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THE LOGARITHMIC DERIVATIVE AND A HOMOGENEOUS DIFFERENTIAL POLYNOMIAL OF A MEROMORPHIC FUNCTION

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1. Introduction

In this note, by a meromorphic function we mean a function meromorphic in the complex plane $\boldsymbol{C}$. We shall here assume that the reader is familiar with the standard notation and terminology of value distribution theory (see for example, Hayman [1] or [3]). For a meromorphic function $g(z)$, which does not vanish identically, we can consider the logarithmic derivative $g'(z)/g(z)$. It plays an important role in Nevanlinna's theory of meromorphic functions. The following occupies the main part.

LEMMA. *Let $g(z)$ be a meromorphic function. If $g(z)$ is transcendental, we have*

$$m\left(r, \frac{g'}{g}\right)=O(\log^+ T(r, g)+\log r) \tag{1.1}$$

as $r\to\infty$ through all values if $g(z)$ has finite order and as $r\to\infty$ outside a set of r of finite linear measure otherwise. If $g(z)$ is a rational function and not identically equal to zero,

$$m\left(r, \frac{g'}{g}\right)=o(1) \tag{1.2}$$

as $r\to\infty$ through all values.

For the sake of simplicity, we shall use the symbol "n. e. (nearly everywhere)" instead of tediously saying that possibly outside a set of r of finite linear measure.

W. K. Hayman pointed out the necessity of treating a homogeneous differential polynomial $g''g-2g'^2$ of an entire function $g(z)$ in his famous book [1; § 3.6, p. 77]. Concerning this proposal E. Mues [4] studied an influence of the zeros of $g''g-ag'^2$ with a complex number a on the entire function $g(z)$ itself.

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He proved that if $g''g-ag'^2$ has no zero, then $g(z)=\exp(\alpha z+\beta)$ are the only transcendental functions with this property if $a\neq 1$. This result settled a question of Hayman made in [1]. Our purpose of this note is to give an estimate of the zeros of $g(z)$ by those of the homogeneous differential polynomial $g''g-ag'^2$. With the equation

$$T\left(r, \frac{g'}{g}\right)=m\left(r, \frac{g'}{g}\right)+\bar{N}(r, 0, g)+\bar{N}(r, g)$$

for a meromorphic function $g(z)\not\equiv 0$ and the above lemma, it gains our purpose to estimate the characteristic function of a logarithmic derivative by the counting function with respect to the zeros of a homogeneous differential polynomial. Our method to obtain such a result is based on arriving at a homogeneous linear equation in g' and g after a linearization of $g''g-ag'^2$. The term "linearization" was introduced by M. Ozawa [5], and Mues [4], Ozawa, G. Frank and others have made frequent use of this method. We now represent the differential polynomial $g''g-ag'^2$ by means of a Wronskian determinant

$$W(f_1, f_2)=f_1f_2'-f_1'f_2 .$$

We have indeed for a constant $a\in \boldsymbol{C}$

$$W((a-1)zg'(z)+g(z),\ g'(z))=g''(z)g(z)-ag'(z)^2 , \tag{1.3}$$

which we denote by $W_a(z)$. That is a reason why we can treat this homogeneous differential polynomial $W_a(z)$.

We shall naturally consider only the case where $W_a(z)$ does not vanish identically. Because if $W_a(z)\equiv 0$, two functions $(a-1)zg'(z)+g(z)$ and $g'(z)$ are linearly dependent over $\boldsymbol{C}$. Then there exist two constants C_1 and C_2, at least one of which is different from zero, such that an equation

$$(C_1(a-1)z+C_2)g'(z)+C_1g(z)=0 \tag{1.4}$$

holds. If $C_1(a-1)z+C_2\equiv 0$, we have $C_2=0$ and $\alpha=1$. By (1.4) it thus follows $g(z)\equiv 0$, which is a contradiction. Hence unless $g(z)\equiv 0$, it is equal to $\exp(-C_1z/C_2+\text{const.})$ if $a=1$ (and thus $C_2\neq 0$), and $g(z)=C_3(C_1(a-1)z+C_2)^{-1/(a-1)}$, $C_3\in -\boldsymbol{C}-\{0\}$, if $a\neq 1$. If $C_1\neq 0$ in the latter case, the exponent $-1/(a-1)$ must be an integer m $(\neq 0)$, say. The following is a summary of this trivial observation:

The meromorphic functions $g(z)$ with the property $W_a(z)\equiv 0$ are reduced to the next three: for $\alpha(\neq 0)$, $\beta\in\boldsymbol{C}$,

1°. *$g(z)\equiv\beta$, when a is any complex number;*
2°. *$g(z)=\exp(\alpha z+\beta)$, when $a=1$;*
3°. *$g(z)=(\alpha z+\beta)^m$, when $a=(m-1)/m$ with a non-zero integer m.*

2. Results

We shall prove the following theorem which gives a desired estimate of the logarithmic derivative $g'(z)/g(z)$.

THEOREM. *Let $g(z)$ be a non-constant meromorphic function and define a homogeneous differential polynomial $W_a(z)$ in $g(z)$ for a complex number a by* (1.3). *If $W_a(z)$ does not vanish identically, then an inequality*

$$T\left(r, \frac{g'}{g}\right) \leqq A_a m\left(r, \frac{g'}{g}\right) + B_a m\left(r, \frac{W_a'}{W_a}\right) + C_a \{\bar{N}(r, 0, W_a) + \bar{N}(r, g)\} + U_a(r) \tag{2.1}$$

holds as $r \to \infty$, except for two cases (i), (ii) *below. Here the constants A_a, B_a, C_a depend only on the number a and satisfy*

$$0 \leqq A_a \leqq \begin{cases} 4, & \text{if } a \neq 1,\ 1/2, \\ 2, & \text{if } a=1, \\ 1, & \text{if } a=1/2, \end{cases} \qquad 0 \leqq B_a \leqq \begin{cases} 5, & \text{if } a \neq 1,\ 1/2,\ 0, \\ 2, & \text{if } a=1, \\ 4, & \text{if } a=1/2, \\ 1, & \text{if } a=0, \end{cases} \qquad 0 \leqq C_a \leqq 5$$

for any a, and also $U_a(r)$ is a real-valued function on $[0, \infty)$ such that if we fix the number a, then it satisfies

$$U_a(r) = \begin{cases} O\left[\log^+ T\left(r, \frac{g'}{g}\right) + \log^+ m\left(r, \frac{W_a'}{W_a}\right) + \log^+ \{\bar{N}(r, 0, W_a) + \bar{N}(r, g)\} + \log r\right], & \\ \qquad\qquad\qquad\qquad \text{if } a \neq 1/2,\ 0, & \\ O\left[\log^+ T\left(r, \frac{g'}{g}\right) + \log^+ \{\bar{N}(r, 0, W_0) + \bar{N}(r, g)\} + \log r\right], & \text{if } a=0, \\ O(1), \quad \text{if } a=1/2, & \end{cases}$$

as $r \to \infty$ possibly outside a set E_a of r of finite linear measure depending on the number a.

(i) *When $a=1/2$, $g(z)=\alpha z^2+\beta z+\gamma$, where α, β, γ are complex constants with $\beta^2-4\alpha\gamma \neq 0$; and*

(ii) *when $a=1$, $g(z)=C_1 e^{\lambda_1 z}+C_2 e^{\lambda_2 z}$, where λ_1, λ_2, C_1, C_2 are complex constants with $\lambda_1 \neq \lambda_2$ and $C_1 \cdot C_2 \neq 0$;*
are the exceptions as mentioned above.

Remark. It is easy to see that $g(z)$ as in the cases (i) and (ii) indeed fails to satisfy the inequality (2.1). In fact:

(i). $g(z)=\alpha z^2+\beta z+\gamma$ gives $W_{1/2}(z) \equiv -(1/2)(\beta^2-4\alpha\gamma)$ $(\neq 0)$. Then we deduce

$$m(r, g'/g)=o(1),$$

$$m(r, W_{1/2}'/W_{1/2})=\bar{N}(r, 0, W_{1/2})=\bar{N}(r, g) \equiv 0,$$

and

$$U_{1/2}(r)=O(1)$$

as $r\to\infty$, while

$$T(r, g'/g)=m(r, g'/g)+N(r, g'/g)=\varepsilon \log r+o(1)$$

as $r\to\infty$, with $\varepsilon=1$ if $\alpha=0$ and $\varepsilon=2$ if $\alpha\neq 0$.

(ii). In this case, $W_1(z)=(\lambda_1-\lambda_2)^2 C_1 C_2 e^{(\lambda_1+\lambda_2)z}$, and that $g(z)$ is an entire function of order 1. Thus we have

$$m(r, g'/g)=O(\log r),$$

$$m(r, W_1'/W_1)=O(\log r),$$

$$\overline{N}(r, 0, W_1)=\overline{N}(r, g)\equiv 0,$$

and

$$U_1(r)=O(\log r) \qquad \text{as } r\to\infty .$$

Using an expression $g(z)=C_1 e^{\lambda_2 z}\{e^{(\lambda_1-\lambda_2)z}+C_2/C_1\}$, we are led to

$$T(r, g'/g)=\overline{N}(r, 0, g)+O(\log r)=\frac{|\lambda_1-\lambda_2|}{\pi}r+O(\log r)$$

as $r\to\infty$ (see for example, Hayman [1: p. 7]), however.

Our way to prove this theorem also applies to the following

COROLLARY. *Besides the hypothesis of our theorem we assume that $g(z)$ is an entire function and that as $r\to\infty$, n.e.,*

$$m(r, W_a)=o\{m(r, g)\}. \tag{2.2}$$

Then $W_a(z)$ must be a constant ($\neq 0$) and $g(z)$ is at least one of the following;

(i) *when $a=1/2$, $g(z)=\alpha z^2+\beta z+\gamma$, where $\alpha, \beta, \gamma\in \boldsymbol{C}$ with $\beta^2-4\alpha\gamma\neq 0$;*

(ii) *when $a=1$, $g(z)=C_1 e^{\lambda z}+C_2 e^{-\lambda z}$, where $\lambda, C_1, C_2\in \boldsymbol{C}-\{0\}$; and*

(iii) *when $a\neq 0, 1/2$, $g(z)=\alpha z+\beta$, where $\alpha(\neq 0), \beta\in \boldsymbol{C}$.*

Remarks 1°. If we further suppose that $g(z)$ is of finite order ρ and $W_a(z)$ has the order λ satisfying $\lambda<\rho$, the case (ii) is the only possible one and then $\rho=1$ and $\lambda=0$ (in particular, $W_1(z)$ is a constant). We may regard it as a partial answer to a problem of A. Edrei (see Hayman [2: Problem 2.25]) when $f=g'$ there.

2°. Replacing the condition (2.2) by

$$T(r, W_a)=o\{T(r, g)\} \qquad \text{as } r\to\infty, \text{ n.e.},$$

we can prove this result for meromorphic functions with the property such that

$$N(r, g)=o\{T(r, g)\} \qquad \text{as } r\to\infty, \text{ n.e.}.$$

3. Proof of Theorem: A Preparation

Because of $W_a(z)\not\equiv 0$, we can consider a second order linear ordinary differential equation

$$w''+G_a(z)w'+H_a(z)w=0 \tag{3.1}$$

with the coefficients

$$G_a(z)=-\frac{W_a'(z)}{W_a(z)}, \tag{3.2}$$

and

$$H_a(z)=\frac{W(\{(a-1)zg'(z)+g(z)\}',\ g''(z))}{W_a(z)}. \tag{3.3}$$

Since (3.1) is written as $(W[w, (a-1)zg'+g, g']/W_a)=0$, two functions $(a-1)zg'(z)+g(z)$ and $g'(z)$ form a fundamental system of this equation. Firstly for $w=(a-1)zg'(z)+g(z)$ Equation (3.1) gives

$$(a-1)z\{g'''(z)+G_a(z)g''(z)+H_a(z)g'(z)\}$$
$$+(2a-1)g''(z)+aG_a(z)g'(z)+H_a(z)g(z)=0.$$

Also for $w=g'(z)$,

$$g'''(z)+G_a(z)g''(z)+H_a(z)g'(z)=0. \tag{3.4}$$

Together with (3.4) the first equation is reduced to

$$(2a-1)g''(z)+aG_a(z)g'(z)+H_a(z)g(z)=0. \tag{3.5}$$

The two, (3.4) and (3.5), are called a linearization of the differential polynomial $W_a(z)$. Eliminating $g'''(z)$ and $g''(z)$ from them we shall obtain an equation in $g'(z)$ and $g(z)$. To do this, we differentiate both sides of (3.5) with respect to z and get

$$(2a-1)g'''(z)+aG_a(z)g''(z)+(aG_a'(z)+H_a(z))g'(z)+H_a'(z)g(z)=0. \tag{3.6}$$

Using (3.5) and (3.6) we reduce Equation (3.4) to

$$\{a(2a-1)G_a'(z)+a(a-1)G_a(z)^2-2(a-1)(2a-1)H_a(z)\}\,g'(z) \tag{3.7}$$
$$=-\{(2a-1)H_a'(z)+(a-1)G_a(z)H_a(z)\}\,g(z),$$

which is the homogeneous linear equation in $g'(z)$ and $g(z)$ as desired. For the sake of simplicity, we denote the coefficients by $\phi_a(z)$ and $\psi_a(z)$, i.e.,

$$\phi_a(z):=a(2a-1)G_a'(z)+a(a-1)G_a(z)^2-2(a-1)(2a-1)H_a(z), \tag{3.8}$$

$$(3.9)\qquad \psi_a(z) := -(2a-1)H_a'(z)-(a-1)G_a(z)H_a(z).$$

Then the above equation (3.7) is of the form

$$(3.10)\qquad \phi_a\cdot g'=\psi_a\cdot g.$$

Now it makes all the difference in methods whether or not $\phi_a\equiv 0$.

4. Proof of Theorem: Case I where $\phi_a\not\equiv 0$

Since $g\not\equiv 0$, Equation (3.10) gives an expression of the logarithmic derivative, i. e.,

$$(4.1)\qquad \frac{g'}{g}=\frac{\psi_a}{\phi_a}.$$

We shall now distinguish the cases about the value of a to study the value distribution of meromorphic functions G_a, H_a, ϕ_a, and ψ_a.

Subcase i. *a is different from* 0, 1/2, 1.

Using Equation (3.5) we have

$$(4.2)\qquad H_a=-(2a-1)\frac{g''}{g}-aG_a\frac{g'}{g}$$

$$=-(2a-1)\left\{\left(\frac{g'}{g}\right)'+\left(\frac{g'}{g}\right)^2\right\}-aG_a\frac{g'}{g}.$$

Thus we apply Lemma to (4.2) and obtain

$$(4.3)\qquad m(r, G_a)=m\left(r, \frac{W_a'}{W_a}\right),$$

and

$$(4.4)\qquad m(r, H_a)=m\left(r, \frac{g'}{g}\left\{-(2a-1)\frac{(g'/g)'}{(g'/g)}-(2a-1)\frac{g'}{g}-aG_a\right\}\right)$$

$$\leqq 2m\left(r, \frac{g'}{g}\right)+m(r, G_a)+m\left(r, \frac{(g'/g)'}{(g'/g)}\right)+O(1)$$

$$\leqq 2m\left(r, \frac{g'}{g}\right)+m\left(r, \frac{W_a'}{W_a}\right)+O\left\{\log^+T\left(r, \frac{g'}{g}\right)+\log r\right\},$$

as $r\to\infty$, n. e.. Since the poles of W_a occur possibly at those of g, we have

$$\bar{N}(r, W_a)\leqq\bar{N}(r, g),$$

and thus

$$(4.5)\qquad N(r, G_a)=N\left(r, \frac{W_a'}{W_a}\right)=\bar{N}(r, W_a)+\bar{N}(r, 0, W_a)$$

$$\leqq\bar{N}(r, 0, W_a)+\bar{N}(r, g).$$

Whenever H_a as well as G_a has a pole, W_a has a zero or g has a pole. Its multiplicity is at most two as we see from (4.2). Thus

$$N(r, H_a) \leqq 2\{\bar{N}(r, 0, W_a)+\bar{N}(r, g)\}. \tag{4.6}$$

If G_a does not vanish identically, an application of Lemma to (3.8) implies

$$\begin{aligned} m(r, \phi_a) &\leqq m\left(r, aG_a\left\{(2a-1)\frac{G_a'}{G_a}+(a-1)G_a\right\}\right)+m(r, H_a)+O(1) \\ &\leqq 2m(r, G_a)+m\left(r, \frac{G_a'}{G_a}\right)+m(r, H_a)+O(1) \\ &\leqq 2m(r, G_a)+m(r, H_a)+O\{\log^+T(r, G_a)+\log r\} \end{aligned} \tag{4.7}$$

as $r\to\infty$, n.e.. This is also valid when $G_a\equiv 0$ so that $\phi_a=-2(a-1)(2a-1)H_a$. We see that H_a is not constantly equal to zero. In fact otherwise, $\psi_a\equiv 0$ by (3.9) and therefore $g'(z)\equiv 0$ by (4.1), which is a contradiction. It follows from (3.9)

$$\begin{aligned} m(r, \psi_a) &= m\left(r, -H_a\left\{(2a-1)\frac{H_a'}{H_a}+(a-1)G_a\right\}\right) \\ &\leqq m(r, H_a)+m(r, G_a)+O\{\log^+T(r, H_a)+\log r\} \end{aligned} \tag{4.8}$$

as $r\to\infty$, n.e.. We can observe the poles of ϕ_a and ψ_a similarly to those of G_a and H_a as in (4.5) and (4.6), respectively, i.e.,

$$N(r, \phi_a) \leqq 2\{\bar{N}(r, 0, W_a)+\bar{N}(r, g)\}, \tag{4.9}$$

and

$$N(r, \psi_a) \leqq 3\{\bar{N}(r, 0, W_a)+\bar{N}(r, g)\}. \tag{4.10}$$

From the estimates (4.7), (4.8), (4.9) and (4.10) we arrive at

$$\begin{aligned} T(r, \phi_a) \leqq\ & 2m(r, G_a)+m(r, H_a)+2\{\bar{N}(r, 0, W_a)+\bar{N}(r, g)\} \\ & +O\{\log^+T(r, G_a)+\log r\} \end{aligned}$$

and

$$\begin{aligned} T(r, \psi_a) \leqq\ & m(r, G_a)+m(r, H_a)+3\{\bar{N}(r, 0, W_a)+\bar{N}(r, g)\} \\ & +O\{\log^+T(r, H_a)+\log r\} \end{aligned}$$

as $r\to\infty$, n.e.. Also from (4.3), (4.4), (4.5) and (4.6),

$$T(r, G_a) \leqq m(r, W_a'/W_a)+\bar{N}(r, 0, W_a)+\bar{N}(r, g),$$

and

$$\begin{aligned} T(r, H_a) \leqq\ & 2m(r, g'/g)+m(r, W_a'/W_a)+2\{\bar{N}(r, 0, W_a)+\bar{N}(r, g)\} \\ & +O\{\log^+T(r, g'/g)+\log r\} \end{aligned}$$

as $r\to\infty$, n.e.. Combining them with the former two, we obtain the following two estimates:

$$T(r,\phi_a)\leqq 2m\Big(r,\frac{g'}{g}\Big)+3m\Big(r,\frac{W_a'}{W_a}\Big)+2\{\bar N(r,0,W_a)+\bar N(r,g)\}+O\Big[\log^+T\Big(r,\frac{g'}{g}\Big)+\log^+m\Big(r,\frac{W_a'}{W_a}\Big)+\log^+\{\bar N(r,0,W_a)+\bar N(r,g)\}+\log r\Big] \tag{4.11}$$

and

$$T(r,\psi_a)\leqq 2m\Big(r,\frac{g'}{g}\Big)+2m\Big(r,\frac{W_a'}{W_a}\Big)+3\{\bar N(r,0,W_a)+\bar N(r,g)\}+O\Big[\log^+T\Big(r,\frac{g'}{g}\Big)+\log^+m\Big(r,\frac{W_a'}{W_a}\Big)+\log^+\{\bar N(r,0,W_a)+\bar N(r,g)\}+\log r\Big] \tag{4.12}$$

as $r\to\infty$, n.e.. By virtue of (4.1) the characteristic function of g'/g is now given by

$$T\Big(r,\frac{g'}{g}\Big)\leqq T(r,\phi_a)+T(r,\psi_a)+O(1).$$

Hence from (4.11) and (4.12) we conclude that

$$T\Big(r,\frac{g'}{g}\Big)\leqq 4m\Big(r,\frac{g'}{g}\Big)+5m\Big(r,\frac{W_a'}{W_a}\Big)+5\{\bar N(r,0,W_a)+\bar N(r,g)\}+O\Big[\log^+T\Big(r,\frac{g'}{g}\Big)+\log^+m\Big(r,\frac{W_a'}{W_a}\Big)+\log^+\{\bar N(r,0,W_a)+\bar N(r,g)\}+\log r\Big]$$

as $r\to\infty$, n.e., which is the inequality as claimed.

Subcase ii. $a=0$.

Then (3.8) becomes $\phi_0=-2H_0$. Refining (4.4) in this case we have

$$m(r,H_0)=m\Big(r,\frac{g'}{g}\Big\{\frac{(g'/g)'}{(g'/g)}+\frac{g'}{g}\Big\}\Big)\leqq 2m\Big(r,\frac{g'}{g}\Big)+O\Big\{\log^+T\Big(r,\frac{g'}{g}\Big)+\log r\Big\} \tag{4.13}$$

as $r\to\infty$, n.e.. Therefore this together with (4.6) leads to an estimate

$$T(r,\phi_0)\leqq 2m\Big(r,\frac{g'}{g}\Big)+2\{\bar N(r,0,W_0)+\bar N(r,g)\}+O\Big\{\log^+T\Big(r,\frac{g'}{g}\Big)+\log r\Big\}$$

as $r\to\infty$, n.e.. On the other hand (3.9) becomes $\psi_0=H_0'+G_0H_0$. It gives

$$m(r,\psi_0)\leqq m(r,H_0)+m(r,G_0)+m\left(r,\frac{H_0'}{H_0}\right)+O(1)$$

$$\leqq 2m\left(r,\frac{g'}{g}\right)+m\left(r,\frac{W_0'}{W_0}\right)+O\left[\log^+T\left(r,\frac{g'}{g}\right)\right.$$

$$\left.+\log^+\{\bar{N}(r,0,W_0)+\bar{N}(r,g)\}+\log r\right],$$

as $r\to\infty$, n.e. by (4.3) and (4.13), because of $H_0\not\equiv 0$. Estimation of (4.10) is now valid as well, so we have

$$T(r,\psi_0)\leqq 2m\left(r,\frac{g'}{g}\right)+m\left(r,\frac{W_0'}{W_0}\right)+3\{\bar{N}(r,0,W_0)+\bar{N}(r,g)\}$$

$$+O\left[\log^+T\left(r,\frac{g'}{g}\right)+\log^+\{\bar{N}(r,0,W_0)+\bar{N}(r,g)\}+\log r\right]$$

as $r\to\infty$, n.e.. Hence we can estimate the logarithmic derivative in terms of W_0 by an inequality

$$T\left(r,\frac{g'}{g}\right)\leqq 4m\left(r,\frac{g'}{g}\right)+m\left(r,\frac{W_0'}{W_0}\right)+5\{\bar{N}(r,0,W_0)+\bar{N}(r,g)\}$$

$$+O\left[\log^+T\left(r,\frac{g'}{g}\right)+\log^+\{\bar{N}(r,0,W_0)+\bar{N}(r,g)\}+\log r\right]$$

as $r\to\infty$, n.e..

Subcase iii. $a=1/2$.

Then (3.8) becomes $\phi_{1/2}=-(1/4)G_{1/2}^2$. Since we have

$$m(r,\phi_{1/2})\leqq 2m(r,G_{1/2})=2m\left(r,\frac{W_{1/2}'}{W_{1/2}}\right)$$

and

$$N(r,\phi_{1/2})=2N(r,G_{1/2})\leqq 2\{\bar{N}(r,0,W_{1/2})+\bar{N}(r,g)\}$$

by (4.3) and (4.5), it follows

$$T(r,\phi_{1/2})\leqq 2m\left(r,\frac{W_{1/2}'}{W_{1/2}}\right)+2\{\bar{N}(r,0,W_{1/2})+\bar{N}(r,g)\}.$$

Similarly (3.9) becomes $\psi_{1/2}=(1/2)G_{1/2}H_{1/2}$. Reconsidering (4.4) as $a=1/2$ we refine it by

$$m(r,H_{1/2})\leqq m\left(r,\frac{g'}{g}\right)+m(r,G_{1/2})$$

$$=m\left(r,\frac{g'}{g}\right)+m\left(r,\frac{W_{1/2}'}{W_{1/2}}\right).$$

By this and (4.3),

$$m(r, \psi_{1/2}) \leqq m(r, G_{1/2}) + m(r, H_{1/2})$$
$$\leqq m\left(r, \frac{g'}{g}\right) + 2m\left(r, \frac{W_{1/2}'}{W_{1/2}}\right),$$

which together with (4.10) gives

$$T(r, \psi_{1/2}) \leqq m\left(r, \frac{g'}{g}\right) + 2m\left(r, \frac{W_{1/2}'}{W_{1/2}}\right) + 3\{\bar{N}(r, 0, W_{1/2}) + \bar{N}(r, g)\}.$$

Hence we are led to

$$T\left(r, \frac{g'}{g}\right) \leqq T(r, \phi_{1/2}) + T(r, \psi_{1/2}) + O(1)$$
$$\leqq m\left(r, \frac{g'}{g}\right) + 4m\left(r, \frac{W_{1/2}'}{W_{1/2}}\right) + 5\{\bar{N}(r, 0, W_{1/2}) + \bar{N}(r, g)\} + O(1).$$

Subcase iv. $a=1$.

In this case Estimates (3.8) and (3.9) become $\phi_1 = G_1'$ and $\psi_1 = -H_1'$, respectively. Because of $G_1' \not\equiv 0$ and $H_1' \not\equiv 0$ we deduce that

$$m(r, \phi_1) \leqq m(r, G_1) + O[\log^+ T(r, G_1) + \log r],$$
$$m(r, \psi_1) \leqq m(r, H_1) + O[\log^+ T(r, H_1) + \log r],$$

as $r \to \infty$, n.e., and

$$N(r, \phi_1) = N(r, G_1) + \bar{N}(r, G_1) = 2\bar{N}(r, G_1),$$
$$N(r, \psi_1) = N(r, H_1) + \bar{N}(r, H_1) = 3\bar{N}(r, H_1).$$

Using (4.3), (4.4), (4.5) and (4.6) we get

$$m(r, \phi_1) \leqq m\left(r, \frac{W_1'}{W_1}\right) + O\left[\log^+ m\left(r, \frac{W_1'}{W_1}\right) + \log^+\{\bar{N}(r, 0, W_1) + \bar{N}(r, g)\} + \log r\right],$$

$$m(r, \psi_1) \leqq 2m\left(r, \frac{g'}{g}\right) + m\left(r, \frac{W_1'}{W_1}\right)$$
$$+ O\left[\log^+ T\left(r, \frac{g'}{g}\right) + \log^+ m\left(r, \frac{W_1'}{W_1}\right)\right.$$
$$\left. + \log^+\{\bar{N}(r, 0, W_1) + \bar{N}(r, g)\} + \log r\right]$$

as $r \to \infty$, n.e., and

$$N(r, \phi_1) \leqq 2\{\bar{N}(r, 0, W_1) + \bar{N}(r, g)\},$$
$$N(r, \psi_1) \leqq 3\{\bar{N}(r, 0, W_1) + \bar{N}(r, g)\}.$$

Then our desired estimate is of an inequality

$$T\left(r, \frac{g'}{g}\right) \leqq 2m\left(r, \frac{g'}{g}\right) + 2m\left(r, \frac{W_1'}{W_1}\right) + 5\{\bar{N}(r, 0, W_1) + \bar{N}(r, g)\}$$

$$+ O\left[\log^+ T\left(r, \frac{g'}{g}\right) + \log^+ m\left(r, \frac{W_1'}{W_1}\right) + \log^+\{\bar{N}(r, 0, W_1) + \bar{N}(r, g)\} + \log r\right],$$

as $r \to \infty$, n. e..

Hence in Case I the inequality (2.1) never fails to hold for any fixed number a.

5. Proof of Theorem: Case II where $\phi_a \equiv 0$

Since $g \not\equiv 0$, (3.10) is reduced to $\psi_a \equiv 0$. Therefore the following two equations are given;

(5.1) $$a(2a-1)G_a'(z) + a(a-1)G_a(z)^2 - 2(a-1)(2a-1)H_a(z) \equiv 0\,,$$

(5.2) $$(2a-1)H_a'(z) + (a-1)G_a(z)H_a(z) \equiv 0\,.$$

We now distinguish the cases with respect to the value of a and determine all the forms of $g(z)$ to satisfy the two equations above.

Subcase i. $a=0$.

Then these become the equations $H_0(z) \equiv 0$ and $H_0'(z) + G_0(z)H_0(z) \equiv 0$, which we can reduce to $H_0(z) \equiv 0$. Applying this to (3.5) we see that $g''(z) \equiv 0$. This is however the $g(z)$ listed in § 1, 3°, so that $W_0(z) \equiv 0$. Hence ϕ_0 cannot vanish identically if $a=0$.

Subcase ii. $a=1/2$.

Then Equations (5.1) and (5.2) become $G_{1/2}(z)^2 \equiv 0$ and $G_{1/2}(z)H_{1/2}(z) \equiv 0$. Using Equation (3.5) with $a=1/2$ and $G_{1/2}(z) \equiv 0$ we get $H_{1/2}(z) \equiv 0$ by $g \not\equiv 0$. Applying these to (3.4) we see that $g'''(z) \equiv 0$, so that

(1) $$g(z) = \alpha z^2 + \beta z + \gamma, \qquad \text{where } \alpha, \beta, \gamma \in \boldsymbol{C}.$$

For this $g(z)$ we find

$$W_{1/2}(z) = W\left(\frac{1}{2}\beta z + \gamma,\ 2\alpha z + \beta\right) \equiv -\frac{1}{2}(\beta^2 - 4\alpha\gamma).$$

Further the constants α, β, γ must be taken as $\beta^2 - 4\alpha\gamma \neq 0$ in (1) (and then clearly $G_{1/2} = H_{1/2} \equiv 0$). It is such a condition that immediately follows from the negation of that in § 1, 3° with $m=2$ as well.

Subcase iii. $a=1$.

Then Equations (5.1) and (5.2) become $G_1'(z) \equiv 0$ and $H_1'(z) \equiv 0$. Therefore we can obtain the functions $g(z)$ to be determined as entire solutions of a second order linear differential equation

$$w''+k_1w'+k_0w=0 \tag{5.3}$$

with the constant coefficients k_1 and k_0. Let λ_1 and λ_2 be the roots of its characteristic equation $\lambda^2+k_1\lambda+k_0=0$. If $\lambda_1\neq\lambda_2$, a general solution of this (5.3) is given by

$$w=C_1e^{\lambda_1 z}+C_2e^{\lambda_2 z} \tag{2}$$

for arbitrary constants C_1, C_2. In order that $W_1(z)=W(w, w')=C_1C_2(\lambda_1-\lambda_2)^2\cdot\exp\{(\lambda_1+\lambda_2)z\}$ does not vanish identically, both C_1 and C_2 should differ from zero. If $\lambda_1=\lambda_2=\lambda$, say, a general solution to (5.3) has a form

$$w=(C_2z+C_1)e^{\lambda z}, \tag{3}$$

where C_1 and C_2 are arbitrary constants. For the w we have $W(w, w')=-C_2^2\exp(2\lambda z)$. Hence it is sufficient for our purpose to choose a non-zero constant C_2 in (3). In this subcase $g(z)$ must be of the form either (2) or (3) for suitable constants C_1 and C_2.

Subcase iv. *a is different from* 0, 1/2, 1.

Firstly suppose that $H_a(z)\equiv 0$. Then Equation (3.5) gives

$$(2a-1)g''(z)=-ag'(z)G_a(z) \tag{5.4}$$

and (5.1) also gives

$$(2a-1)G_a'(z)+(a-1)G_a(z)^2=0\,. \tag{5.5}$$

If $G_a(z)\equiv 0$, $g''(z)\equiv 0$ by (5.4), so that

$$g(z)=\alpha z+\beta \tag{4}$$

for $\alpha(\neq 0)$, β, $\in \boldsymbol{C}$. Then $W_a(z)\equiv -a\alpha^2\neq 0$. Unless $G_a\equiv 0$, Equation (5.5) leads us to

$$G_a(z)=\frac{2a-1}{a-1}\cdot\frac{1}{z-z_0},\qquad z_0\in\boldsymbol{C},$$

and then Equation (5.4) gives

$$\frac{g''(z)}{g'(z)}=-\frac{a}{2a-1}G_a(z)=-\frac{a}{a-1}\cdot\frac{1}{z-z_0}.$$

Therefore it follows

$$g'(z)=C(z-z_0)^{-a/(a-1)},\qquad C\in\boldsymbol{C}-\{0\},$$

so that $-a/(a-1)=m$, say, is an integer different from 0, -1, 1, and

(5) $$g(z)=\frac{C}{m+1}\{(z-z_0)^{m+1}-\zeta\}, \qquad \zeta\in\boldsymbol{C}.$$

Noting $a=m/(m+1)$ we see that $\zeta\neq 0$ in order that $W_a(z)=-aC^2\zeta(z-z_0)^{m-1}$ should not vanish identically.

After this we may suppose that $H_a(z)\not\equiv 0$. By Equation (5.2) we obtain

(5.6) $$G_a(z)=-\frac{2a-1}{a-1}\cdot\frac{H_a'(z)}{H_a(z)}.$$

Then (5.1) gives

$$2(a-1)(2a-1)H_a(z)=-\frac{a(2a-1)^2}{a-1}\Big(\frac{H_a'(z)}{H_a(z)}\Big)'+\frac{a(2a-1)^2}{a-1}\Big(\frac{H_a'(z)}{H_a(z)}\Big)^2$$
$$=-\frac{a(2a-1)^2}{a-1}\cdot\frac{H_a''(z)H_a(z)-2H_a'(z)^2}{H_a(z)^2},$$

and thus

$$\Big(\frac{1}{H_a(z)}\Big)''=-\frac{H_a''(z)H_a(z)-2H_a'(z)^2}{H_a(z)^3}\equiv\frac{2(a-1)^2}{a(2a-1)}.$$

Hence we deduce

$$H_a(z)=\frac{a(2a-1)}{(a-1)^2}\cdot\frac{1}{(z-\alpha)(z-\beta)}, \qquad \alpha,\ \beta\in\boldsymbol{C}$$

In virtue of this expression we get

$$G_a(z)=\frac{2a-1}{a-1}\Big\{\frac{1}{z-\alpha}+\frac{1}{z-\beta}\Big\}$$

from Equation (5.6) and thus it follows from (3.2) that

(5.7) $$W_a(z)=C\{(z-\alpha)(z-\beta)\}^{-(2a-1)/(a-1)}$$

for a non-zero constant C. Here $-(2a-1)/(a-1)=m$, say, is a number different from -2, -1, and 0, which is equal to an integer if $\alpha\neq\beta$ and to half an integer if $\alpha=\beta$. Then we can transform (3.5) into an equation

(5.8) $$g''(z)-(m+1)\Big(\frac{1}{z-\alpha}+\frac{1}{z-\beta}\Big)g'(z)+\frac{(m+1)(m+2)}{(z-\alpha)(z-\beta)}g(z)=0.$$

Therefore $g(z)$ is able to posess the poles possibly at $z=\alpha$ or β. Let $f(z)$ be an entire function with $f(\alpha)\neq 0$ and $f(\beta)\neq 0$, and both k and l be integers if $\alpha\neq\beta$ and half integers with $k=l$ if $\alpha=\beta$, such that

$$g(z)=(z-\alpha)^k(z-\beta)^l f(z).$$

Using Equation (5.8) we write

$$\frac{g''(z)}{g(z)}=(m+1)\Big(\frac{1}{z-\alpha}+\frac{1}{z-\beta}\Big)\frac{g'(z)}{g(z)}-\frac{(m+1)(m+2)}{(z-\alpha)(z-\beta)}$$

and thus an expression

$$\frac{W_a(z)}{g(z)^2}=\frac{g''(z)}{g(z)}-\frac{m+1}{m+2}\Big(\frac{g'(z)}{g(z)}\Big)^2$$

$$=-\frac{m+1}{m+2}\Big\{\Big(\frac{g'(z)}{g(z)}\Big)^2-(m+2)\Big(\frac{1}{z-\alpha}+\frac{1}{z-\beta}\Big)\frac{g'(z)}{g(z)}+\frac{(m+2)^2}{(z-\alpha)(z-\beta)}\Big\}$$

$$=-\frac{m+1}{m+2}\Big(\frac{g'(z)}{g(z)}-\frac{m+2}{z-\alpha}\Big)\Big(\frac{g'(z)}{g(z)}-\frac{m+2}{z-\beta}\Big)$$

$$=-\frac{m+1}{m+2}\Big(\frac{k-m-2}{z-\alpha}+\frac{l}{z-\beta}+\frac{f'(z)}{f(z)}\Big)\Big(\frac{k}{z-\alpha}+\frac{l-m-2}{z-\beta}+\frac{f'(z)}{f(z)}\Big).$$

On the other hand, (5.7) gives

$$\frac{W_a(z)}{g(z)^2}=\frac{C}{(z-\alpha)^{2k-m}(z-\beta)^{2l-m}f(z)^2}.$$

In the case where $\alpha\neq\beta$, $g(z)$ is expressed by

$$g(z)=c_\alpha(z-\alpha)^k\{1+O(z-\alpha)\},\qquad c_\alpha\in\boldsymbol{C}-\{0\}$$

in a neighborhood of $z=\alpha$. Substituting this into the equation (5.8) and comparing the coefficients of the term $(z-\alpha)^{k-2}$, we get a characteristic equation $k(k-m-2)=0$, so that k can be of the value 0 or $m+2$. Then it immediately follows that $m=-3$ and $k=-1$, when we compare the behavior of two expressions above for $W_a(z)/g(z)^2$ in a neighborhood of $z=\alpha$. In fact we see that $2k-m=1$ in both cases of $k=0$ and $k=m+2$. We have $m=-3$ when $k=m+2$, while $m=-1$ when $k=0$. The latter is now excluded. The same is true of the number l. Hence $a=2$ and

$$g(z)=\frac{f(z)}{(z-\alpha)(z-\beta)}$$

if $\alpha\neq\beta$. Concerning $f(z)$ we have

$$\frac{C}{(z-\alpha)(z-\beta)f(z)^2}=-2\Big\{-\frac{1}{z-\beta}+\frac{f'(z)}{f(z)}\Big\}\Big\{-\frac{1}{z-\alpha}+\frac{f'(z)}{f(z)}\Big\}$$

and thus

$$\{(z-\beta)f'(z)-f(z)\}\{(z-\alpha)f'(z)-f(z)\}\equiv-\frac{C}{2}\quad(\neq 0).$$

Differentiating both sides of this, we get an identity

$$(z-\beta)f''(z)\{(z-\alpha)f'(z)-f(z)\}=-(z-\alpha)f''(z)\{(z-\beta)f'(z)-f(z)\}.$$

Unless $f''(z)\equiv 0$,

$$(z-\beta)\{(z-\alpha)f'(z)-f(z)\}=-(z-\alpha)\{(z-\beta)f'(z)-f(z)\}$$

and therefore

$$(\alpha-\beta)f(\alpha)=0\,.$$

This is impossible, so that $f''(z)\equiv 0$, i.e., $f(z)=D(z-\gamma)$ where $D\in \boldsymbol{C}-\{0\}$ and $\gamma\in \boldsymbol{C}-\{\alpha, \beta\}$. Then

$$g(z)=\frac{D(z-\gamma)}{(z-\alpha)(z-\beta)}\,, \tag{6}$$

which satisfies the condition (5.7) with $C=-2D^2(\gamma-\alpha)(\gamma-\beta)$.

Next we shall consider the case where $\alpha=\beta$. Equation (5.8) is then equal to

$$g''(z)-\frac{2(m+1)}{z-\alpha}g'(z)+\frac{(m+1)(m+2)}{(z-\alpha)^2}g(z)=0$$

Here we make a similar discussion to the above with

$$g(z)=(z-\alpha)^{2k}f(z)\,,$$

and obtain a characteristic equation

$$2k(2k-1)-2k\cdot 2(m+1)+(m+1)(m+2)=0\,,$$

and so $2k=m+1$ or $2k=m+2$. Since we now have

$$\frac{W_a(z)}{g(z)^2}=\frac{C}{(z-\alpha)^{2(2k-m)}f(z)^2}=-\frac{m+1}{m+2}\left\{\frac{2k-m-2}{z-\alpha}+\frac{f'(z)}{f(z)}\right\}^2,$$

the latter, $2k=m+2$, gives immediately a contradiction as $2k-m\neq 0$. For the former case where $2k=m+1$ the behavior of two expressions above for $W_a(z)/g(z)^2$ is compatible. Then $f(z)$ satisfies the relation

$$\{(z-\alpha)f'(z)-f(z)\}^2\equiv -\frac{m+2}{m+1}\cdot C(\neq 0)\,.$$

Differentiating this we have $f''(z)\equiv 0$, or $f(z)=D(z-\gamma)$, $D\in \boldsymbol{C}-\{0\}$, $\gamma\in \boldsymbol{C}-\{\alpha\}$ again. Hence if $\alpha=\beta$, $a=(m+1)/(m+2)$ and

$$g(z)=D(z-\alpha)^{m+1}(z-\gamma)\,, \tag{7}$$

provided that m is an integer different from 0, -1, and -2. In order that $g(z)$ may satisfy (5.7), i.e.,

$$W_a(z)=C(z-\alpha)^{2m}\,,$$

we choose the constant $C=-aD^2(\gamma-\alpha)^2$.

We have discussed all the possible forms that $g(z)$ has in Case II:

(1) when $a=1/2$, $g(z)=\alpha z^2+\beta z+\gamma$, where $\beta^2-4\alpha\gamma\neq 0$;

(2) when $a=1$, $g(z)=C_1e^{\lambda_1 z}+C_2e^{\lambda_2 z}$, where $\lambda_1 \neq \lambda_2$ and $C_1C_2 \neq 0$;
(3) when $a=1$, $g(z)=(C_2z+C_1)e^{\lambda z}$, where $C_2 \neq 0$;
(4) when $a \neq 0$, $1/2$, 1, $g(z)=C_1(z-\alpha)$, where $C_1 \neq 0$;
(5) when $a=(m-1)/m$, $g(z)=C_2\{(z-\alpha)^m-C_1\}$, where $C_1C_2 \neq 0$ and $m \neq 0$, 1, 2;
(6) when $a=2$, $g(z)=(C_1(z-\gamma)/(z-\alpha)(z-\beta))$, where $C_1 \neq 0$, and α, β, γ are mutually distinct;
(7) when $a=(m+1)/(m+2)$, $g(z)=C_1(z-\alpha)^{m+1}(z-\gamma)$, where $C_1 \neq 0$, $\alpha \neq \gamma$ and $m \neq 0$, -1, -2,

provided that C_1, C_2, λ_1, λ_2, λ, α, β, $\gamma \in \boldsymbol{C}$ and m is an integer. As their $W_a(z)$ we obtain also

(1) $W_{1/2}(z) \equiv -(1/2)(\beta^2-4\alpha\gamma)$;
(2) $W_1(z)=C_1C_2(\lambda_1-\lambda_2)^2e^{(\lambda_1+\lambda_2)z}$;
(3) $W_1(z)=-C_2{}^2e^{2\lambda z}$;
(4) $W_a(z) \equiv -a\alpha^2$;
(5) $W_{(m-1)/m}(z)=-m(m-1)C_1C_2(z-\alpha)^{m-2}$;
(6) $W_2(z)=(-2C_1{}^2(\gamma-\alpha)(\gamma-\beta))/((z-\alpha)^3(z-\beta)^3)$;
(7) $W_{(m+1)/(m+2)}(z)=-\{(m+1)/(m+2)\}C_1{}^2(\gamma-\alpha)^2(z-\alpha)^{2m}$.

Finally we need to examine whether the inequality (2.1) holds or not in each case above. The function $U_a(r)$ in (2.1) grows at least as rapidly as $O(\log r)$ for $r \to \infty$, n. e.. Therefore (2.1) is satisfied by $g(z)$ given in (4), (5), (6) and (7) as rational functions. As proved in *Remark* 2° in § 2, two possibilities (1) and (2) are the very exceptions. With $g(z)$ as in (3) it is easily shown that

$$m(r, g'/g)=O(1), \qquad m(r, W_1'/W_1)=O(1),$$

$$\bar{N}(r, 0, W_1)=\bar{N}(r, g) \equiv 0,$$

and

$$T(r, g'/g)=\log r+O(1),$$

as $r \to \infty$. Then $U_a(r)=O(\log r)$ as r tends to infinity, so Inequality (2.1) also holds. This completes the proof of the theorem.

Remark. Mention needs to be made of rational functions. Reconsidering the above proof in Case I as a rational function $g(z)$, we see that all of $m(r, G_a)$, $m(r, H_a)$, $m(r, \phi_a)$ and $m(r, \psi_a)$ grow possibly in the degree of $o(1)$ with the aid of (1.2) in Lemma. Inequality (2.1) can be therefore sharpened by

$$T\left(r, \frac{g'}{g}\right) \leqq 5\{\bar{N}(r, 0, W_a)+\bar{N}(r, g)\}+O(1). \tag{5.9}$$

Then there exist such the rational functions $g(z)$ as never satisfy (5.9) only in (3) with $\lambda=0$, (4), (5) with $m>5$ or $m<-4$, as well as (1) of Case II. In fact, since Inequality (5.9) equals

$$\bar{N}(r, 0, g) \leqq 5\bar{N}(r, 0, W_a) + 4\bar{N}(r, g) + O(1)$$

in virtue of the equation

$$T\left(r, \frac{g'}{g}\right) = m\left(r, \frac{g'}{g}\right) + N\left(r, \frac{g'}{g}\right) = \bar{N}(r, 0, g) + \bar{N}(r, g) + o(1),$$

this fact can be shown by studying these counting functions in each occasion. In (5) for example, if $m > 2$,

$$\bar{N}(r, 0, g) = m \log r, \quad \bar{N}(r, 0, W_a) = \log r, \quad \bar{N}(r, g) \equiv 0$$

and if $m < 0$,

$$\bar{N}(r, 0, g) = -m \log r, \quad \bar{N}(r, 0, W_a) \equiv 0, \quad \bar{N}(r, g) = \log r$$

for sufficiently large r. The equality of (5.9) occurs if $m=5$ or $m=-4$.

6. Proof of Corollary

In order to prove this result we shall return to the proof of Theorem. At first we are concerned about Case I in Section 4. Assume that $\varphi_a(z) \not\equiv 0$. The present assumption (2.2) reduces the equations (4.3) and (4.4) to

$$m(r, G_a) = m\left(r, \frac{W_a'}{W_a}\right) = S(r, W_a) = S(r, g)$$

and

$$m(r, H_a) = S(r, g) + S(r, W_a) = S(r, g),$$

respectively. Similarly (4.5) and (4.6) become

$$N(r, G_a) \leqq \bar{N}(r, 0, W_a) = m(r, W_a) + O(1) = S(r, g)$$

and

$$N(r, H_a) \leqq 2\bar{N}(r, 0, W_a) = S(r, g).$$

All of them hold independently of the value of $a \in \boldsymbol{C}$. Therefore it follows also for both ϕ_a and ψ_a that

$$T(r, \phi_a) = S(r, g) \quad \text{and} \quad T(r, \psi_a) = S(r, g),$$

so that

$$T\left(r, \frac{g'}{g}\right) = S(r, g). \tag{6.1}$$

Using a relation

$$g^2 = \frac{W_a}{\left(\frac{g'}{g}\right)' - (a-1)\left(\frac{g'}{g}\right)^2},$$

we obtain

$$2T(r, g)\leqq T(r, W_a)+T\left(r, \left(\frac{g'}{g}\right)'-(a-1)\left(\frac{g'}{g}\right)^2\right)+O(1)$$

$$\leqq m(r, W_a)+4T\left(r, \frac{g'}{g}\right)+S\left(r, \frac{g'}{g}\right)+O(1).$$

Then from (2.2) and (6.1) we conclude

$$T(r, g)=S(r, g),$$

which is impossible. Hence it must be hold $\varphi_a(z)\equiv 0$.

Concerning the possibilities in Case II we have made a list in Section 5. We shall pick out those what give entire functions $g(z)$ with the property (2.2). Evidently (5), (6) and (7) are beside our object. If $g(z)$ is a polynomial, $W_a(z)$ must be a constant. Possibilities (1) and (4) come under this heading. If $\lambda_1+\lambda_2 \neq 0$ in (2), then $W_1(z)=C_1C_2(\lambda_1-\lambda_2)^2e^{(\lambda_1+\lambda_2)z}$ is an entire function of order one and

$$m(r, W_1)=\frac{|\lambda_1+\lambda_2|}{\pi}r+O(1) \qquad \text{as } r\to\infty .$$

(See Hayman [1], p. 7.) A similar observation shows

$$m(r, g)\leqq(|\lambda_1|+|\lambda_2|)\frac{r}{\pi}+O(1) \qquad \text{as } r\to\infty .$$

Therefore (2.2) fails to hold since

$$\lim_{r\to\infty}\frac{m(r, W_1)}{m(r, g)}\geqq\frac{|\lambda_1+\lambda_2|}{|\lambda_1|+|\lambda_2|}>0.$$

When $\lambda_1+\lambda_2=0$, $W_1(z)$ is a constant and $g(z)$ is such a transcendental entire function that satisfies all the assumptions in Corollary. Functions in the last remaining (3) can satisfy Condition (2.2) only if $\lambda=0$. We have thus proved the corollary.

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