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COMPLEX HYPERSURFACES DIFFEOMORPHIC TO AFFINE SPACES

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1. Introduction and main results

This note is related to the following two basic problems in Algebraic Geometry.

PROBLEM A. Topological characterization of the complex affine space $\boldsymbol{C}^n$.

Let X be an n-dimensional smooth affine variety over $\boldsymbol{C}$. Is it possible to impose some *topological* conditions on X which imply that X is isomorphic to the affine space $\boldsymbol{C}^n$ as algebraic varieties?

Such isomorphic varieties are denoted in this note by $X \cong \boldsymbol{C}^n$.

PROBLEM B. Existence of exotic embeddings of $\boldsymbol{C}^n$ into $\boldsymbol{C}^{n+1}$ (The Abhyankar-Sathaye Conjecture).

Assume that $X: f=0$ is a smooth hypersurface in $\boldsymbol{C}^{n+1}$ such that $X \cong \boldsymbol{C}^n$. Does there exist an algebraic automorphism $h: \boldsymbol{C}^{n+1} \to \boldsymbol{C}^{n+1}$ such that $f \circ h$ is a linear form? This can be restated as whether any embedding of $\boldsymbol{C}^n$ into $\boldsymbol{C}^{n+1}$ is equivalent to a linear embedding.

For Problem A one has the following answers.

LEMMA 1. *If X is a smooth affine connected curve with Euler number $E(X) = E(\boldsymbol{C}) = 1$, then $X \cong \boldsymbol{C}$.*

Proof. Any such curve X is obtained from a smooth connected projective curve of genus g by deleting k points. Since the Euler number can be computed by the formula

$$E(X) = 2 - 2g - k$$

it follows that $g=0$ and $k=1$.

THEOREM 2. *If X is a smooth affine surface homeomorphic to the affine plane $\boldsymbol{C}^2$, then $X \cong \boldsymbol{C}^2$.*

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Proof. This follows from Ramanujam's main result in [Ra], the property of simply-connectedness at infinity being clearly invariant under homeomorphisms.

Remark 3. As already noticed by Ramanujam, it is not enough to require that the surface X is contractible. Here is a beautiful example of this phenomenon due to tom Dieck-Petrie [DP2].

Let a and d be positive integers such that $1<a<d$ and $(a, d)=1$.

Consider the surface $V_{d,a}$ defined in $\boldsymbol{C}^3$ by the equation

$$f_{d,a}(x, y, z)-1=0$$

where

$$f_{d,a}(x, y, z)=((xz+1)^d-(yz+1)^a)/z\,.$$

Then $V_{d,a}$ is a smooth contractible surface with logarithmic Kodaira dimension $\kappa(V_{d,a})=1$.

Recall that the logarithmic Kodaira dimension is invariant under algebraic isomorphisms and $\kappa(\boldsymbol{C}^n)=-\infty$ for all n, see [I].

In the case of dimensions $n>2$ the picture is completely different. A smooth affine variety X which is diffeomorphic to the affine space $\boldsymbol{C}^n$ but nonisomorphic to it algebraically is called an *exotic algebraic structure* on $\boldsymbol{C}^n$.

Examples of such exotic structures were constructed by Petrie -tom Dieck [DP2] (implicitly) and explicitly by Zaidenberg [Z 1-3] and Kaliman [K 1-2]. Moreover, by the work of Flenner-Zaidenberg [FZ] and Kaliman [K2], we have now families of such exotic structures depending on moduli.

On the other hand there is the interesting algebro-topological characterization of the affine 3-dimensional space by Miyanishi [M]. One may ask whether such 'mixed' characterizations exist in higher dimensions.

Concerning the Problem B our knowledge consists essentially only of the next result, due to Abhyankar-Moh [AM], saying that the answer is "Yes" when $n=1$.

THEOREM 4. *Any embedding of the line $X=\boldsymbol{C}$ into the plane $\boldsymbol{C}^2$ is equivalent to a linear embedding.*

For interesting higher dimensional partial results see [DP1]. There are also deep relations of Problem B to the *Jacobian Conjecture*, see for instance Jelonek [J].

In this note we introduce a (discrete) family of odd dimensional hypersurfaces which have interesting properties related to the problems A and B above.

THEOREM 5. *For positive integers m, d and a such that*

$$m>1, \quad 0<a<d-1 \quad \text{and} \quad (a, d)=(a, d-1)=1$$

let $X_{d,a}$ *be the hypersurface in* C^{2m} *defined by the equation* $f(x)=0$, *where*

$$f(x)=x_0^a x_1^{d-a}+x_1 x_2^{d-1}+\cdots+x_{2m-3}x_{2m-2}^{d-1}+x_{2m-2}+x_{2m-1}^d.$$

Then

(i) $X_{d,a}$ *is a smooth hypersurface diffeomorphic to* C^{2m-1};

(ii) *for* $a=1$, *the polynomial map* $f: C^{2m}\to C$ *is a topological trivial fibration. In particular, all the fibers* $Y_t=f^{-1}(t)$ *are smooth hypersurfaces diffeomorphic to* C^{2m-1};

(iii) *for* $a>1$, *the fibers* Y_t *are not contractible for* $t\neq 0$.

In the 3-fold case, we can strengthen (ii) above.

PROPOSITION 6. *For* $m=2$ *and* $a=1$, *the 3-fold* $X_{d,1}$ *is isomorphic to* C^3 *and its embedding into* C^4 *is equivalent to a linear one.*

The following questions are still open.

QUESTION 1. Does Proposition 6. hold for all $m>2$?

QUESTION 2. Does there exist a triple (m, d, a) as above with $a>1$ such that the corresponding hypersurface $X_{d,a}$ is isomorphic to C^{2m-1}?

Such a triple would give a negative answer to Problem B.

QUESTION 3. Determine the exotic algebraic structures on C^{2m-1} coming from the hypersurfaces $X_{d,a}$?

We note that Kaliman [K1] has constructed a larger class of hypersurfaces diffeomorphic to affine spaces and showed that *some* of them are not isomorphic to affine spaces. However, this latter class of hypersurfaces is disjoint from ours.

Before starting the proofs, we would like to explain why our hypersurfaces are 'natural', e.g. how we have arrived at the polynomial f in Thm. 5.

Let Z be the closure of the hypersurface $X=X_{d,a}$ in the projective space P^{2m}. Let $V=Z\setminus X$ be the part at infinity of Z.

Hypersurfaces of type Z and V have been considered in [BD] as examples of projective hypersurfaces with isolated singularities having the same *integral* homology as a projective space. See also [D3], p. 167.

Now, if one looks for such examples in the class of surfaces in P^3 having in addition a C^*-action, then the varieties V above (for $m=2$) are the only possible cases [BD]. Indeed, all our hypersurfaces X, Z and V have obvious C^*-actions, coming from diagonal C^*-actions on the ambient affine or projective spaces (the corresponding weights are both negative and positive!).

Projective hypersurfaces of type Z and V for the special value $a=1$ have been considered for the first time by Libgober [L], and then rediscovered in the surface case by Barthel [B] and Choudary-Dimca [CD].

2. The proofs

Proof of Theorem 5. (i) STEP 1. The hypersurface $X=X_{d,a}$ is acyclic: $\tilde{H}_j(X)=0$ for $j\geqq 0$. Here and in the sequel homology and cohomology are with integral coefficients.

Using the Alexander-Lefschetz duality, it is enough to show that the morphism

$$j^k : H^k(Z) \longrightarrow H^k(V)$$

induced by the inclusion $j: V\to Z$ is an isomorphism for $k<4m-2$. When $k<2m-1$ this follows directly from the Lefschetz hyperplane section Theorem, see for instance [D3], p. 25.

For $k>2m-2$ and odd, both groups are trivial so there is nothing to prove. Finally, for $k>2m-2$ and even, one uses the result (2.11) in [D3], p. 144 (and of course the fact that V and Z are integral homology projective spaces as noted above).

STEP 2. The hypersurface X is contractible.

Since X has the homotopy type of a CW-complex, one can apply Whitehead Theorem, see [Sp], p. 399 and p. 405, and reduce the question to showing that X is simply-connected. But this follows directly from [D2].

STEP 3. The hypersurface X is diffeomorphic to $\boldsymbol{C}^{2m-1}$.

Let B be an open, large ball centered at the origin of $\boldsymbol{C}^{2m}$. Then X is diffeomorphic to the intersection $X\cap B$, see [D3], p. 26. Since this intersection is the interior of the manifold with boundary $M=X\cap Closure(B)$, it is enough to show that M is diffeomorphic to a closed ball. By a result due to Smale, see [S], Thm. 5.1, it is enough to show that the boundary of M is simply-connected. This in turn follows from [D3], p. 28 since the hypersurface X is simply-connected.

(ii) Let Z_t be the projective closure of the fibre Y_t. Any hypersurface Z_t has exactly one singularity, namely the point $p_1=(1:0:\cdots:0)$.

Moreover a direct calculation (see the proof of Prop. 6. below) shows that the family of isolated hypersurface singularities (Z_t, p_1) is μ^*-constant. The result follows from general properties of polynomial functions, see the proof of (4.1) in [D3], pp. 20–21. (For the definition of the μ^k invariants of Teissier one can see [D3], pp. 11–12.)

(iii) When $a>1$ the projective hypersurfaces Z_t have two singularities, namely p_1 and $p_2=(0:1:0:\cdots:0)$. The new family of hypersurface singularities (Z_t, p_2) has a jump in Milnor number for $t=0$. Using standard formulas relating Milnor numbers and Euler numbers of hypersurfaces, see for instance [D3], p. 162, we get that $E(Y_t)\neq 1$. In particular, Y_t is not contractible in this case.

Proof of Proposition 6. To simplify notation, let us denote by x, y, z and t the coordinates on $\boldsymbol{C}^4$. Hence our 3-fold X is given by the equation

$$x+x^{d-1}y+y^{d-1}z+t^d=0$$

Consider the associated surface S in $\boldsymbol{C}^3$ defined by the equation

$$x+x^{d-1}y+y^{d-1}z=0$$

To complete the proof it is enough to establish the following.

LEMMA 7. *The surface S is isomorphic to the plane $\boldsymbol{C}^2$ and the embedding of S into $\boldsymbol{C}^3$ is equivalent to a linear embedding.*

Proof. STEP 1. The surface S is contractible.

To show this, consider the function $g:\boldsymbol{C}^3\to\boldsymbol{C}$ given by

$$g(x, y, z)=x+x^{d-1}y+y^{d-1}z\,.$$

Let $F_t=g^{-1}(t)$ be the fibers of g, and G_t be their projective closure in $\boldsymbol{P}^3$. Then the surface G_t has just one singularity, namely $q=(0:0:1:0)$.

The family of singularities (G_t, q) has the following local equation.

$$g_t(x, y, u)=xu^{d-1}+x^{d-1}y+y^{d-1}-tu^d=0$$

As in the proof of Theorem 5 (ii), it is enough to show that this is a μ^*-constant family of hypersurface singularities.

First we show that g_t is a μ-constant family. For this, note that g_t is a semi weighted homogeneous polynomial relative to the weights

$$wt(x)=(d-1)(d-2),\quad wt(y)=(d-1)^2 \text{ and } wt(u)=d^2-3d+3\,.$$

In fact, one has $\deg(xu^{d-1})=\deg(x^{d-1}y)=\deg(y^{d-1})<\deg(u^d)$.

It follows that (see for instance [D1], p. 116)

$$\mu^3(g_t):=\mu(g_t)=constant.$$

On the other hand

$$\mu^1(g_t):=multiplicity(g_t)-1=d-2=constant.$$

Hence it remains to look at $\mu^2(g_t)$. To do this, note that a generic plane section is given by $x=Au+By$, with A and B nonzero constants.

Since $wt(x)<wt(u)<wt(y)$, it follows that the terms of lowest degree relative to our weights in $g_t(Au+By, y, u)$ are $(A-t)u^d+y^{d-1}$. One may choose

$A\neq t$ and the resulting curve singularity is again semi weighted homogeneous (relative to new obvious weights). It follows as above that

$$\mu^2(g_t)=\mu(Au^d+y^{d-1})=(d-1)(d-2)=const.$$

As a result of all this, S is the fiber of the topological fibration induced by g. The long exact sequence of homotopy groups of a fibration shows that all the homotopy groups of S are trivial. Since S has the homotopy type of a CW-complex, this implies that S is contractible.

STEP 2. The surface S is isomorphic to the affine plane $\boldsymbol{C}^2$.

In addition to being contractible, the surface S has the following properties

(a) S has an obvious $\boldsymbol{C}^*$ action;

(b) S contains two distinct lines, its intersection with the plane $x=0$.

Anyone of these two conditions together with contractibility imply that S is isomorphic to the plane, see Rynes [Ry] or Zaidenberg [Z1].

STEP 3. The embedding of S into $\boldsymbol{C}^3$ is equivalent to a linear one.

One can apply Theorem 2.3 in Russell [Ru] to complete the proof.

Remarks 8. (i) It can be shown that the surfaces

$$S_a: x+x^{d-1}y+y^{d-a}z^a=0$$

are not contractible for $a>1$, see [D3], p. 174.

(ii) Nagata has considered in [N], p. 16 an automorphism $h=(h^1, h^2, h^3)$ of $\boldsymbol{C}^3$ such that (up to a change in notation)

$$h^1(x, y, z)=x+x^2y+y^2z.$$

This automorphism h gives therefore an explict linearization of the surface S in the case $d=3$. It would be interesting to have such explicit linearizations for $d>3$ as well.

(iii) Note that the $(2m-1)$-fold $X_{d,a}$ contains a copy of the affine $(2m-2)$-dimensional space, namely the trace of the hyperplane $x_{2m-3}=0$. This is not the case with Zaidenberg examples [Z3].

(iv) There is a key difference between our hypersurfaces $X_{d,a}$ and the surfaces $V_{d,a}$ constructed by tom Dieck-Petrie, see Remark 3 above. In both cases one looks at the fiber of a weighted homogeneous polynomial. However, we consider the *special* fiber over 0, while tom Dieck and Petrie consider the *generic* fiber over a nonzero complex number.

Using the Sebastiani-Thom construction due in this context to Oka [O], see also [D3], p. 88, one can easily see that the hypersurface in $\boldsymbol{C}^6$ given by the equation

$$f_{d,a}(x_1, y_1, z_1)+f_{e,b}(x_2, y_2, z_2)=1$$

where the pairs (d, a) and (e, b) satisfy the conditions from Remark 3 is diffeomorphic to C^5.

It would be very interesting to have a Sebastiani-Thom formula for the logarithmic Kodaira dimension of such hypersurfaces. This would lead in particular to new ways of constructing exotic algebraic structures on affine spaces.

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