R.K. NAGAICH
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CONSTANCY OF HOLOMORPHIC SECTIONAL CURVATURE IN INDEFINITE ALMOST HERMITIAN MANIFOLDS

BY NAGAICH R.K.

1. Introduction.

For an almost Hermitian manifold (M, g, J) of dimension. $m \geqq 4$, satisfying the property $R(X, Y, X, Z)=R(JX, JY, JX, JZ)$, S. Tanno [3] has proved the following:

THEOREM A. *Let* $m \geqq 4$. *Assume that an almost Hermitian manifold* (M^m, g, J) *satisfies*

$$R(X, Y, X, Z)=R(JX, JY, JX, JZ) \tag{1}$$

for every tangent vectors X, Y *and* Z. *Then* (M, g, J) *is of constant holomorphic sectional curvature at* x, *if and only if*

$$R(X, JX)\, X \text{ is proportional to } JX, \tag{2}$$

for every tangent vector X *at* x.

The purpose of this paper is to generalize the above theorem for an indefinite almst Hermitian manifold by proving the following:

MAIN THEOREM. *Let* (M^m, g, J) $(m \geqq 4)$ *be an indefinite almost Hermitian manifold satisfying* (1). *Then* (M^m, g, J) *is of constant holomorphic sectional curvature at* x, *if and only if* $R(X, JX)X$ *is proportional to* JX *for every tangent vector* X *at* x.

By an indefinite almost Hermitian manifold we mean a semiRiemannian manifold (M, g) with almst complex structure J which preserves the metric g i.e.

$$g(JX, JY)=g(X, Y), \qquad X, Y \in \mathfrak{X}(M).$$

If X is a vector field on M, we shall say that X is space like, time like and null if $g(X, X)>0$, $g(X, X)<0$ and $g(X, X)=0$, $X \neq 0$, respectively. The

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metric g is said to be degenerate if there exists a non-zero vector $X \in \mathfrak{X}(M)$ such that $g(X, Y)=0$ for all $Y \in \mathfrak{X}(M)$. It is well known, see for instance [2], that the plane $p=sp\{X, Y\}$ is degenerate if and only if

$$g(X, X)g(Y, Y)-g(X, Y)^2=0\,. \tag{3}$$

For a non-degenerate plane plane $p=sp\{X, Y\}$, the sectional curvature is defined as usual by

$$K(X, Y)=\frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y)-g(X, Y)^2}\,.$$

The holomorphic sectional curvature $H(X)$ for a unit tangent vector is the sectional curvature $K(X, JX)$. If $H(X)$ is constant for every tangent vector X at x, then (M, g, J) is said to be of constant holomorphic sectional curvature at x.

Before we proceed further, we shall give some examples of indefinite almost Hermitian manifolds.

Example 1. Let TM be the tangent bundle of an n-dimensional almost Hermitian manifold (M, g, J), where g is definite or indefinite. Let J^c (resp. g^c) be the complete lift to TM of J (resp. g). Then (TM, g^c, J^c) is an indefinite almost Hermitian manifold with index $2n$. If J is integrable then J^c is also integrable (See [4]).

Example 2. Let (M, g) be an n-dimensional indefinite—Riemannian manffold where the metric g has index m. Then, the metric of Sasaki $\bar{g}$ is an indefinite metric on TM with index $2m$. Let $\bar{J}$ be the natural almost complex structure on TM. Then $(TM, \bar{g}, \bar{J})$ is an indefinite almost Kaehler manifold. Moreover, it is indefinite Kaehler if and only if M is locally flat [1].

Example 3. For an indefinite Kaehler manifold (M, g, J), $(TM, \bar{g}, J^h)$ is an indefinite Hermitian manifold where J^h denotes the horizontal lift of J.

2. Proof of the theorem. Let (M, g, J) be an indefinite almost Hermitian manifold of dimension $\geqq 4$. Assume that (M, g, J) has the property

$$R(X, Y, X, Z)=R(JX, JY, JX, JZ)$$

for every unit vectors X, Y and Z. If $R(X, JX)X=cJX$, then it follows obviously that $K(X, JX)=c$ for all X. To prove the converse, we shall consider the following cases:

(i) $g(X, X)=g(Y, Y)$ and

(ii) $g(X, X)=-g(Y, Y)$.

For the first case the proof is similar as given by Tanno [3], so we drop it here. Therefore, we consider the case when $g(X, X)=-g(Y, Y)$. Let $\{X, Y, JX\}$ be an orthonormal set and assume dimension. $m\geqq 6$. Define X' and Z' by

$$X'=\frac{X+iY}{\sqrt{2}} \quad \text{and} \quad Z'=\frac{iJX+JY}{\sqrt{2}}. \tag{4}$$

Then, $\{X', JX', Z'\}$ also form an orthonormal triplet. By the argument as in [3], we have

$$R(X', JX', X', Z')=0. \quad \text{i. e.} \tag{5}$$

$$0=(1/4)R(X+iY, JX+iJY, X+iY, iJX+JY). \tag{6}$$

From the last relation, it is easy to get $R(X, JX, X, JX)=R(Y, JY, Y, JY)$. This shows that

$$H(X)=H(Y). \tag{7}$$

Now, if $sp\{U, V\}$ is holomorphic, then $JU=aU+bV$ for some scalers a and b. Then

$$sp\{U, JU\}=sp\{U, aU+bV\}=sp\{U, V\}.$$

Similarly, $sp\{V, JV\}=sp\{U, V\}$ i. e. $sp\{U, JU\}=sp\{V, JV\}$. This shows that

$$H(U)=H(V). \tag{8}$$

If $sp\{U, V\}$ is not a holomorphic section, then we can always choose unit vectors $X\in sp\{U, JU\}^{\perp}$ and $Y\in sp\{V, JV\}^{\perp}$ which determine a holomorphic section $\{X, Y\}$. Consequently, we get

$$H(U)=H(X)=H(Y)=H(V). \tag{9}$$

This shows that any holomorphic section has same sectional curvature.

Next, we assume that dimension of $M=4$ and $g(X, X)=1$, $g(Y, Y)=-1$, $g(X, Y)=0$ and $g(X, JY)=0$. Using the properties of curvature tensor R, we get the following relations:

$$R(X, JX)X=H(X)JX,$$

$$R(X, JX)Y=-R(X, JX, Y, JY)JY,$$

$$R(X, JY)X=-R(X, JY, X, Y)Y-R(X, JY, X, JY)JY,$$

$$R(X, JY)Y=R(X, JY, Y, X)X+R(X, JY, Y, JX)JX,$$

$$R(Y, JY)X=R(Y, JY, X, JX)JX,$$

$$R(Y, JX)Y=R(Y, JX, Y, X)X+R(Y, JX, Y, JX)JX,$$

$$R(Y, JX)X=-R(Y, JX, X, Y)Y-R(Y, JX, X, JY)JY,$$

$$R(Y, JY)Y=-H(Y)JY=-H(X)JY .$$

Now, define $X'=aX+bY$ with $a^2-b^2=1$. Then, using above relations we get

(10) $$R(X', JX')X'=C_1X+C_2Y+C_3JX+C_4JY$$

where C_1 and C_2 are not necessary for our argument and

$$C_3=a^3H(X)+ab^2C_5 ,$$

$$C_4=-b^3H(X)-a^2bC_5 ,$$

where

(11) $$C_5=R(X, JX, Y, JY)+R(X, JY, Y, JX)+R(Y, JX, Y, JX).$$

On the other hand,

(12) $$R(X', JX')X'=H(X')JX'=H(X') \quad (aJX+bJY) .$$

Comparing (10) and (12) we get

(13) $$a^2H(X)+b^2C_5=H(X') ,$$

(14) $$-b^2H(X)-a^2C_5=H(X') .$$

From last two equations, we get

(15) $$C_5=-H(X) .$$

Thus, $H(X')=(a^2-b^2)H(X)=H(X)$. Similarly we can prove $H(Y')=H(Y)$ and thus M is of constant holomorphic sectional curvature at x.

3. Remarks. An indefinite almost Hermitian manifold (M, g, J) is called indefinite Kaehler manifold if J is parallel. Obviously, the property (1) is satisfied by every indefinite Kaehler manifold. Hence, the following corollary is consequence of our main theorem:

COROLLARY B. *An indefinite Kaehler manifold (M, g, J) of dimension $\geqq 4$ is of constant holomorphic sectional curvature, if and only if $R(X, JX)\, X$ is proportional to JX, for every vector field X on M.*

In an indefinite almost Hermitian manifold (M, g, J), if $(\nabla_X J)Y+(\nabla_Y J)X=0$ holds for every X and Y, then (M, g, J) is called an indefinite K-space. For an indefinite K-space, the property (1) holds good. Therefore, as an immediate consequence of our theorem, we have the following:

COROLLARY C. *An indefinite K-space (M, g, J) of dimension $\geqq 4$ is of constant holomorphic sectional curvature, if and only if, $R(X, JX)X$ is proportional to JX for every vector field X on M.*

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DEPARTMENT OF MATHEMATICS
PUNJABI UNIVERSITY
PATIALA-147002
INDIA