'}}\}$ and $v\in W_0^{1,p}(\Omega)$ satisfying

$$\begin{aligned} \tilde{u}_{\varepsilon_{n'}} &\longrightarrow v \qquad \text{weakly in } W_0^{1,p}(\Omega) \\ \tilde{u}_{\varepsilon_{n'}} &\longrightarrow v \qquad \text{strongly in } L^{q+1}(\Omega) \\ \tilde{u}_{\varepsilon_{n'}} &\longrightarrow v \qquad \text{a.e. } \Omega\,. \end{aligned} \tag{3.12}$$

Since $\tilde{u}_{\varepsilon_{n'}}\geqq 0$ a.e. Ω, $v\geqq 0$ a.e. Ω. From (3.8) and (3.9),

$$\|\tilde{u}_{\varepsilon_{n'}}\|_{q+1}\longrightarrow 1 \quad \text{and} \quad \|\nabla\tilde{u}_{\varepsilon_{n'}}\|_p^p\longrightarrow\|\nabla u_0\|_p^p=\lambda(0) \quad \text{as } n'\to\infty\,.$$

Thus, by (3.12), we have $\|v\|_{q+1}=1$ and

$$\|\nabla v\|_p\leqq \liminf_{n'\to\infty}\|\nabla\tilde{u}_{\varepsilon_{n'}}\|_p\leqq\|\nabla u_0\|_p=\lambda(0)^{1/p}\,.$$

Here we used the lower semicontinuity of the $W_0^{1,p}(\Omega)$-norm. Therefore we have $v\in X_0$ and $\lambda(0)\leqq\|\nabla v\|_p^p\leqq\|\nabla u_0\|_p^p=\lambda(0)$. Hence v is a minimizer of $(1.1)_0$. Since the minimizer u_0 of $(1.1)_0$ is unique by the assumption, $v=u_0$ must hold. Letting $n=n'\to\infty$ in (3.11), we have $0=|F(v)-F(u_0)|\geqq\eta$. This contradicts $\eta>0$. Thus we get (3.10).

From (3.8) and (3.9) we can see that

$$\|\tilde{u}_\varepsilon\|_{W_0^{1,p}(\Omega)}\longrightarrow\|u_0\|_{W_0^{1,p}(\Omega)} \qquad \text{as } \varepsilon\to 0\,. \tag{3.13}$$

By (3.10), (3.13) and the uniform convexity of $W_0^{1,p}(\Omega)$,

$$\tilde{u}_\varepsilon\longrightarrow u_0 \quad \text{strongly in } W_0^{1,p}(\Omega) \qquad \text{as } \varepsilon\to 0 \tag{3.14}$$

holds.

We put $\tilde{u}_\varepsilon = u_0 + v_\varepsilon$. Then, $v_\varepsilon \to 0$ strongly in $W_0^{1,p}(\Omega)$ as $\varepsilon \to 0$. We have

$$\int_\Omega |\nabla \tilde{u}_\varepsilon|^{p-2} \sum_{j,k=1}^N \frac{\partial S_k}{\partial x_j} \frac{\partial \tilde{u}_\varepsilon}{\partial x_j} \frac{\partial \tilde{u}_\varepsilon}{\partial x_k} dx \tag{3.15}$$

$$= \int_\Omega |\nabla u_0|^{p-2} \sum_{j,k=1}^N \frac{\partial S_k}{\partial x_j} \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial x_k} dx + I_1(\varepsilon) + I_2(\varepsilon),$$

where

$$I_1(\varepsilon) = \int_\Omega (|\nabla \tilde{u}_\varepsilon|^{p-2} - |\nabla u_0|^{p-2}) \sum_{j,k=1}^N \frac{\partial S_k}{\partial x_j} \frac{\partial \tilde{u}_\varepsilon}{\partial x_j} \frac{\partial \tilde{u}_\varepsilon}{\partial x_k} dx$$

$$I_2(\varepsilon) = \int_\Omega |\nabla u_0|^{p-2} \sum_{j,k=1}^N \frac{\partial S_k}{\partial x_j} \left(\frac{\partial u_0}{\partial x_j} \frac{\partial v_\varepsilon}{\partial x_k} + \frac{\partial v_\varepsilon}{\partial x_j} \frac{\partial u_0}{\partial x_k} + \frac{\partial v_\varepsilon}{\partial x_j} \frac{\partial v_\varepsilon}{\partial x_k} \right) dx.$$

It is easy to see that

$$I_2(\varepsilon) = o(1). \tag{3.16}$$

On the other hand, by using Lemma A.4 in the Appendix with $w_1 = \nabla u_0$ and $w_2 = \nabla \tilde{u}_\varepsilon$, we see

$$|I_1(\varepsilon)| \leqq C \int_\Omega \left| |\nabla \tilde{u}_\varepsilon|^{p-2} - |\nabla u_0|^{p-2} \right| |\nabla \tilde{u}_\varepsilon|^2 dx$$

$$\leqq \begin{cases} C \int_\Omega |\nabla v_\varepsilon|^{p-2} |\nabla \tilde{u}_\varepsilon|^2 dx & \text{(if } 2 < p \leqq 3) \\ C \int_\Omega (|\nabla u_0| + |\nabla v_\varepsilon|)^{p-3} |\nabla v_\varepsilon| \, |\nabla \tilde{u}_\varepsilon|^2 dx & \text{(if } p > 3) \end{cases}$$

$$\leqq \begin{cases} C \|\nabla v_\varepsilon\|_p^{p-2} \|\nabla \tilde{u}_\varepsilon\|_p^2 & \text{(if } 2 < p \leqq 3) \\ C \left(\int_\Omega (|\nabla u_0| + |\nabla v_\varepsilon|)^p dx \right)^{(p-3)/p} \|\nabla v_\varepsilon\|_p \|\nabla \tilde{u}_\varepsilon\|_p^2 & \text{(if } p > 3). \end{cases}$$

Notice that $I_1(\varepsilon) = 0$ if $p = 2$. Thus we have

$$I_1(\varepsilon) = o(1). \tag{3.17}$$

From (3.7), (3.14), (3.15), (3.16) and (3.17), we see

$$\int_\Omega |\nabla \tilde{u}_\varepsilon|^p dx = \lambda(\varepsilon) - \varepsilon \int_\Omega |\nabla u_0|^p \sum_{i=1}^N \frac{\partial S_i}{\partial x_i} dx \tag{3.18}$$

$$+ \varepsilon p \int_\Omega |\nabla u_0|^{p-2} \sum_{j,k=1}^N \frac{\partial S_k}{\partial x_j} \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial x_k} dx + o(\varepsilon).$$

Furthermore, since $\tilde{u}_\varepsilon \to u_0$ strongly in $L^{q+1}(\Omega)$ as $\varepsilon \to 0$, the following follows easily from (3.6).

$$\int_\Omega \tilde{u}_\varepsilon^{q+1} dx = 1 - \varepsilon \int_\Omega u_0^{q+1} \sum_{i=1}^N \frac{\partial S_i}{\partial x_i} dx + o(\varepsilon) \tag{3.19}$$

From (3.5), (3.18) and (3.19), we have

$$\begin{aligned}\lambda(0) \leqq \lambda(\varepsilon) &- \varepsilon \int_\Omega |\nabla u_0|^p \sum_{i=1}^N \frac{\partial S_i}{\partial x_i} dx \\ &+ \varepsilon p \lambda(\varepsilon)(q+1)^{-1} \int_\Omega u_0^{q+1} \sum_{i=1}^N \frac{\partial S_i}{\partial x_i} dx \\ &+ \varepsilon p \int_\Omega |\nabla u_0|^{p-2} \sum_{j,k=1}^N \frac{\partial S_k}{\partial x_j} \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial x_k} dx + o(\varepsilon) .\end{aligned}$$

Using (3.9) in the third term of the right hand side of the above inequality, we get the following.

LEMMA 3.2. *For any sufficiently small* $\varepsilon > 0$

$$\lambda(0) \leqq \lambda(\varepsilon) - \mu\varepsilon + o(\varepsilon) \tag{3.20}$$

holds, where μ *is defined as in Lemma* 3.1.

Now we are in a position to prove Theorem 1. Since $u_0 \in C^1(\bar{\Omega})$ and $u_0 = 0$ on $\partial\Omega$, we have the following by the divergence theorem.

$$\begin{aligned}(q+1)^{-1} \int_\Omega u_0^{q+1} \sum_{i=1}^N \frac{\partial S_i}{\partial x_i} dx + \int_\Omega u_0^q (\nabla u_0 \cdot S) dx \\ = \int_\Omega \operatorname{div}((q+1)^{-1} u_0^{q+1} S) dx \\ = \int_{\partial\Omega} (q+1)^{-1} u_0^{q+1} (S \cdot \nu_x) d\sigma_x = 0\end{aligned} \tag{3.21}$$

We recall (2.3) and (2.4). Then we have the following by the divergence theorem.

$$\begin{aligned}\int_\Omega |\nabla u_0|^p \sum_{i=1}^N \frac{\partial S_i}{\partial x_i} dx + \int_{\Omega \setminus \Omega''} S \cdot \nabla(|\nabla u_0|^p) dx \\ = \int_{\Omega \setminus \Omega''} \operatorname{div}(|\nabla u_0|^p S) dx \\ = \int_{\partial\Omega} |\nabla u_0|^p (S \cdot \nu_x) d\sigma_x = \int_{\partial\Omega} |\nabla u_0|^p \rho(x) d\sigma_x\end{aligned} \tag{3.22}$$

(3.23) $$\int_{\Omega\setminus\Omega''}(\operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0))(\nabla u_0\cdot S)dx+\int_{\Omega\setminus\Omega''}(|\nabla u_0|^{p-2}\nabla u_0\cdot\nabla(\nabla u_0\cdot S))dx$$
$$=\int_{\Omega\setminus\Omega''}\operatorname{div}((\nabla u_0\cdot S)|\nabla u_0|^{p-2}\nabla u_0)dx$$
$$=\int_{\partial\Omega}(\nabla u_0\cdot S)|\nabla u_0|^{p-2}\frac{\partial u_0}{\partial\nu_x}d\sigma_x=\int_{\partial\Omega}|\nabla u_0|^{p-2}\left|\frac{\partial u_0}{\partial\nu_x}\right|^2\rho(x)d\sigma_x$$

It is easy to see that

(3.24) $$p|\nabla u_0|^{p-2}\nabla u_0\cdot\nabla(\nabla u_0\cdot S)$$
$$=S\cdot\nabla(|\nabla u_0|^p)+p|\nabla u_0|^{p-2}\sum_{j,k=1}^{N}\frac{\partial S_k}{\partial x_j}\frac{\partial u_0}{\partial x_j}\frac{\partial u_0}{\partial x_k}$$

holds in $\Omega\setminus\Omega''$.

From (2.4), (3.4), (3.21), (3.22), (3.23) and (3.24), we can easily get the following.

$$\mu=\int_{\partial\Omega}\left(|\nabla u_0|^p-p|\nabla u_0|^{p-2}\left|\frac{\partial u_0}{\partial\nu_x}\right|^2\right)\rho(x)d\sigma_x$$
$$+p\int_{\Omega\setminus\Omega''}(\operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0)+\lambda(0)u_0^q)(\nabla u_0\cdot S)dx$$

Since $u_0=0$ on $\partial\Omega$, $|\nabla u_0|=|\partial u_0/\partial\nu_x|$ on $\partial\Omega$. Furthermore, by (2.4), u_0 satisfies

$$-\operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0)=\lambda(0)u_0^q \qquad \text{in } \Omega\setminus\Omega''$$

in the strong sense. Hence we have

(3.25) $$\mu=-(p-1)\int_{\partial\Omega}\left|\frac{\partial u_0}{\partial\nu_x}\right|^p\rho(x)d\sigma_x\,.$$

From Lemmas 3.1, 3.2 and (3.25) we get the desired Theorem 1.

4. Appendix

In this section we refer to the regularity of a solution u_ε of (1.2). Furthermore we give some inequalities. At first we have the following.

LEMMA A.1. *Let G be a bounded domain in $\boldsymbol{R}^N$ ($N\geqq 2$) with a smooth boundary ∂G. Assume that $p>1$ and g is continuous in $\bar{G}\times\boldsymbol{R}$ and satisfies*

$$|g(x,t)|\leqq C|t|^r+D \qquad (x,t)\in\bar{G}\times\boldsymbol{R}\,,$$

where C and D are real positive constants and $r\in(0,p^*-1)$. If $u\in W_0^{1,p}(G)$ satisfies

(A.1) $$-\operatorname{div}(|\nabla u|^{p-2}\nabla u)=g(\cdot,u) \qquad \text{in } G$$

$$u=0 \quad on\ \partial G,$$

then $u \in C^{1+\alpha}(\bar{G})$ *for some* $\alpha \in (0, 1)$.

Proof. When $p>N$, $u \in L^\infty(G)$ follows by the Sobolev embedding: $W_0^{1,p}(G) \hookrightarrow C^{1-N/p}(\bar{G})$. Therefore the above assertion easily follows from Corollary 1.1 and Remark 1.2 in Guedda-Veron [6, p. 884]. q. e. d.

From Lemma A.1 $u_\varepsilon \in C^{1+\alpha}(\bar{\Omega}_\varepsilon)$ holds for some $\alpha \in (0, 1)$. Furthermore we have the following.

LEMMA A.2. *Assume that* $q \geqq p-1$. *Then there exists a neighbourhood* O *of* $\partial\Omega$ *in* Ω *such that*

$$u_0 \in C^2(\bar{O}). \tag{A.2}$$

Proof. We recall $u_0 \in W_0^{1,p}(\Omega) \cap C^{1+\alpha}(\bar{\Omega})$ satisfies

$$\begin{aligned} -\operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) &= a(x)u_0^{p-1} \quad \text{in } \Omega \\ u_0 &= 0 \quad \text{on } \partial\Omega \\ u_0 &\geqq 0 \quad \text{a. e. } \Omega, \end{aligned} \tag{A.3}$$

where $a(x)=u_0^{q-(p-1)}(x)$. Thus $a(x) \in L^\infty(\Omega)$. Therefore the following follows from Harnack's inequality due to Trudinger [18, Theorem 1.1, p. 724].

$$u_0>0 \quad \text{in } \Omega \tag{A.4}$$

From (A.3), (A.4) and Hopf's lemma due to Sakaguchi [14, Lemma A.3, p. 417], we have

$$\partial u_0/\partial \nu_x<0 \quad \text{on } \partial\Omega.$$

Since $u_0 \in C^1(\bar{\Omega})$, there exist a neighbourhood O of $\partial\Omega$ in Ω and $\eta>0$ such that

$$|\nabla u_0| \geqq \eta>0 \quad \text{in } \bar{O}.$$

Therefore (A.2) follows from the regularity theory of the elliptic partial differential equation (see, for example, Gilbarg-Trudinger [5], Ladyzhenskaja-Ural'tseva [8]). q. e. d.

Next we give the following inequalities.

LEMMA A.3. *Assume that* $p \geqq 2$. *Then*

$$\begin{aligned} &||w_2|^p-|w_1|^p-p|w_1|^{p-2}w_1\cdot(w_2-w_1)| \\ &\leqq p(p-1)(|w_1|+|w_2-w_1|)^{p-2}|w_2-w_1|^2 \end{aligned} \tag{A.5}$$

holds for any w_1, $w_2 \in \boldsymbol{R}^N$.

Proof. We fix w_1, $w_2 \in \boldsymbol{R}^N$. At first we assume that $w_1+t(w_2-w_1)\neq 0$ for any $t\in[0, 1]$. We put

$$g(t)=|w_1+t(w_2-w_1)|^p \qquad t\in[0, 1]\,.$$

Then

$$g(1)=g(0)+g'(0)+\int_0^1(1-t)g''(t)dt\,,$$

where

$$g'(t)=p|w_1+t(w_2-w_1)|^{p-2}(w_1+t(w_2-w_1))\cdot(w_2-w_1)$$

$$g''(t)=p|w_1+t(w_2-w_1)|^{p-2}|w_2-w_1|^2$$
$$+p(p-2)|w_1+t(w_2-w_1)|^{p-4}((w_1+t(w_2-w_1))\cdot(w_2-w_1))^2\,.$$

Using Schwarz's inequality, we have

$$\begin{aligned}|g''(t)|&\leqq p(p-1)|w_1+t(w_2-w_1)|^{p-2}|w_2-w_1|^2\\&\leqq p(p-1)(|w_1|+t|w_2-w_1|)^{p-2}|w_2-w_1|^2\\&\leqq p(p-1)(|w_1|+|w_2-w_1|)^{p-2}|w_2-w_1|^2\end{aligned}$$

for $t\in[0, 1]$. Summing up these facts, we get (A.5).

Next we assume that $w_1+t(w_2-w_1)=0$ for some $t\in[0, 1]$. When $t=0$ (i.e. $w_1=0$), (A.5) is equivalent to $1\leqq p(p-1)$. Since $p\geqq 2$, $p(p-1)\geqq 1$ holds. When $t\in(0, 1]$, we put $s=t^{-1}$. Then $w_2=(1-s)w_1$ and (A.5) is equivalent to

(A.6) $$(s-1)^p+sp-1\leqq p(p-1)(1+s)^{p-2}s^2 \qquad (s\geqq 1)\,.$$

Since $s^2\geqq(s^2+1)/2$ for $s\geqq 1$,

(A.7) $$\begin{aligned}p(p-1)(1+s)^{p-2}s^2&\geqq(p(p-1)/2)(1+s)^{p-2}s^2+(p(p-1)/2)(1+s)^{p-2}\\&\geqq s^p+p-1 \qquad (s\geqq 1)\end{aligned}$$

hold for $p\geqq 2$. On the other hand,

(A.8) $$s^p+p-1\geqq(s-1)^p+sp-1 \qquad (s\geqq 1)$$

holds for $p\geqq 2$, since

$$s^p=(s-1+1)^p\geqq(s-1)^p+p(s-1) \qquad (p\geqq 2,\ s\geqq 1)\,.$$

From (A.7) and (A.8) we get (A.6). Therefore we get (A.5).

Thus the proof is complete. q.e.d.

LEMMA A.4. *Assume that* $p\geqq 2$. *Then*

$$\text{(A.9)} \qquad ||w_2|^{p-2}-|w_1|^{p-2}| \leqq \begin{cases} |w_2-w_1|^{p-2} & (\textit{if } 2\leqq p\leqq 3) \\ (p-2)(|w_1|+|w_2-w_1|)^{p-3}|w_2-w_1| & (\textit{if } p>3) \end{cases}$$

hold for any $w_1,\ w_2\in \boldsymbol{R}^N$.

Proof. We fix $w_1,\ w_2\in\boldsymbol{R}^N$. If $p\in[2,3]$, then we see

$$|w_1|^{p-2}\leqq(|w_2|+|w_2-w_1|)^{p-2}\leqq|w_2|^{p-2}+|w_2-w_1|^{p-2}$$

and

$$|w_2|^{p-2}\leqq(|w_1|+|w_2-w_1|)^{p-2}\leqq|w_1|^{p-2}+|w_2-w_1|^{p-2}.$$

Hence we get (A.9) for $p\in[2,3]$.

Hereafter we assume $p>3$. When $w_1+t(w_2-w_1)=0$ for some $t\in[0,1]$, we can easily get (A.9) as in the proof of Lemma A.3. Therefore we may assume that $w_1+t(w_2-w_1)\neq 0$ for any $t\in[0,1]$. We put

$$h(t)=|w_1+t(w_2-w_1)|^{p-2} \qquad t\in[0,1].$$

Then

$$h(1)=h(0)+\int_0^1 h'(t)dt,$$

where

$$\begin{aligned}|h'(t)|&=(p-2)|w_1+t(w_2-w_1)|^{p-4}|(w_1+t(w_2-w_1))\cdot(w_2-w_1)|\\ &\leqq(p-2)(|w_1|+|w_2-w_1|)^{p-3}|w_2-w_1|\end{aligned}$$

hold for $t\in[0,1]$. Summing up these facts, we get (A.9).

Thus the proof is complete. q. e. d.

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