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DEFORMATIONS OF A HYPERBOLIC 3-MANIFOLD NOT AFFECTING ITS TOTALLY GEODESIC BOUNDARY

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Introduction.

By a hyperbolic manifold, we will mean a Riemannian manifold with constant sectional curvature -1. In this paper, we study complete oriented hyperbolic 3-manifolds with totally geodesic boundary (possibly with cone type singularities along some closed geodesics) and deformations of hyperbolic structures on such 3-manifolds. A smooth totally geodesic boundary of such a 3-manifold becomes a hyperbolic surface.

Let g be an integer greater than or equal to 2 and $\boldsymbol{M}_g$ be the moduli space of closed Riemann surfaces of genus g. Let $\boldsymbol{S}_g$ be the subset of $\boldsymbol{M}_g$ consisting of those hyperbolic surfaces which are boundary components of *compact* oriented hyperbolic 3-manifolds with totally geodesic boundary. It is well known that $\boldsymbol{S}_g$ is a countable subset of $\boldsymbol{M}_g$. Moreover, we can show that $\boldsymbol{S}_g$ is dense in $\boldsymbol{M}_g$, by making use of the theory of hyperbolic Dehn surgery due to Thurston [5] and a theorem of Brooks [1], which states that closed hyperbolic surfaces that can be filled by circle packings (see §1 for terminology) form a dense subset of $\boldsymbol{M}_g$. This fact can be proved roughly as follows. First of all, following a method of Brooks [1] (see §1 for its precise description), for each closed hyperbolic surface filled by a circle packing, we can construct a complete hyperbolic 3-manifold with torus cusps such that each boundary component of it is isomorphic to the given hyperbolic surface. Then, by deforming the hyperbolic structure of the 3-manifold by means of the hyperbolic Dehn surgery at every toral end of the 3-manifold, we obtain a compact hyperbolic 3-manifold with totally geodesic boundary such that each boundary component is arbitrarily close to the initial closed hyperbolic surface in the moduli space $\boldsymbol{M}_g$. Thus, by combining the above construction with Brooks' theorem, we can show the required fact (I owe this argument to T. Soma).

Now the argument above suggests a relationship between the set $\boldsymbol{S}_g$ and Dehn surgery. In particular it naturally raises a question of whether there is a difference between the initial closed hyperbolic surface filled by the circle packing and the hyperbolic surface of the boundary component of the Dehn surgered compact hyperbolic 3-manifold or not. In our previous paper [2], we

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gave an example of a complete hyperbolic 3-manifold with one torus cusp whose boundary is totally geodesic and is isomorphic to a closed hyperbolic surface which can be filled by a circle packing such that a boundary of a surgered hyperbolic 3-manifold is different from the initial hyperbolic surface. (Note that our construction of this 3-manifold is different from that of Brooks which is mentioned in § 1 of this paper.) Having this example, it seems to be natural to expect that the surface obtained after the deformation, in general, is different from the initial surface (cf. Kapovich [3] for a general discussion of this phenomenon).

In this paper however, in contrast to the above, we will show the existence of an opposite case, using the hyperbolic surface which is the boundary of the hyperbolic 3-manifold given in [2] and mentioned above, under the situation where we permit the surgered hyperbolic 3-manifold to have cone type singularities. Namely, we give a concrete example of a hyperbolic 3-manifold M with torus cusps and totally geodesic boundary (although the 3-manifold M is different from the example in [2], each boundary component of M is isomorphic to the boundary hyperbolic surface of the example in [2]) which has the following property: the hyperbolic structure of the totally geodesic boundary remains to be constant under a deformation of the hyperbolic structure which makes cone type singularities on the toral ends.

Our original hyperbolic 3-manifold M will be constructed by using the method of Brooks mentioned above: given any closed hyperbolic surface on which a circle packing exists, we can construct a hyperbolic 3-manifold with some torus cusps and totally geodesic boundary consisting of four components such that each of its boundary components is isometric to the given hyperbolic surface. By applying this construction to the closed hyperbolic surface mentioned above on which we can write a concrete circle packing consisting of two isometric circles, we obtain the hyperbolic 3-manifold M (see § 1). Then the deformation is explicitly accomplished by adequately enlarging the radii of circles of the packing and making cone type singularities on all toral ends of M with the same varying angles (see § 2). At every total end, the coefficients of the hyperbolic Dehn surgery of our deformation have the forms $(p, 0)$ $(p \in \boldsymbol{R})$ with respect to an appropriate choice of generators of the fundamental group of the toral end, so that the deformed hyperbolic 3-manifolds can be completed to yield hyperbolic 3-orbifolds with cone type singularities along the toral ends (see [5]). This deformation is easy to be seen and quite clear. Also, it seems to be possible to generalize the construction above to the case where there are given hyperbolic surfaces filled by circle packings each of which consists of several isometric circles rather than just two, by applying the method of Brooks to the given hyperbolic surfaces.

Neumann-Reid [4] also constructed concrete examples of hyperbolic 3-manifolds, each having one torus cusp and totally geodesic boundary, which admit deformations which do not affect the boundaries. Their method is different from ours. More precisely, they take certain finite cyclic coverings of a complete hyperbolic 3-orbifold having a torus cusp and a totally geodesic boundary whose hyperbolic structure is rigid, namely whose Teichmüller space is trivial. Then, by using the triviality of the Teichmüller space, they showed that hyperbolic 3-manifolds, which is the total spaces of the cyclic coverings, admit deformations leaving the boundaries invariant. The hyperbolic Dehn surgery coefficients of the deformations of these examples of Neumann-Reid have the forms (p, q)'s where each (p, q) is a coprime pair of integers, so that the deformed hyperbolic 3-manifolds can be compactified to yield smooth closed hyperbolic 3-manifolds by the Dehn surgery (see [5]).

1. A hyperbolic 3-manifold M with totally geodesic boundary and torus cusps.

In this paper we use the Poincaré model of the hyperbolic 3-space $\boldsymbol{H}^3$. Namely, let $\boldsymbol{H}^3$ be the space $\{x=(x_1, x_2, x_3)\in\boldsymbol{R}^3 ; |x|<1\}$ endowed with the Riemannian metric ds given by $ds=2|dx|/1-|x|^2$ and let $\boldsymbol{H}^2(\subset\boldsymbol{H}^3)$ be the subspace $\{x=(x_1, x_2, x_3)\in\boldsymbol{H}^3 ; x_3=0\}$. Denote by S_∞ the sphere at infinity of $\boldsymbol{H}^3$. In this paper, we use notations $\boldsymbol{H}^3_+$ and S^+_∞ to indicate the spaces $\{x=(x_1, x_2, x_3)\in\boldsymbol{H}^3 ; x_3\geqq 0\}$ and $\{x=(x_1, x_2, x_3)\in S_\infty ; x_3>0\}$ respectively (see Fig. 1). Let us denote by pr the orthogonal projection from $\boldsymbol{H}^2$ onto S^+_∞ along geodesics each of which starts from $\boldsymbol{H}^2$ in the orthogonal direction (see Fig. 2).

Let $\mathcal{R}$ be a hyperbolic surface without cusps uniformized by $\boldsymbol{H}^2$ so that $\mathcal{R}=\boldsymbol{H}^2/\Delta$, for some Fuchsian group Δ. A configuration of circles on $\mathcal{R}$ is a collection of simple closed curves on $\mathcal{R}$ which bound disks, such that the lifts of these curves to $\boldsymbol{H}^2$ are Euclidean circles. A configuration on $\mathcal{R}$ is said to

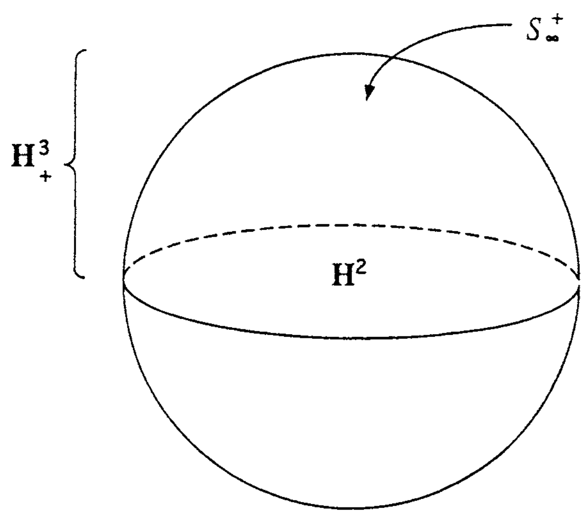

Fig. 1.

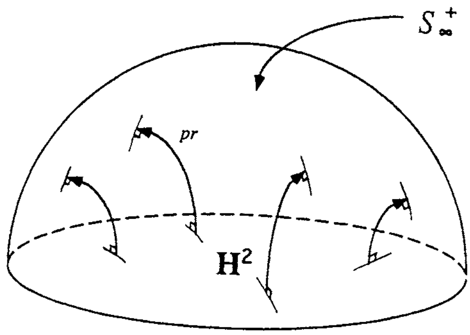

Fig. 2.

be a *circle packing* if the interiors of disks are all disjoint and the region complementary to the interiors of the disks consists only of curvilinear triangles. Now suppose, in general, that there are given three circles which are either mutually tangent to one another (Fig. 3*a*) or have a hyperbolic triangle region between them (Fig. 3*b*). Then there exists a unique circle which meets each of the three circles perpendicularly. Let us call this the *perpendicular circle* for the triple of circles.

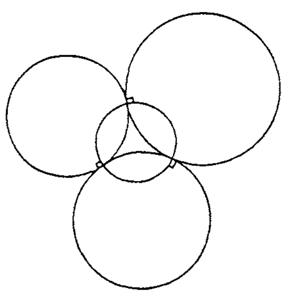

Fig. 3(*a*)

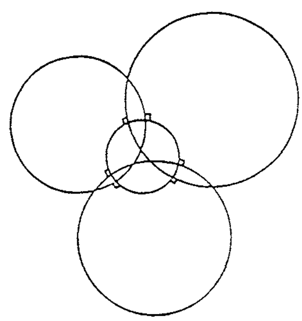

Fig. 3(*b*)

In this section, following the method of Brooks [1], we construct a hyperbolic 3-manifold with totally geodesic boundary and torus cusps using a circle packing of a hyperbolic surface. First consider two copies of a regular dodecagon in $\boldsymbol{H}^2$ with interior angles $2\pi/3$. Identify the edges as in Fig. 4. Then we obtain a closed hyperbolic surface $\mathcal{S}$ of genus 2. Let Γ be the corresponding Fuchsian group (i. e. $\mathcal{S}=\boldsymbol{H}^2/\Gamma$). As indicated in Fig. 4, the fundamental domain for Γ is packed by two circles C^1, C^2 and $\boldsymbol{H}^2$ is packed by a configuration of circles made of the Γ-orbit of C^1 and C^2. Denote this circle packing of $\boldsymbol{H}^2$ by $\mathcal{C}$. Now draw the union of all perpendicular circles $\mathcal{P}$ for all the triples of the circle packing $\mathcal{C}$ of $\boldsymbol{H}^2$.

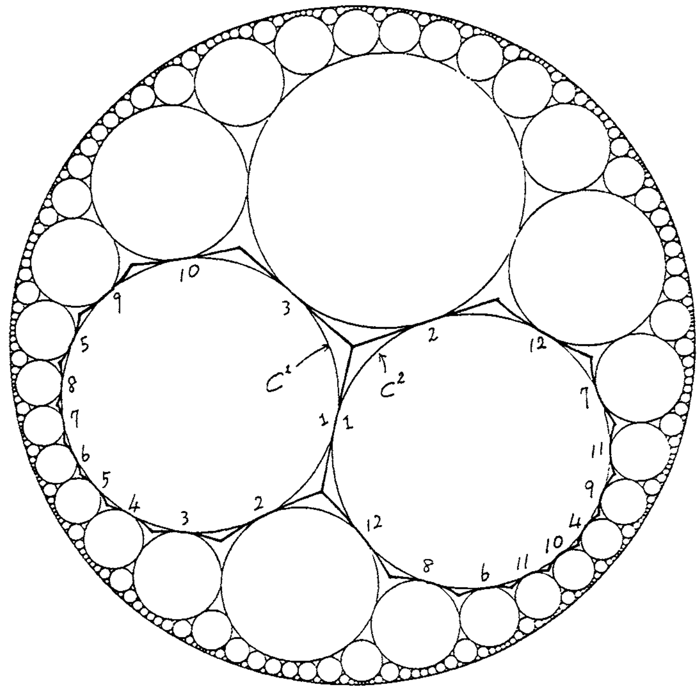

Fig. 4. Glue together the numbered sides of two pieces of a regular 12-gon with angles $2\pi/3$ so that the numbers are matched.

Map all of the circles of C and $\mathcal{P}$ to S^+_∞ by the orthogonal projection pr. Let C' and $\mathcal{P}'$ be the images of C and $\mathcal{P}$ respectively, i. e., $C'=pr(C)$, $\mathcal{P}'=pr(\mathcal{P})$. The configuration C' in S^+_∞ can be identified with C in $\boldsymbol{H}^2$ by the orthogonal projection pr. Then we can regard Fig. 4 as a picture of C' in S^+_∞, provided that S^+_∞ is identified with the interior of the outer circle in this figure.

Let $\tilde{N}$ be the space obtained by removing from $\boldsymbol{H}^3_+$ the regions interior to all hemispheres lying over all the circles of C' and $\mathcal{P}'$. The 3-dimensional space $\tilde{N}$ is a geodesic polyhedron with ideal vertices and has two boundary components $\tilde{\partial}_1$, $\tilde{\partial}_2$. One is $\boldsymbol{H}^2(=\tilde{\partial}_1)$. The other one $\tilde{\partial}_2$ comes from the boundaries of the hemispheres above, so that it consists of infinitely many ideal dodecagons and ideal triangles which meet each other at right angles along all edges of polygons (see Fig. 6).

Let Γ act on $\boldsymbol{H}^3$ as a Kleinian group with the limit set of Γ equal to $\partial\boldsymbol{H}^2$. Now note that Γ acts on S^+_∞ via Möbius transformations and the configurations C' and $\mathcal{P}'$ are invariant by the action of Γ. Let N be the quotient space $\tilde{N}/\Gamma$. Let $\partial_1=\boldsymbol{H}^2/\Gamma$ and $\partial_2=\tilde{\partial}_2/\Gamma$. Then $\partial N=\partial_1\cup\partial_2$. Since the hyperbolic surface $\boldsymbol{H}^2/\Gamma(=S)$ is obtained as in Fig. 4 and Γ acts on ∂_2 in the same way as on $\boldsymbol{H}^2$, the boundary component ∂_2 of N is illustrated as in Fig. 7. Now we can

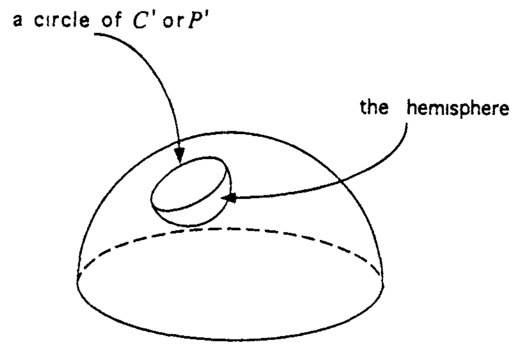

Fig. 5. A hemisphere lying over a circle of C' or $\mathcal{P}'$.

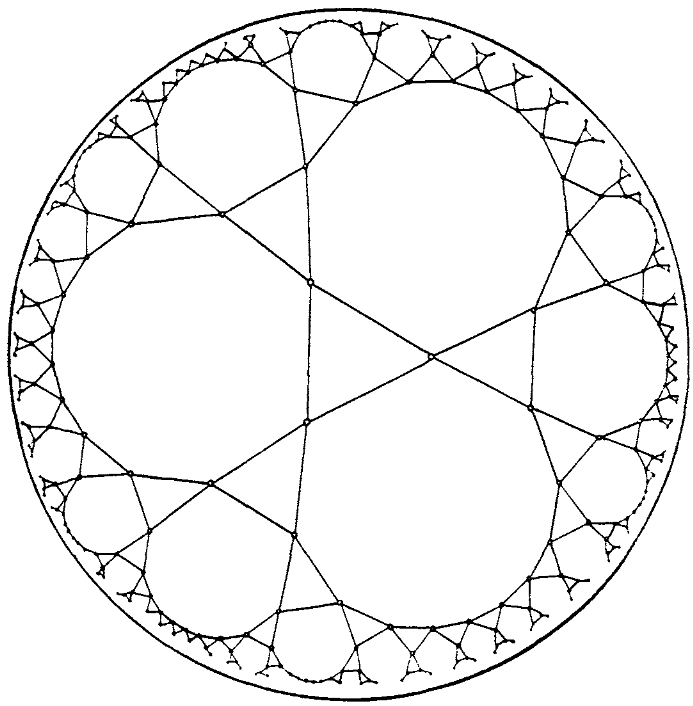

Fig. 6. This is a picture of $\tilde{\partial}_2$ orthogonally projected to S^+_∞. The points $\circ$ are ideal vertices.

check that ∂_2 consists of two ideal dodecagons and eight ideal triangles and has twelve ends. A section $\mathcal{O}$ of each end of N by cutting at a horosphere centered at the infinity point of the end forms a rectangle in $\boldsymbol{R}^2$ and the end is isomorphic to $[0, \infty)\times\mathcal{O}$ (see Fig. 8).

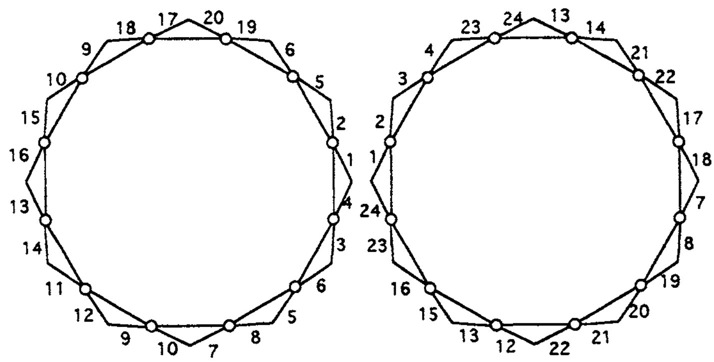

Fig. 7. Each piece is made of one ideal dodecagon and twelve triangles with two ideal vertices. All dihedral angles between the triangles and the dodecagon are $\pi/2$. The vertices $\circ$ are ideal ones. All numbered sides of triangles have the same length. The interior angles of each triangle are 0, 0 and $2\pi/3$. The boundary component ∂_2 can be obtained by gluing together the numbered sides of the triangles so that the numbers are matched.

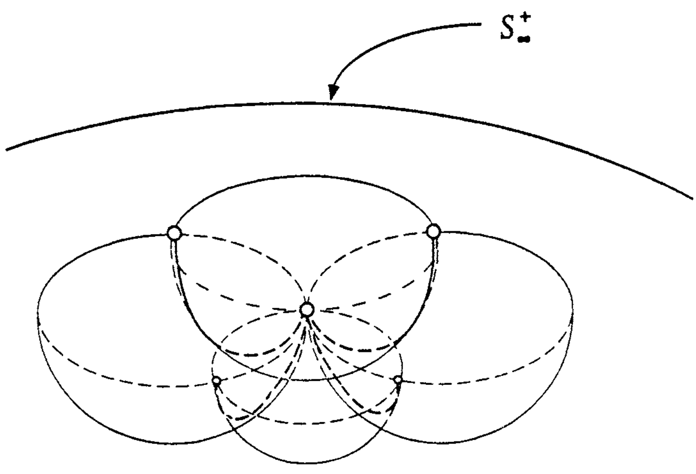

Fig. 8. Each end of N.

Let L be the double of N along the eight ideal triangles of ∂_2. Since each ideal dodecagon intersects adjacent ideal triangles at right angle and do not meet the other ideal dodecagon in N, there exist two boundary components in L each of which is a twelve-punctured hyperbolic surface. Besides the two boundary components, L has two other boundary components each of which is isomorphic to the hyperbolic surface $\mathcal{S}$.

Let M be the double of L along the twelve-punctured spheres of ∂L. The boundary ∂M of M consists of four components each of which is totally geodesic and isomorphic to $\mathcal{S}$. Since each end of N is isomorphic to $[0, \infty)\times\mathcal{O}$ and each end of M is the double of the double of the corresponding end of N, M is a hyperbolic 3-manifold with twelve *torus* cusps (see Fig. 9). This is the desired manifold.

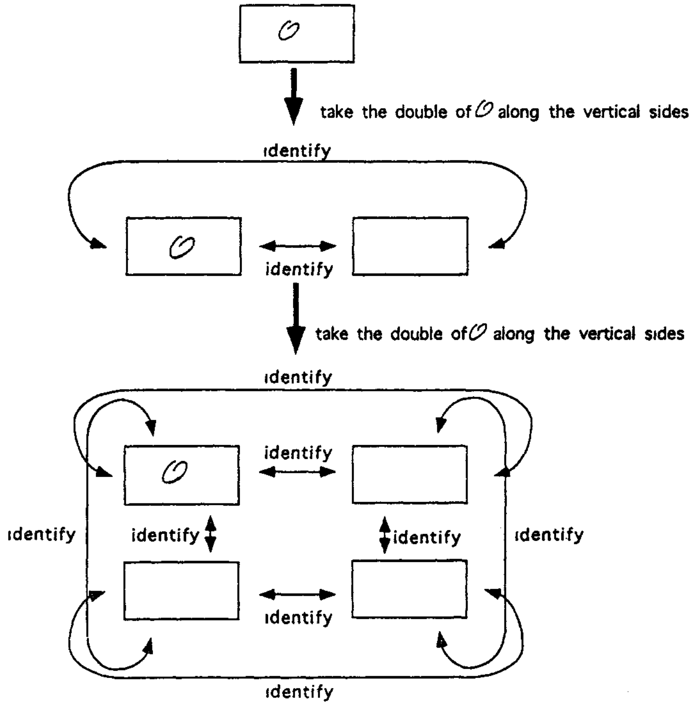

Fig. 9. This is a picture showing that each end of M is a torus cusp. We see it by cutting each end along a horosphere.

2. A deformation of the hyperbolic structure on M.

In this section, we explicitly deform the hyperbolic structure on M, which was constructed in §1, by enlarging the radii of the circles of C in §1 to obtain the following:

THEOREM. *Let M be the hyperbolic 3-manifold with four totally geodesic boundary components and twelve torus cusps as was constructed in* §1. *Let α be any real number satisfying $0\leqq\alpha<\pi/6$. Then there is a deformation of the hyperbolic structure on M with cone type singularities of angle 4α at the twelve torus ends keeping the moduli of the totally geodesic boundary components of M*

constant.

Proof. By a Möbius transformation g, we translate the configuration C of circles which was given in § 1, so as a center of some circle of C to be the origin (0, 0, 0) in $\boldsymbol{H}^2$. Consider a 1-parameter family C_α^1 $(0 \leqq \alpha < \pi/6)$ of concentric circles centered at the origin in $\boldsymbol{H}^2$ (see Fig. 10). Let C_α^2 be the reflected image of C_α^1 by a reflection along one of the edges of the regular dodecagon with interior angles $2\pi/3$. For a while, assume that $0 < \alpha < \pi/6$. Two circles $g^{-1}C_\alpha^1$, $g^{-1}C_\alpha^2$ cover the fundamental domain of Γ with overlapping each other. Consider the Γ-orbit of these circles. Then we have a configuration of infinite many circles $\gamma g^{-1}C_\alpha^1$, $\gamma g^{-1}C_\alpha^2 (\gamma \in \Gamma)$ on $\boldsymbol{H}^2$. Denote this configuration by C_α.

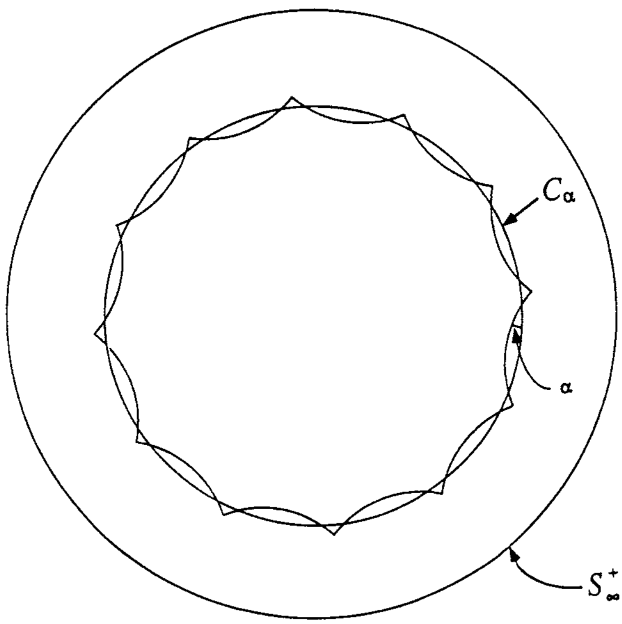

Fig. 10. Parameter α gives the corresponding angle.

Now we apply, after a slight modification, the procedure which was used for constructing the hyperbolic 3-manifold M in § 1 to the configuration C_α of circles. Each triple of the configuration C_α has a hyperbolic triangle region between them. Draw the union of all perpendicular circles $\mathcal{P}_\alpha$ for all triple of the configuration C_α of circles. Now map all of the circles of C_α and $\mathcal{P}_\alpha$ to S_∞^+ by the orthogonal projection pr. Let C'_α and $\mathcal{P}'_\alpha$ be the images of C_α and $\mathcal{P}_\alpha$ respectively, i. e., $C'_\alpha = pr(C_\alpha)$ and $\mathcal{P}'_\alpha = pr(\mathcal{P}_\alpha)$.

Let $\tilde{N}_\alpha$ be the space obtained by removing from $\boldsymbol{H}_+^3$ the regions interior to all hemispheres lying over all the circles of C'_α and $\mathcal{P}'_\alpha$. The 3-dimensional space $\tilde{N}_\alpha$ is an infinite-sided geodesic polyhedron. The boundary of $\tilde{N}_\alpha$ consists of two components $\tilde{\partial}_1^\alpha$, $\tilde{\partial}_2^\alpha$. One is $\boldsymbol{H}^2(=\tilde{\partial}_1^\alpha)$. The other one $\tilde{\partial}_2^\alpha$ is made of in-

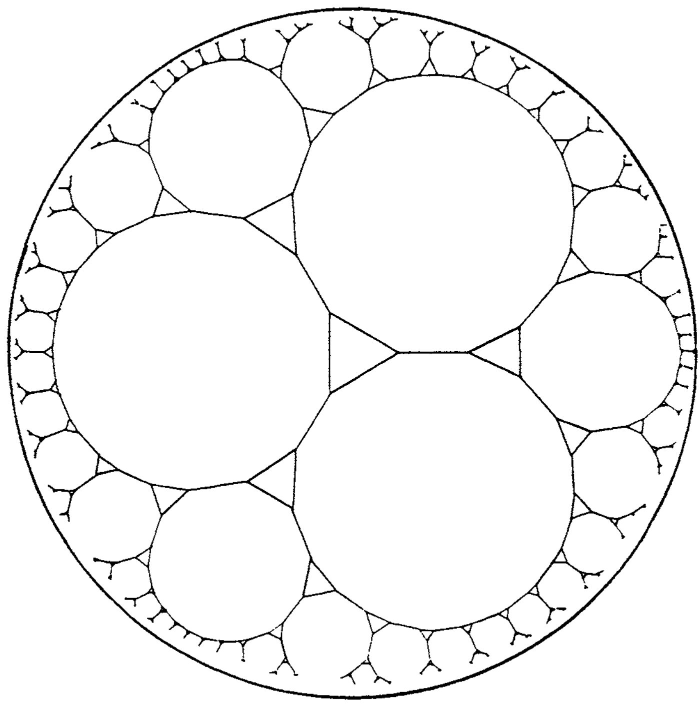

Fig. 11. This is a picture of $\tilde{\partial}_2^\alpha$ orthogonally projected to S_∞^+.

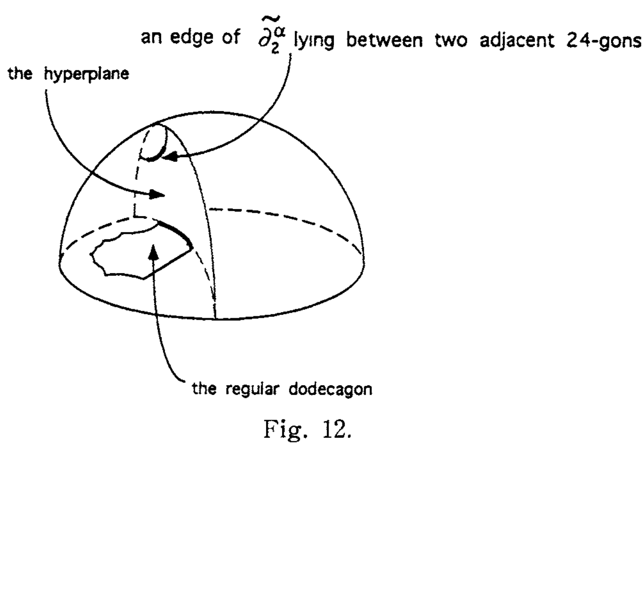

Fig. 12.

finitely many regular triangles with interior angles 2α and infinitely many right-angled 24-gons (see Fig. 11). All edges of $\tilde{\partial}_2^\alpha$ lying between the adjacent 24-gons have the same length. It can be seen that all such edges lie on hyperplanes, each of which orthogonally intersects $\boldsymbol{H}^2(\subset \boldsymbol{H}^3)$ along a geodesic including a segment which is a part of the tessellation of $\boldsymbol{H}^2$ made of the regular dodecagons (see Fig. 12). Also we can see that all such edges cut the triangles on $\tilde{\partial}_2^\alpha$ into three parts each of which is a triangle with interior angles α, α and $2\pi/3$ (see Fig. 13).

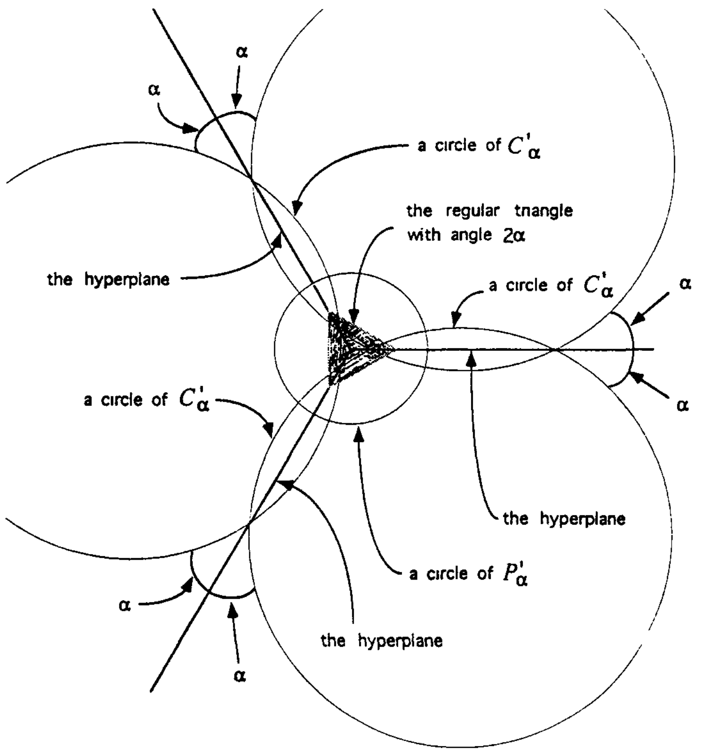

Fig. 13. A picture othogonally projected to S^+_∞.

Let N_α be the quotient space $\tilde{N}_\alpha/\Gamma$. Let $\partial_1^\alpha = \boldsymbol{H}^2/\Gamma$ and $\partial_2^\alpha = \tilde{\partial}_2^\alpha/\Gamma$. Then $\partial N_\alpha = \partial_1^\alpha \cup \partial_2^\alpha$. In the same way as we have obtained the picture of the boundary component ∂_2 of N in § 1, we have a picture of ∂_2^α as shown in Fig. 14. It can be seen that ∂_2^α consists of two right-angled 24-gons and eight regular triangles with angles 2α. Each dihedral angle between the triangle and the 24-gon is $\pi/2$ and one between the 24-gon and another 24-gon is 2α.

Let L_α be the double of N_α along the triangles on $\tilde{\partial}_2^\alpha$. Two of boundary components of L_α are hyperbolic 2-orbifolds bent along twenty-four geodesics

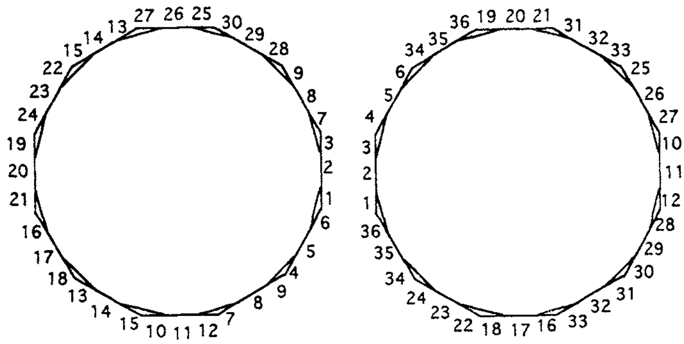

Fig. 14. Each piece is made of one right-angled 24-gon and twelve triangles with interior angles α, α and $2\pi/3$. All dihedral angles between the triangles and the 24-gon are $\pi/2$. The boundary component ∂_2^g can be obtained by identifying the numbered sides of the triangles so that the numbers are matched.

with angle 2α. There are two other boundary components of ∂N_α. Each of them is isomorphic to $\mathcal{S}$.

Let M_α be the double of L_α along the hyperbolic 2-orbifolds. The boundary ∂M_α consists of disjoint four hyperbolic surfaces. Any of them is isomorphic to $\mathcal{S}$. M_α has twelve simple closed geodesics along each of which there are cone type singularities with angle 4α.

Now consider the case where $\alpha=0$. If α goes to 0, the hyperbolic 3-orbifold M_α converges to the hyperbolic 3-manifold M in Gromov's sense. Namely, we can obtain M_α with the continuous deformation of the hyperbolic structure on M at all torus cusps to make cone singularities of angle 4α. This deformation affects no boundary component of M, which is isomorphic to $\mathcal{S}$. ■

Finally, consider the case where α goes to $\pi/6$. If α goes to $\pi/6$, the right-angled 24-gons on $\tilde{\partial}_2^g$ converge to ideal dodecagons and the regular hyperbolic triangles with angles 2α shrink to points on the sphere at infinity S_∞ (see Fig. 15). The shape of each triangle becomes closer to a Euclidean triangle. The limit Euclidean triagle is a regular one with angles $\pi/3$. Let us see the phenomenon above on the hyperbolic 3-orbifold M_α. Then it can be seen that in M_α there is a hyperbolic 2-orbifold with underlying surface S^2 and three cone points with the same angle 2α. Thus we obtain a family of the structures of the hyperbolic 2-orbifolds parametrized by α, which begins with the complete hyperbolic structures with cone angles zero and shrinks to points as α goes to $\pi/6$. Thus M_α splits open into two parts at the limit Euclidean triangles when α goes to $\pi/6$.

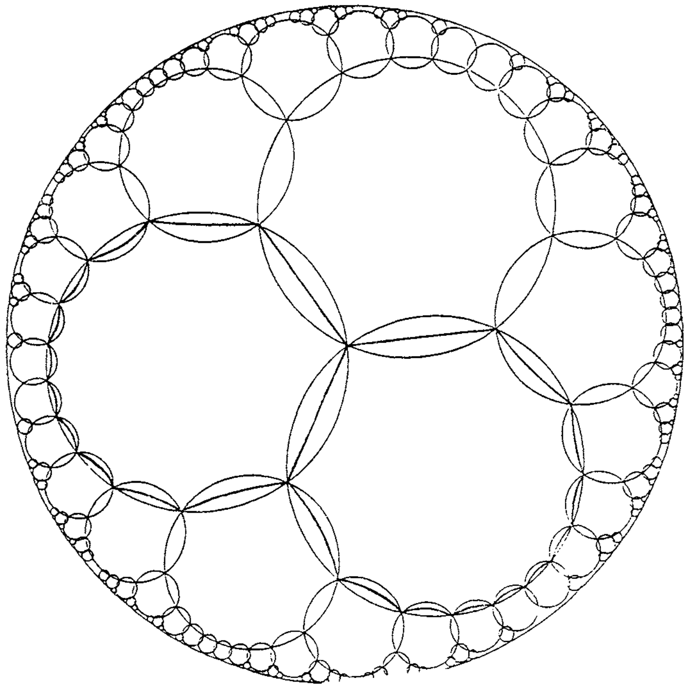

Fig. 15.

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REFERENCES

[1] R. BROOKS, Circle packings and co-compact extensions of Kleinian groups, Invent. Math., **86** (1986), 461-469.

[2] M. FUJII, On totally geodesic boundaries of hyperbolic 3-manifolds, Kodai Math. J., **15** (1992), 244-257

[3] M. KAPOVICH, Eisenstein series and Dehn surgery, Preprint.

[4] W. D. NEUMANN AND A. W. REID, Rigidity of cusps in deformations of hyperbolic 3-orbifolds, Preprint.

[5] W. P. THURSTON, The Geometry and Topology of 3-Manifolds, Lecture Notes, Princeton, Princeton University Press, 1978/79.

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