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KODAI MATH. J.
17 (1994), 125—132

GAP THEOREMS FOR ENTIRE FUNCTIONS

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Abstract

In this paper, we give a necessary and sufficient condition that there exists an entire function represented by a gap power series with an appropriate growth condition for the maximum modulus, and the entire function is bounded and not identically zero in an angular domain.

1. Introduction

We consider a nonconstant entire function $f(z)$, which is represented by a gap power series

$$f(z)=\sum_{n=1}^{+\infty} a_n z^{\lambda_n} \tag{1}$$

and is bounded in the angular domain $B(a)=\{z=re^{i\theta}: |\theta| \leqq a\}$, where $0 \leqq a < \pi$, $\Lambda=\{\lambda_n\}(n=1, 2, 3, \cdots)$ is a strictly increasing sequence of positive intergers. We denote the set of such functions $f(z)$ by $E(\Lambda, a)$. It is well-known by A. J. Macintyre's theorem ([7]) that when $a=0$, a necessary and sufficient condition that there exists an entire function $f(z) \in E(\Lambda, 0)$ is that $\lambda(r) \to +\infty$, as $r \to +\infty$, where

$$\lambda(r)=\sum_{\lambda_n \leqq r} \frac{2}{\lambda_n}, \quad \text{if } r \geqq \lambda_1; \quad \text{and} \quad \lambda(r)=0, \text{ if } r < \lambda_1. \tag{2}$$

J. M. Anderson and K. G. Binmore ([1]) also proved a necessary and sufficient condition for the existence of an entire function $f(z) \in E(\Lambda, 0)$ with a growth condition concerning the maximum modulus of $f(z)$.

Suppose that $\varphi(t)$ be a convex function defined on $[0, +\infty)$ such that

$$\lim_{t \to +\infty} t^{-1}\varphi(t)=+\infty. \tag{3}$$

It is well-known that there exists a constant $t_1 \geqq 0$ such that $\varphi(t)$ is strictly

* The Project supported by National Science Foundation of China.
Key Words: Entire Function. Gap Power Series.
AMS Subject classifications number. 30D10, 30D20.
Received September 29, 1992; revised March 8, 1993.

increasing and continuous on $[t_1, +\infty)$, so $s=\varphi(t)$ has the inverse function $t=\tilde{\varphi}(s)$. We can define the φ-order ρ $(0\leqq\rho\leqq+\infty)$ and the lower φ-order λ of an entire function $f(z)$ by

$$\rho=\limsup_{t\to+\infty} t^{-1}\tilde{\varphi}[\log M(e^t, f)], \tag{4}$$

$$\lambda=\liminf_{t\to+\infty} t^{-1}\tilde{\varphi}[\log M(e^t, f)], \tag{5}$$

where $M(r, f)$ is the maximum modudus of $f(z)$ on the ciricle $|z|=r$, i.e.

$$M(r, f)=\max\{|f(z)|: |z|=r\}, \qquad (r\geqq 0) \tag{6}$$

The φ-order and the lower φ-order reduce to the usual order notion in the case where $\varphi(t)$ is the exponential function. ([2], [3])

Utilizing Carlemen's formula ([2]) for analytic functions, the present article proves some results which are similar to those in [1] for the angular domain and obtains a necessary and sufficient condition as follows.

THEOREM 1. *Let $0\leqq a<\pi$, $\Lambda=\{\lambda_n\}$ be a strictly increasing sequence of positive integers and $\varphi(t)$ be an convex function defined on $[0, +\infty)$ satisfying* (3). *Then there exists an entire function $f(z)\in E(\Lambda, a)$ of the finite φ-order at most ρ, if and only if*

$$\liminf_{r\to+\infty} r^{-1}k(\varphi(r))\geqq\frac{1}{\rho}, \tag{7}$$

where

$$k(r)=\lambda(r)-\frac{2a}{\pi}\log^+ r \tag{8}$$

and $\lambda(r)$ is defined by (2).

Remark 1. When $a=0$, the sufficiency is proved by J.M. Anderson and K.G. Binmore ([1]), and when $a=0$ and $\log\varphi(s)$ is a convex function of s, they also gave a necessity. So our result is a generalization and an improvement of a result in [1] even in the case $a=0$. The method of proofs in this paper is quite different from that in [1].

THEOREM 2. *Let a, $\Lambda=\{\lambda_n\}$, $\lambda(r)$ and $\varphi(t)$ be the same as those in Theorem* 2. *Then there exists $f(z)\in E(\Lambda, a)$ of finite lower φ-order at most λ, if and only if*

$$\limsup_{r\to+\infty} r^{-1}k.(\varphi(r))\geqq\frac{1}{\lambda}, \tag{9}$$

where

$$k.(r)=\inf\{k(r'): r'\geqq r\}, \tag{10}$$

and $k(r)$ is defined by (8).

Remark 2. The letter A will be used for absolute and positive constants, not necessarily the same at each occurrence.

Remark 3. We notice that, if $f(z)\in E(\Lambda, a)$, then the function

$$\varphi_0(t)=\log M(e^t, f) \tag{11}$$

is a convex function satisfying (3) and $f(z)$ is of finite φ_0-order 1. So we have the following corollary.

COROLLARY 1. *Let* $0\leqq a<\pi$ *and* $\Lambda=\{\lambda_n\}$ *be a strictly increasing sequence of positive integers. Then there exists an entire function* $f(z)\in E(\Lambda, a)$, *if and only if*

$$\lim_{r\to+\infty} k(r)=+\infty, \tag{12}$$

where $k(r)$ *is defined by* (8).

Remark 4. In the papers ([4], [5]), the following similar assertion is shown: There exists an function $f(z)\not\equiv 0$, which is represented by a gap power series (1) in a neighbourhood of zero, analytic and bounded in the angular domain $B(a)$ $(0<a<\pi)$, if and only if $k(r)$ has a finite lower bound in $(0, +\infty)$.

2. Proofs of Theorems 1 and 2

W. H. J. Fuchs' lemma ([2], [6]) shows that the function

$$G(z)=\prod_{n=1}^{+\infty}\left(\frac{z-\lambda_n}{z+\lambda_n}\right)\exp\left(\frac{2z}{\lambda_n}\right) \tag{13}$$

is analytic in the half-plane $x=\mathrm{Re}z>-\lambda_1$, and

$$|G(z)|\leqq\exp\{x\lambda(r)+Ax\} \qquad (x\geqq 0); \tag{14}$$

$$|G(z)|\geqq\exp\{x\lambda(r)-Ax\} \qquad (z\in D(\Lambda)); \tag{15}$$

$$|G'(\lambda_n)|\geqq\exp\{\lambda_n\lambda(\lambda_n)-A\lambda_n\} \qquad (n=1, 2, 3, \cdots), \tag{16}$$

where $D(\Lambda)=\{z=x+iy: x\geqq 0,\ |z-\lambda_n|\geqq 8^{-1},\ n=1, 2, 3, \cdots\}$, $r=|z|$ and $\lambda(r)$ is defined by (2). Stirling's formula implies that the Gamma function $\Gamma(z)$ satisfies

$$\left|\Gamma\left(\frac{1}{2}+\frac{2}{\pi}az\right)\right|=\exp\left\{\frac{2}{\pi}ax\log^+|z|-a|y|+c(x)\right\} \tag{17}$$

where $|c(x)|\leqq A+Ax$ for $x\geqq 0$. So the function

$$g_0(z)=\frac{\Gamma((1/2)+(2/\pi)az)}{(1+z)^2G(z)} \tag{18}$$

is meromorphic and satisfies

$$|g_0(z)|\leqq(1+y^2)^{-1}\exp\{-xk(r)-a|y|+Ax+A\} \tag{19}$$

for $z\in D(A)$, where $k(r)$ is defined by (8).

Proof of the necessity of the conditions in Theorems 1 and 2. We assume that a transcendental entire function $f(z)\not\equiv 0$, which is represented by a gap power series (1), is bounded in the angular domain $B(a)=\{z=re^{i\theta}: |\theta|\leqq a\}$. Let $M(r, f)$ be defined by (6). Then the function $\varphi_0(t)$ defined by (11) is a convex function of t and satisfies (3) with $\varphi(t)$ replaced by $\varphi_0(t)$. The coefficients $\{a_n\}$ of (1) satisfy

$$|a_n|\leqq\exp\{\varphi_0(\sigma)-\lambda_n\sigma\}, \qquad (n=1, 2, \cdots) \tag{20}$$

for any real number σ.

The function $h_0(z)$ defined by

$$h_0(z)=G(z)\int_0^{+\infty}f(w)w^{-z-1}dw \tag{21}$$

is analytic in the band $0<x<\lambda_1$, and for any real number σ,

$$h_0(z)=G(z)\int_{e^\sigma}^{+\infty}f(w)w^{-z-1}dw+\sum_{n=1}^{+\infty}\frac{G(z)}{\lambda_n-z}a_n\exp\{\sigma(\lambda_n-z)\}.$$

This implies that the function $h_0(z)$ admits an analytic continuation (the continuation is again denoted by $h_0(z)$.) on the half-plane $\{z: x=\mathrm{Re}z>0\}$.

By Cauchy's formula,

$$h_0(z)=\sum_{n=1}^{+\infty}\frac{G(z)}{\lambda_n-z}a_n\exp\{(\sigma+it)(\lambda_n-z)\}+G(z)\int_{L(\sigma,t)}f(w)w^{-z-1}dw$$

for $x>0$ and for any real numbers σ and t such that $|t|\leqq a$, where

$$L(\sigma, t)=\{re^{it}: e^\sigma\leqq r<+\infty\}.$$

Taking $t=-a$ if $y\geqq 0$, and taking $t=a$ if $y<0$, we obtain that

$$|h_0(z)|\leqq x^{-1}\exp\{\varphi(\sigma)-\sigma x+x\lambda(r)-a|y|+Ax+A\}$$

for $x>0$ and any real number σ. So the function $h(z)=h_0(8^{-1}+z)$ satisfies

$$|h(z)|\leqq\exp\{-\psi_0(x)+x\lambda(r)-a|y|+Ax+A\}$$

for $x\geqq 0$, where

$$\psi_0(x)=\sup\{x\sigma-\varphi_0(\sigma): \sigma\in(-\infty, +\infty)\}. \tag{22}$$

Applying Carleman's formula ([2]) for the semi-ring $\{z: 1\leqq|z|\leqq R, x\geqq 0\}(R>1)$ to the function $h(z)=h_0(8^{-1}+z)$, we have

$$-A\leqq\frac{1}{2\pi}\int_1^R\left(\frac{1}{y^2}-\frac{1}{R^2}\right)\log|h(iy)h(-iy)|dy+\frac{1}{\pi R}\int_{-\pi/2}^{\pi/2}\log|h(Re^{i\theta})|\cos\theta d\theta.$$

Therefore

$$-A-\lambda(R)\leqq-\frac{2a}{\pi}\log R+\frac{4}{\pi R}\int_0^{\pi/2}(-\psi_0(R\cos\theta)\cos\theta)d\theta\,.$$

By (22), for any real number σ,

$$\int_0^{\pi/2}\psi_0(R\cos\theta)\cos\theta d\theta\geqq\int_0^{\pi/2}[\sigma R\cos\theta-\varphi_0(\sigma)]\cos\theta d\theta=\frac{\pi}{4}R\sigma-\varphi_0(\sigma),$$

and again by (22),

$$\frac{2a}{\pi}\log R-\lambda(R)+\frac{4}{\pi R}\psi_0\Big(\frac{\pi}{4}R\Big)\leqq A\,. \tag{23}$$

The inequality

$$\psi_0\Big(\frac{\pi R}{4}\Big)\geqq\frac{\pi R}{4}\tilde{\varphi}_0(R)-R \tag{24}$$

is obtained from (22) by taking $\sigma=\tilde{\varphi}_0(R)$. As $\tilde{\varphi}_0(s)$ is increasing for s large enough. From (23) and (24), we have

$$k(R)\geqq k.(R)\geqq\tilde{\varphi}_0(R)-A\,. \tag{25}$$

If $f(z)$ is of finite φ-order at most ρ, then for any $\rho'>\rho$,

$$\varphi_0(t)\leqq\varphi(\rho' t) \tag{26}$$

for t large enough, and so

$$k(R)\geqq k.(R)\geqq\tilde{\varphi}_0(R)-A\geqq\frac{1}{\rho'}\tilde{\varphi}(R)-A \tag{27}$$

for sufficiently large R. We obtain the inequality (7) by taking $R=\varphi(r)$ and letting $\rho'\to\rho$.

If $f(z)$ is of finite lower φ-order at most λ, then for any $\rho'>\lambda$, (27) holds for a set of values of R which is unbounded above. We obtain the inequlity (9) by taking $R=\tilde{\varphi}(r)$ and letting $\rho'\to\lambda$.

Proof of the sufficiency of the conditions in Theorems 1 and 2. Suppose that one of (7) and (9) holds, then $k.(r)$ in (10) is well-defined and unbounded on $[0,+\infty)$. Since $\Lambda=\{\lambda_n\}$ is a strictly increasing sequence of positive integers, there exists a twice continuously differentiable function $p(r)$ defined on $[0,+\infty)$ such that

$$|p(r)-\lambda(r)|\leqq A\,;\qquad 0\leqq rp'(r)\leqq 1\,.$$

Let $q(r)=p(r)-(2a/\pi)\log^+ r$, $q.(r)=\inf\{q(r')\colon r'\geqq r\}$ and $r_2>r_1$. If $q.(r_2)\geqq q.(r_1)$, then

$$\begin{aligned}0<q.(r_2)-q.(r_1)&=\sup\{q.(r_2)-q(r')\colon r_1\leqq r'\leqq r_2\}\\&\leqq\log r_2-\log r_1\,,\end{aligned}$$

which implies that there exists a differentiable and strictly increasing function $l(r)$ defined on $[0, +\infty)$ such that

$$l(0)=0\,;\ |l(r)-q.(r)|\leqq A\,;\ 0\leqq rl'(r)\leqq 1\,;\ |k.(r)-l(r)|\leqq A\,. \tag{28}$$

So the function $f(z)$ defined by

$$f(z)=-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} g_0(w)z^w dw \tag{29}$$

is continuous, bounded and not identically zero in the angular domain $B(a)$, where $g_0(w)$ is defined by (18). By Cauchy's formula,

$$f(z)=\sum_{k=1}^{n} a_k z^{\lambda} k+f_n(z) \tag{30}$$

for $n=1, 2, 3, \cdots$, where the coefficients

$$a_k=\Gamma\Big(\frac{1}{2}+\frac{2}{\pi}a\lambda_k\Big)(1+\lambda_k)^{-2}(G'(\lambda_k))^{-1} \tag{31}$$

are the residues of $g_0(z)$ at the poles λ_k, and

$$f_n(z)=-\frac{1}{2\pi i}\int_{\lambda_n+(1/2)-i\infty}^{\lambda_n+(1/2)+i\infty} g_0(w)z^w dw\,. \tag{32}$$

From (11-17) and (28-32), we have

$$|f_n(z)|\leqq\exp\{A+(A+\log^+|z|-k.(\lambda_n))(\lambda_n+2^{-1})\} \tag{33}$$

for $z\in B(a)$ and that the coefficients $\{a_n\}$ satisfy

$$|a_n|\leqq\exp\{-\lambda_n l(\lambda_n)+A\lambda_n+A\} \tag{34}$$

for $n=1, 2, 3, \cdots$. Therefore, from (33) and (34), we see that the function $f(z)$ is an entire function represented by a gap power series (1) and bounded in the angular domain $B(a)$. So $f(z)$ belongs to $E(\Lambda, a)$ and its maximum modulus $M(r, f)$ satisfies

$$M(e^t, f)\leqq\sum_{n=1}^{+\infty}|a_n|\exp(\lambda_n t)\leqq\exp\{\varphi_1(t+A)\}, \tag{35}$$

where

$$\begin{aligned}\varphi_1(t)&=\sup\{tu-ul(u):\ u\geqq 0\}\\&=\sup\{u(t-l(u)):\ u\geqq 0,\ 1(u)\leqq t\}.\end{aligned} \tag{36}$$

There exists a positive function $u(t)$ defined on $(0, +\infty)$ such that

$$0\leqq t-1(u(t))=u(t)l'(u(t))\leqq 1\,;$$

$$\varphi_1(t)=u(t)[t-l(u(t))]\leqq u(t)\leqq\tilde{l}(t),$$

where $t=\tilde{l}(s)$ is the inverse function of $s=l(t)$.

If (9) holds, then by (28), we have

$$\limsup_{r\to+\infty} r^{-1}l(\varphi(r))=\limsup_{r\to+\infty} r^{-1}k.(\varphi(r))\geqq\frac{1}{\lambda}.$$

So, for any $\varepsilon>0$, there exists an increasing and unbounded sequence $\{r_n\}$ such that

$$l(\varphi(r_n))\geqq\frac{r_n}{\lambda+\varepsilon}=\rho_n,\ \varphi_1(\rho_n)\leqq\tilde{l}(\rho_n)\leqq\varphi[(\lambda+\varepsilon)\rho_n].$$

Therefore, by (35) and (36), for given any $\varepsilon>0$,

$$\liminf_{t\to+\infty} t^{-1}\tilde{\varphi}[\log M(e^t,\ f)]=\limsup_{t\to+\infty} t^{-1}\tilde{\varphi}[\varphi((\lambda+\varepsilon)(t+A))]=\lambda+\varepsilon,$$

which implies that the entire function $f(z)\in E(\Lambda,\ a)$ is of finite lower φ-order at most λ.

If (7) holds, we have

$$\liminf_{r\to+\infty} r^{-1}l[\varphi(r)]\leqq\liminf_{r\to+\infty} r^{-1}k.[\varphi(r)]\geqq\frac{1}{\rho}.$$

So, for any $\varepsilon>0$, there exists a positive constant $A(\varepsilon)$ such that

$$l[\varphi(t)]\geqq\frac{t}{\rho+\varepsilon},\qquad \varphi_1(t)\leqq\tilde{l}(t)\leqq\varphi[(\rho+\varepsilon)t]$$

for $t\geqq A(\varepsilon)$. Therefore, by (35) and (36), for any $\varepsilon>0$,

$$\limsup_{t\to+\infty} t^{-1}\tilde{\varphi}[\log M(e^t,\ f)]\leqq\limsup_{t\to+\infty} t^{-1}\tilde{\varphi}[\varphi((\rho+\varepsilon)(t+A))]=\rho+\varepsilon,$$

which implies that $f(z)\in E(\Lambda,\ a)$ is of finite φ-order at most ρ.

Proof of the necessity of the condition in Corollary 1. If there exists an entire function $f(z)\not\equiv 0$, which is represented by (1) and is bounded in the angular domain $B(a)$, then the function $\varphi_0(x)$ defined by (11) is convex and satisfies (3). So $f(z)$ is of φ_0-order 1 and consequently by Theorem 1, we have the necessary condition (12).

Proof of the sufficiency of the condition in Corollary 1. If (12) holds, then there exists an increasing, unbouded and concave function $\tilde{\varphi}(x)$ defined on $[0,\ +\infty)$ satisfying $k.(x)\geqq\tilde{\varphi}(x)$ such that the inverse function $x=\varphi(r)$ of $r=\tilde{\varphi}(x)$ is convex and satisfies (3). We see from Theorem 1 that there exists an entire function $f(z)\not\equiv 0$, which is represented by (1) and is bounded in the angular domain $B(a)$. This completes the proof of Corollary 1.

In conclusion, it is my pleasure to thank referee for his very helpful comments.

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