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A THEOREM ON THE GROWTH OF ENTIRE FUNCTIONS ON ASYMPTOTIC PATHS AND ITS APPLICATION TO THE OSCILLATION THEORY OF $w''+Aw=0$

Dedicated to Professor Nobuyuki Suita on the occasion of his sixtieth birthday

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1. Introduction.

Let $A=A(z)$ be a transcendental entire function and let w_1, w_2 be two linearly independent entire solutions of the differential equation

$$(1)\qquad w''+Aw=0.$$

It is known that any non-zero solution of (1) is an entire function of infinite order ([1]). Put

$$E=w_1w_2.$$

It then holds ([1], p. 354) that

$$(2)\qquad 4A=(E'/E)^2-2E''/E-(c/E)^2,$$

where c is the Wronskian of w_1 and w_2, which is a non-zero constant in this case.

For an entire function f we denote the order of f by $\rho(f)$, the lower order of f by $\mu(f)$ and the order of $N(r, 1/f)$ by $\lambda(f)$.

S. B. Bank and I. Laine ([1], Theorem 2, (A)) proved from (2) that $\rho(A)<1/2$ implies $\lambda(E)=+\infty$. They also gave examples of (1) with two linearly independent entire solutions each having no zeros, in each case of which, $\rho(A)$ is either a positive integer or $+\infty$ ([1], p. 356).

It is conjectured that if $\rho(A)$ is finite and not a positive integer, then we always have $\lambda(E)=+\infty$ (see [2], p. 164). In this direction J. Rossi ([12]) and L.-C. Shen ([13]) proved some results which contains that $\rho(A)\leqq 1/2$ implies $\lambda(E)=+\infty$. Recently, C.-Z. Huang ([9]) proved the following result which generalizes them.

THEOREM A. *If $\mu(A)<1$, then either $\lambda(E)=+\infty$ or*

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$$\mu(A)^{-1}+\lambda(E)^{-1}\leqq 2 \qquad ([9],\ \textit{Theorem } 1).$$

One of our main purpose of this paper is to give a result which contains Theorem A. To prove it, we need a growth property of $A(z)$ in the set

$$\{z:\ |A(z)|>1\}$$

and so we shall first give a result on the growth of entire functions along asymptotic paths. We shall assume that the reader is familiar with the satndard notation of the Nevanlinna theory of meromorphic functions ([5]).

2. Growth of entire functions along asymptotic paths.

A few years ago J. Rossi and A. Weitsman ([11]) proved the following.

THEOREM B. *Let $f(z)$ be a transcendental entire function. Suppose that for some constant K the set*

$$\{z:\ |f(z)|>K\}$$

contains at least two components. Then there exists a path Γ from 0 to ∞ such that for $z\varepsilon\Gamma$

$$(3)\qquad \log|f(z)|>|z|^{\rho(f)/(2\rho(f)-1)-\varepsilon(z)} \qquad (0\leqq\varepsilon(z)\to 0\ \textit{as}\ z\to\infty).$$

(We consider $\rho(f)/(2\rho(f)+k)=1/2$ when $\rho(f)=+\infty$ and k is finite.)

Examples showing that Theorem B is sharp are given in [4]. Besides this result we can find interesting results on the growth of entire and subharmonic functions along asymptotic paths ([3], [4], [10], [11], [14], [15] and Chapter 8 in [8]).

The purpose of this section is to improve Theorem B and to give a subharmonic analogue, which is an improvement of Theorem 1 in [4].

2-1. Lemmas.

We shall give some lemmas for later use. Let D be an unbounded regular plane domain. We put

$$E(r)=\{\theta\varepsilon[0,\ 2\pi):\ re^{i\theta}\varepsilon D\}$$

and

$$(4)\qquad \theta(r)=\begin{cases} +\infty & \text{if}\quad \{|z|=r\}\subset D \\ \text{the measure of } E(r) & \text{otherwise.}\end{cases}$$

It is clear that there is a positive number a such that $\theta(r)>0$ for all $r\geqq a$.

LEMMA 1. *If*

$$\liminf_{r\to\infty}\ (\log r)^{-1}\pi\int_a^r\frac{dt}{t\theta(t)}=\mu \qquad (1/2\leqq\mu<\infty),$$

then there exists $u>0$ *harmonic in* D *such that for all* $z \varepsilon D$

$$u(z) \geqq |z|^{\mu-\varepsilon(z)} \qquad (0 \leqq \varepsilon(z) \to 0 \text{ as } z \to \infty)$$

([11], *Lemma* 1 *and its correction*).

LEMMA 2. *Let* $g(z)$ *be regular in* D *and continuous on the closure of* D *such that*

$$|g(z)| \leqq 1 \qquad (z \varepsilon \partial D).$$

If there exists one point z_0 *in* D *such that*

$$|g(z_0)| > 1,$$

then

$$\log\log M(r, g) \geqq \pi \int_a^{r/2} \frac{dt}{t\theta(t)} + O(1),$$

where $M(r, g) = \sup\{|g(z)| : (|z|=r) \cap D\}$ ([17], p. 117).

LEMMA 3. *Let* $v(z)$ *be a non-constant subharmonic function in* $|z|<\infty$. *Then there exists a path* Γ *tending to* ∞ *such that*

$$v(z) \longrightarrow +\infty \qquad \text{as} \quad z \longrightarrow \infty \quad \text{on} \quad \Gamma$$

([14], *Theorem* 1).

2-2. Theorem.

We shall give a result generalizing Theorem B.

THEOREM 1. *Let* $f(z)$ *be a transcendental entire function with* $\mu(f)<+\infty$. *Suppose that for some constant* K *the set*

$$\{z : |f(z)| > K\}$$

contains at least N *components* $D_1, \cdots, D_N$, *where* $N \geqq 2$. *Then for each* j $(=1, \cdots, N)$ *there exists a path* Γ_j *tending to* ∞ *in* D_j *such that on* Γ_j

$$(5) \qquad \log|f(z)| > |z|^{\rho(f)/(2\rho(f)+1-N)-\varepsilon_j(z)} \quad (0 \leqq \varepsilon_j(z) \to 0 \text{ as } z \to \infty).$$

Proof. It is clear that $D_1, \cdots, D_N$ are mutually disjoint unbounded regular domains in $|z|<\infty$ and there exists an $a>0$ such that for all $r \geqq a$

$$\{|z|=r\} \cap D_j \neq \phi \qquad (j=1, \cdots, N).$$

We here use $\theta_j(r)$ for D_j instead of $\theta(r)$ defined for D in (4). Then

$$\theta_j(r) > 0 \qquad (r \geqq a)$$

and

$$\sum_{j=1}^{N} \theta_j(r) \leqq 2\pi . \tag{6}$$

From (6) we obtain the inequality

$$\sum_{j=1}^{N} \int_a^r \frac{\theta_j(t)}{t} dt \leqq 2\pi \log \frac{r}{a} \tag{7}$$

and by the Cauchy-Schwarz inequality we have

$$\int_a^r \frac{\theta_j(t)}{t} dt \int_a^r \frac{dt}{t\theta_j(t)} \geqq \left(\int_a^r \frac{dt}{t} \right)^2 = \left(\log \frac{r}{a} \right)^2 . \tag{8}$$

From (7) and (8) we have

$$\sum_{j=1}^{N} \frac{1}{\{\log (r/a)\}^{-1} \pi \int_a^r \frac{dt}{t\theta_j(t)}} \leqq 2 . \tag{9}$$

Applying Lemma 2 to $f(z)/K$ in D_j we obtain the following inequalities:

$$\log \log M(2r, f) \geqq \pi \int_a^r \frac{dt}{t\theta_j(t)} + O(1) \tag{10}$$

from which we have

$$\liminf_{r\to\infty} (\log r)^{-1} \pi \int_a^r \frac{dt}{t\theta_j(t)} \leqq \mu(f) < +\infty . \tag{11}$$

From (9) and (10) we have for each $j\,(=1, \cdots, N)$

$$\frac{N-1}{\{\log (r/a)\}^{-1} \{\log \log M(2r, f) + O(1)\}} + \frac{1}{\{\log (r/a)\}^{-1} \pi \int_a^r \frac{dt}{t\theta_j(t)}} \leqq 2$$

and hence

$$\frac{N-1}{\rho(f)} + \frac{1}{\liminf_{r\to\infty} (\log r)^{-1} \pi \int_a^r \frac{dt}{t\theta_j(t)}} \leqq 2 . \tag{12}$$

From (11) and (12) we have for each $j\,(=1, \cdots, N)$

$$\frac{\rho(f)}{2\rho(f)+1-N} \leqq \liminf_{r\to\infty} (\log r)^{-1} \pi \int_a^r \frac{dt}{t\theta_j(t)} \leqq \mu(f) < +\infty . \tag{13}$$

Since $\rho(f)/(2\rho(f)+1-N) \geqq 1/2$ in (13), there exists a positive harmonic function u_j in D_j such that for all $z \in D_j$

$$u_j(z) \geqq |z|^{\rho(f)/(2\rho(f)+1-N)-\varepsilon_0(z)} \qquad (0 \leqq \varepsilon_0(z) \to 0 \text{ as } z \to \infty) \tag{14}$$

by Lemma 1. We can find z_j in D_j for which

$$|f(z_j)|>K$$

and choose a positive constant δ so small that

$$\log|f(z_j)|>\delta u_j(z_j)+\log K.$$

We then define

$$U_j(z)=\begin{cases}\max\{\log(|f(z)|/K)-\delta u_j(z),\ 0\} & (z\in D_j)\\ 0 & (z\notin D_j).\end{cases}$$

Since $U_j(z_j)>0$ and $U_j(z)=0$ for $z\notin D_j$, it is clear that $U_j(z)$ is a non-constant subharmonic function in $|z|<\infty$. Hence by Lemma 3 there exists a path Γ_j tending to ∞ such that

$$U_j(z)\longrightarrow +\infty \quad \text{as} \quad z\longrightarrow\infty \quad \text{on} \quad \Gamma_j.$$

We may assume without loss of generality that

$$U_j(z)>0 \quad \text{on} \quad \Gamma_j$$

so that Γ_j lies in D_j and on Γ_j

$$U_j(z)=\log|f(z)|-\delta u_j(z)-\log K>0.$$

Thus we have by (14)

$$\log|f(z)|>|z|^{\rho(f)/(2\rho(f)+1-N)-\varepsilon_j(z)} \qquad (0\leqq\varepsilon_j(z)\to 0 \text{ as } z\to\infty) \quad \text{on} \quad \Gamma_j.$$

Remark 1. By a well-known Ahlfors' theorem (see [6], p. 255), it is known that $N=1$ when $\mu(f)<1$ and $N\leqq 2\mu(f)$ when $1\leqq\mu(f)<+\infty$.

From (9) and (10) we obtain for $r\geqq a$

$$N\leqq 2\{\log\log M(2r,\ f)+O(1)\}/\log(r/a),$$

which reduces to $N\leqq 2\mu(f)$ when $N\geqq 2$.

Example 1. Let

$$f(z)=\cos hz^{N/2} \qquad (N=2,\ 3,\ \cdots).$$

Then,

$$M(r,\ f)=\frac{\exp(r^{N/2})+\exp(-r^{N/2})}{2} \quad \text{and} \quad \rho(f)=\mu(f)=N/2.$$

It is easily seen that for $k=0,\ 1,\ \cdots,\ N-1$ and for $0\leqq t<+\infty$

$$|f(te^{(2k+1)\pi i/N})|\leqq 1$$

and

$$\log|f(te^{2k\pi i/N})|>t^{N/2-\varepsilon(t)} \qquad (0<\varepsilon(t)\to 0 \quad \text{as} \quad t\to+\infty).$$

Remark 2. This example shows that Theorem 1 is sharp.

Example 2. Let $f(z)$ be an entire function of finite lower order with $N(\geqq 2)$ distinct finite asymptotic values. Then, for a sufficiently large K the set

$$\{z: |f(z)|>K\}$$

has at least N components.

We can find a concrete example of $f(z)$ with N distinct finite asymptotic values in [8], p. 562.

2-3. Subharmonic analogue.

Let $v(z)$ be a non-constant subharmonic function in $|z|<\infty$. Put

$$B(r, v)=\sup_{|z|=r} v(z),$$

$$\rho=\limsup_{r\to\infty} \log B(r, v)/\log r \qquad \text{(the order of } v),$$

$$\mu=\liminf_{r\to\infty} \log B(r, v)/\log r \qquad \text{(the lower order of } v).$$

It is said that $v(z)$ has at least N tracts in $|z|<\infty$ if and only if

$$\{z: v(z)>K\}$$

has at least N components for all sufficiently large K, where N is a positive integer ([7], [8]). When $N\geqq 2$, the following result is given ([8], p. 593).

THEOREM C. *Suppose that $v(z)$ has at least $N(\geqq 2)$ tracts in the finite plane. Then there exist sectionally polygonal paths $\gamma_1, \cdots, \gamma_N$ from 0 to ∞ such that*

1) $\gamma_j\cap\gamma_k=\{0, \infty\}$ $\quad (j\neq k)$,
2) *γ_j and γ_{j+1} bound a domain D_j and $D_j\cap\gamma_k=\phi$ $(\gamma_{N+1}=\gamma_1)$,*
3) *$v(z)$ is bounded on the γ_j and not bounded above in the D_j.*

Put

$$B_j(r, v)=\sup\{v(z): (|z|=r)\cap D_j\}.$$

By 3) in Theorem C, there exists $z_j\in D_j$ for each j such that

$$v(z_j)>0$$

and for all sufficiently large r

$$B_j(r, v)>0.$$

Further there exists a positive number M such that

$$V(z)=v(z)-M$$

is negative on $\gamma_1\cup\cdots\cup\gamma_N$.

LEMMA 4. $\log B_j(r, v)\geqq \pi\int_1^{r/2}\frac{dt}{t\theta_j(t)}+O(1)$

where we use $\theta_j(r)$ *for* D_j *instead of* $\theta(r)$ *defined for* D *in* (4).

We can prove this lemma by applying Theorem 8.3 ([8], p. 548) to $V_{j(z)}=\max\{V_{(z)}, 0\}$ if $z\in D_j$, $=0$ otherwise.

THEOREM 2. *Suppose that* $v(z)$ *has at least* $N(\geqq 2)$ *tracts in the finite plane and* $\mu<+\infty$. *Then there exists a path* Γ_j *tending to* ∞ *in* D_j *such that*

$$v(z)>|z|^{\rho/(2\rho+1-N)-\varepsilon_j(z)} \qquad (0\leqq\varepsilon_j(z)\to 0 \text{ as } z\to\infty)$$

on $\Gamma_j (j=1, \cdots, N)$.

We can prove this theorem as in the case of Theorem 1 using Lemma 4 instead of Lemma 2. We note that $N\leqq 2\mu$ as in Remark 1.

3. Application to the oscillation theory of $w''+Aw=0$.

We shall first give some lemmas for later use. We use the same notation as in the section 1.

LEMMA 5. *If* $\rho(E)<+\infty$, *for a given* $\varepsilon>0$ *there exists a positive number* $d=d(\varepsilon)$ *such that*

$$|(E'/E)^2(re^{i\theta})-2(E''/E)(re^{i\theta})|\leqq r^d$$

for all $r\geqq r_0>1$ *and all* $\theta\notin J(r)$, *where the angular measure of* $J(r)$, $m(J(r))\leqq\varepsilon\pi$ ([12], Lemma 1).

LEMMA 6. *If* $\lambda(E)<\rho(E)$, *then*

$$\mu(E)=\rho(E)=\mu(A)=\rho(A)$$

and these numbers are equal to an integer or $+\infty$.

Proof. From (2) we easily have

$$2T(r, E)=2N(r, 1/E)+T(r, A)+S(r, E). \tag{15}$$

Set

$$E(z)=\Pi(z)e^{P(z)},$$

where $\Pi(z)$ is the Weierstrass product of the zeros of E and $P(z)$ is an entire function. Then, it is known that $\rho(\Pi)=\lambda(E)$ (see [5]).

a) The case $\rho(E)=+\infty$. In this case, $P(z)$ is transcendental and it is easy to see that $\mu(E)=+\infty$. Let α be any number such that $\lambda(E)<\alpha<+\infty$. Then from (15) we have

$$2T_\alpha(r, E)=2N_\alpha(r, 1/E)+T_\alpha(r, A)+S_\alpha(r, E), \tag{16}$$

where

$$T_\alpha(r, E)=\int_1^r \frac{T(t, E)}{t^{1+\alpha}}dt \quad \text{is of lower order } +\infty,$$

$$N_\alpha(r, 1/E)=\int_1^r \frac{N(t, 1/E)}{t^{1+\alpha}}dt \quad \text{is bounded},$$

$$T_\alpha(r, A)=\int_1^r \frac{T(t, A)}{t^{1+\alpha}}dt \leqq T(r, A)/\alpha$$

and

$$S_\alpha(r, E)=\int_1^r \frac{S(t, E)}{t^{1+\alpha}}dt=o(T_\alpha(r, E)) \qquad (r\to\infty)$$

(see [16], Proposition 1 and Lemma 1), so that

$$+\infty=\liminf_{r\to\infty}\frac{\log T_\alpha(r, E)}{\log r}=\liminf_{r\to\infty}\frac{\log T_\alpha(r, A)}{\log r}\leqq \mu(A).$$

We have $\mu(A)=\rho(A)=+\infty$.

b) The case $\rho(E)<+\infty$. In this case, $P(z)$ must be a polynomial and it is easy to see that

(17) $$\mu(E)=\rho(E)=\text{the degree of } P(z)$$

since $\lambda(E)<\rho(E)$. From (15) and (17) we have

$$\mu(A)=\rho(A)=\mu(E)=\rho(E)=\text{an integer}.$$

THEOREM 3. *Suppose that $\mu(A)<+\infty$ and for a positive constant K not smaller than $1/2$ the set*

$$\{z:\ |A(z)|>K\}$$

has at least N components. Then, either $\rho(E)=+\infty$ or

$$\frac{N}{\mu(A)}+\frac{1}{\rho(E)}\leqq 2.$$

Proof. Suppose that $\rho(E)<+\infty$. Let D_0 be a component of the set

$$\{z:\ |E(z)|>|c|\},$$

which is a non-empty unbounded set since E is transcendental by (2).

The set $\{z:\ |A(z)|>K\}$ has at least N components, and since $A(z)$ is transcendental and Theorem 1 holds for $N\geqq 2$, for any positive integer p and for $K_1=\max\{K, M(1, A)\}$ the set

$$\{z:\ \log|A(z)|-p\log|z|-\log K_1>0\}$$

has at least N unbounded components. Let $D_1, \cdots, D_N$ be those N unbounded components. For $j=0, 1, \cdots, N$, put

$$E_j(r)=\{\theta\in[0,\ 2\pi):\ re^{i\theta}\in D_j\}$$

and

$$\theta_j(r)=\begin{cases} +\infty & \text{if}\quad \{|z|=r\}\subset D_j \\ \text{the measure of } E_j(r) & \text{otherwise.}\end{cases}$$

Then there is a positive number a such that $\theta_j(r)>0$ for all $r\geqq a$ and for all j. By Lemma 2 we have

$$\log\log M(r,\ E)\geqq\pi\int_a^{r/2}\frac{dt}{t\theta_0(t)}+O(1) \tag{18}$$

and

$$\log\{\log M(r,\ A)-p\log r-\log K_1\}\geqq\pi\int_a^{r/2}\frac{dt}{t\theta_j(t)}O(1) \tag{19}$$

for $j=1,\ \cdots,\ N$.

For any fixed positive number $\varepsilon<1$, let p be a positive integer such that $p>d$, where d is the constant given in Lemma 5. We define for $j=0,\ 1,\ \cdots,\ N$

$$l_j(t)=\begin{cases} 2\pi & \text{if}\quad \theta_j(t)=+\infty \\ \theta_j(t) & \text{otherwise.}\end{cases}$$

Then applying Lemma 5 to (2) we obtain the inequality

$$\sum_{j=0}^{N} l_j(t)\leqq(2+\varepsilon)\pi \tag{20}$$

for all $r\geqq b=\max(a,\ r_0)$ from which we have

$$\sum_{j=0}^{N}\int_b^r\frac{l_j(t)}{t}dt\leqq(2+\varepsilon)\pi\log(r/b). \tag{21}$$

By the Cauchy-Schwarz inequality

$$\int_b^r\frac{l_j(t)}{t}dt\int_b^r\frac{dt}{tl_j(t)}\geqq\left(\int_b^r\frac{dt}{t}\right)^2=\left(\log\frac{r}{b}\right)^2. \tag{22}$$

From (21) and (22) we obtain the inequality

$$\sum_{j=0}^{N}\frac{\log(r/b)}{\pi\displaystyle\int_b^r\frac{dt}{tl_j(t)}}\leqq 2+\varepsilon. \tag{23}$$

Define

$$B_0=\{r:\ \theta_0(r)=+\infty\}\ .$$

Then, B_0 is a sum of intervals. Let

$$\chi_0(r)=\begin{cases} 1 & \text{if } r \text{ belongs to } B_0 \\ 0 & \text{otherwise.}\end{cases}$$

If r belongs to B_0 and $r \geqq b$, we have

$$\theta_j(r)=l_j(r) \qquad \text{for} \quad j=1, \cdots, N$$

and

$$\theta_1(r)+\cdots+\theta_N(r) \leqq \varepsilon\pi$$

from (20). Thus, if we set

$$F_j=\{r\,;\ \theta_j(r)\leqq \varepsilon\pi\}\,,$$

then

$$B_0 \subset \bigcup_{j=1}^{N} F_j\,. \tag{24}$$

Define

$$\phi_j(r)=\begin{cases} 1 & \text{if } r \text{ belongs to } F_j \\ 0 & \text{otherwise} \end{cases}$$

and put

$$M(r)=\log M(r,\ A)-p\log r-\log K_1$$

We then have from (24)

$$\int_b^r \frac{\chi_0(t)}{t}\,dt \leqq \sum_{j=1}^{N}\int_b^r \frac{\phi_j(t)}{t}\,dt \leqq N\varepsilon \log M(2r)+O(1) \tag{25}$$

since $\varepsilon^{-1}\phi_j(t)\leqq \pi/\theta_j(t)$ and so

$$\varepsilon^{-1}\int_b^r \frac{\phi_j(t)}{t}\,dt \leqq \pi\int_b^r \frac{dt}{t\theta_j(t)} \leqq \log M(2r)+O(1)$$

by (19).

(i) The case $N \geqq 2$. In this case it is clear that for $j=1, \cdots, N$

$$0<\theta_j(r)<2\pi \quad \text{and} \quad \theta_j(r)=l_j(r) \qquad (r\geqq b)\,.$$

Since

$$\pi\int_b^r \frac{dt}{t\theta_0(t)}=\pi\int_b^r \frac{dt}{tl_0(t)}-\frac{1}{2}\int_b^r \frac{\chi_0(t)}{t}\,dt\,, \tag{26}$$

from (18), (19), (23) and (25) we obtain for $r \geqq b$

$$\frac{N\log(r/b)}{\log M(2r)+O(1)}+\frac{\log(r/b)}{\log\log M(2r,\ E)+(N\varepsilon/2)\log M(2r)+O(1)} \leqq 2+\varepsilon\,. \tag{27}$$

Let $\{r_n\}$ be a sequence tending to $+\infty$ such that

$$\lim_{n\to\infty}\frac{\log\log M(2r_n,\ A)}{\log 2r_n}=\mu(A)\,.$$

Put $r=r_n$ in (27) and let n tend to $+\infty$. We then obtain

$$\frac{N}{\mu(A)}+\frac{1}{\rho(E)+N\varepsilon\mu(A)/2}\leqq 2+\varepsilon .$$

Tending $\varepsilon\to 0$, we have

$$\frac{N}{\mu(A)}+\frac{1}{\rho(E)}\leqq 2 . \tag{28}$$

(ii) The case $N=1$. Let

$$B_1=\{r:\ \theta_1(r)=+\infty\} .$$

Then, B_1 is a sum of intervals. Define

$$\chi_1(r)=\begin{cases} 1 & \text{if } r \text{ belongs to } B_1 \\ 0 & \text{otherwise}. \end{cases}$$

If r belongs to B_1 and $r\geqq b$, we have

$$\theta_0(r)\leqq\varepsilon\pi$$

by (20). Put

$$F_0=\{r:\ \theta_0(r)\leqq\varepsilon\pi\}$$

and

$$\psi_0(r)=\begin{cases} 1 & \text{if } r \text{ belongs to } F_0 \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$\int_b^r\frac{\chi_1(t)}{t}dt\leqq\int_b^r\frac{\psi_0(t)}{t}dt\leqq\varepsilon\log\log M(2r,\ E)+O(1) \tag{29}$$

since $B_1\subset F_0$, $\varepsilon^{-1}\psi_0(t)\leqq\pi/\theta_0(t)$ and so

$$\varepsilon^{-1}\int_b^r\frac{\psi_0(t)}{t}dt\leqq\pi\int_b^r\frac{dt}{t\theta_0(t)}\leqq\log\log M(2r,\ E)+O(1)$$

by (18). Since

$$\pi\int_b^r\frac{dt}{t\theta_1(t)}=\pi\int_b^r\frac{dt}{tl_1(t)}-\frac{1}{2}\int_b^r\frac{\chi_1(t)}{t}dt ,$$

from (18), (19), (23) and (25) for $N=1$, (26) and (29), we have

$$\frac{\log(r/b)}{\log M(2r)+\varepsilon\log\log M(2r,\ E)+O(1)} \tag{30}$$
$$+\frac{\log(r/b)}{\log\log M(2r,\ E)+(\varepsilon/2)\log M(2r)+O(1)}\leqq 2+\varepsilon .$$

Then as in the case of $N\geqq 2$ where we obtained (28) from (27), we obtain the inequality

$$\frac{1}{\mu(A)}+\frac{1}{\rho(E)}\leqq 2$$

from (30).

CORLLARY. *Under the same assumption as in Theorem* 3,

1) *If* $\mu(A)<\rho(A)=+\infty$, *then* $\lambda(E)=+\infty$.

2) *When* $\rho(A)<+\infty$, *if* $\mu(A)<\rho(A)$ *or if* A *is of regular growth and* $\rho(A)$ *is not equal to an integer, then either* $\lambda(E)=+\infty$ *or*

$$\frac{N}{\mu(A)}+\frac{1}{\lambda(E)}\leqq 2\,. \tag{31}$$

3) *If* $\mu(A)\leqq 1/2$ *or if* $\mu(A)=N/2$ *in case of* $N\geqq 2$, *then*

$$\lambda(E)=+\infty\,.$$

Proof. 1) We easily have

$$\rho(A)\leqq\rho(E)$$

from (15) and since $\mu(A)<\rho(A)=+\infty$ we have

$$\lambda(E)=\rho(E)=+\infty$$

by Lemma 6.

2) In this case, we have

$$\lambda(E)=\rho(E)$$

by Lemma 6. We obtain (31) from Theorem 3.

3) Noting the fact that

"$N=1$ if $\mu(A)<1$ and $N\leqq 2\mu(A)$ if $1\leqq\mu(A)<+\infty$"

(see Remark 1), we easily obtain $\lambda(E)=+\infty$ when $\mu(A)\leqq 1/2$ or $\mu(A)=N/2$ in case N is odd from 2) of this corollary.

When N is even and positive, $\mu(A)=N/2$ implies $\rho(E)=+\infty$ by Theorem 3. If $\lambda(E)<+\infty$, then $\mu(A)=\rho(E)=+\infty$ by Lemma 6. This is a contradiction. $\lambda(E)$ must be equal to $+\infty$.

Remark 3. The functions of Examples 1 and 2 in the section 2 satisfy the conditions of Theorem 3 for $N\geqq 2$.

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