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ON A CONJECTURE OF C.C. YANG FOR THE CLASS F OF MEROMORPHIC FUNCTIONS

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Abstract

In this paper, we give a positive answer to a conjecture of C.C. Yang for the class F of meromorphic functions, and improve a result of C.C. Yang.

Key words: Meromorphic function, Deficient value, Unicity.

1. Introduction and Main Results.

In this paper, we use the signs as given in Nevanlinna theory [3], let E denote a set of positive real number with a finite linear measure, which is not necessarily the same at each time it occurs. If the two meromorphic functions f and g have the same a-points and multiplicities, we denote it by

$$E(a, f)=E(a, g).$$

In 1977, C. C. Yang proved the following theorem:

THEOREM A ([1]). *Let F denote a class of meromorphic functions with the form as $f=\mu_1 e^{\alpha}+\mu_2$, where α is a nonconstant entire function with finite order, μ_1 ($\not\equiv 0$) and μ_2 ($\not\equiv$const.) are two meromorphic functions with finite order, satisfying*

$$T(r, \mu_i)=o\{T(r, e^{\alpha})\}, \qquad (i=1, 2).$$

Suppose c_1, c_2 are two distinct finite complex numbers, $f\in F$ and $g\in F$. If

$$E(c_i, f)=E(c_i, g), \qquad (i=1, 2)$$

then $f\equiv g$ or

$$f=\frac{c_2-c_1\lambda(z)}{1-\lambda(z)}-\frac{(c_1-c_2)^2\lambda(z)}{1-\lambda(z)}\cdot\frac{1}{h(z)\cdot e^{\phi(z)}},$$

$$g=\frac{c_1-c_2\lambda(z)}{1-\lambda(z)}+\frac{h(z)e^{\phi(z)}}{1-\lambda(z)},$$

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where $\phi(z)$ is a nonconstant entire function, $\lambda(z)$ ($\not\equiv$const.) and $h(z)$ are two meromorphic functions, satisfying

$$T(r, \lambda)=o\{T(r, e^{\phi})\}, \qquad T(r, h)=o\{T(r, e^{\phi})\}.$$

Further, he conjectured that theorem A also holds for the class of meromorphic functions with the form as

$$f=\mu_1 e^{\alpha}+\mu_2,$$

where α is a nonconstant entire function, $\mu_1(\not\equiv 0)$ and $\mu_2(\not\equiv\text{const.})$ are two meromorphic functions, satisfying

$$T(r, \mu_i)=o\{T(r, e^{\alpha})\}. \qquad (i=1, 2)$$

In the present paper, we give a positive answer to C. C. Yang's conjecture. More generally, the following results are obtained.

THEOREM 1. *Let f, g, μ and λ be nonconstant meromorphic functions, satisfying*

$$T(r, \mu)=o\{T(r, f)\}, \qquad T(r, \lambda)=o\{T(r, g)\}.$$

If $E(\infty, f)=E(\infty, g)$, $E(\mu, f)=E(\lambda, g)$, and

$$\delta(0, f)+\Theta(\infty, f)>\frac{3}{2}, \qquad \delta(0, g)+\Theta(\infty, g)>\frac{3}{2},$$

then

$$\frac{f}{\mu}=\frac{g}{\lambda} \quad or \quad f\cdot g=\mu\cdot\lambda.$$

THEOREM 2. *Let f, g, φ_1, φ_2, h_1 and h_2 be nonconstant meromorphic functions, satisfying*

$$T(r, \varphi_i)=o\{T(r, f)\}, \qquad T(r, h_i)=o\{T(r, g)\}, \qquad (i=1, 2).$$

If $E(\infty, f)=E(\infty, g)$, $E(\varphi_i, f)=E(h_i, g)$, $(i=1, 2)$ and

$$\delta(0, f)+\Theta(\infty, f)>\frac{3}{2}, \qquad \delta(0, g)+\Theta(\infty, g)>\frac{3}{2},$$

then

$$\frac{f}{\varphi_1}=\frac{g}{h_1} \quad and \quad \frac{\varphi_1}{h_1}=\frac{\varphi_2}{h_2},$$

or

$$f\cdot g=\varphi_1\cdot h_1 \quad and \quad \varphi_1\cdot h_1=\varphi_2\cdot h_2.$$

COROLLARY. *The conjecture of C.C. Yang is ture.*

2. Some Lemmas.

LEMMA 1 ([2]). *Let f_j $(j=1, 2, \cdots, n)$ be n linearly independent meromorphic functions with* $\sum_{j=1}^{n} f_j \equiv 1$, *then*

$$T(r, f_j) < \sum_{i=1}^{n} N\left(r, \frac{1}{f_i}\right) + N(r, f_j) + N(r, D)$$

$$- \sum_{i=1}^{n} N(r, f_i) + o\{T(r)\}, \qquad (r \notin E;\ j=1, 2, \cdots, n)$$

where $T(r) = \max_{1 \le j \le n} \{T(r, f_j)\}$,

$$D = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

LEMMA 2 ([4]). *Let f_1 and f_2 be two nonconstant meromorphic functions, and let α_1 $(\not\equiv 0)$ and α_2 $(\not\equiv 0)$ be two meromorphic functions, it satisfies*

$$T(r, \alpha_i) = o\{T(r)\}, \qquad (r \notin E;\ i=1, 2)$$

where $T(r) = \max\{T(r, f_1), T(r, f_2)\}$. *If* $\alpha_1 f_1 + \alpha_2 f_2 = 1$, *then*

$$T(r, f_i) < \bar{N}\left(r, \frac{1}{f_1}\right) + \bar{N}\left(r, \frac{1}{f_2}\right) + \bar{N}(r, f_i) + o\{T(r)\}. \quad (r \notin E;\ i=1, 2)$$

LEMMA 3 ([5]). *Let f_j $(j=1, 2, 3)$ be three nonconstant meromorphic functions, satisfying* $\sum_{j=1}^{3} f_j \equiv 1$. *And let* $g_1 = -f_1/f_2$, $g_2 = 1/f_2$, $g_3 = -f_3/f_2$. *If f_j $(j=1, 2, 3)$ are linearly independent, then g_j $(j=1, 2, 3)$ also are linearly independent.*

3. Proof of Theorems and Corollary.

The proof of theorem 1. In fact, let

$$\frac{f-\mu}{g-\lambda} = h \tag{1}$$

and $T(r) = \max\{T(r, f), T(r, g)\}$. Then the poles and zeros of h only occur at the poles of μ and λ at most by $E(\infty, f) = E(\infty, g)$ and $E(\mu, f) = E(\lambda, g)$. Hence

$$N(r, h) + N\left(r, \frac{1}{h}\right) = o\{T(r)\}. \tag{2}$$

Next, from (1) we get

$$f-\mu=gh-\lambda h\,. \tag{3}$$

We complete the proof by the following two cases:

CASE 1. $h\equiv k$ (const.), then when $k\not\equiv\mu/\lambda$ we have from (3)

$$\frac{f}{\mu-\lambda k}-\frac{k}{\mu-\lambda k}g=1\,. \tag{4}$$

From Lemma 2 (by taking $\alpha_1=(1/\mu-\lambda k)\not\equiv 0$, $\alpha_2=-(k/\mu-\lambda k)\not\equiv 0$) we get

$$T(r,\ f)<N\Big(r,\ \frac{1}{f}\Big)+N\Big(r,\ \frac{1}{g}\Big)+\bar{N}(r,\ f)+o\{T(r)\},\qquad (r\notin E).$$

On the other hand, from (4) we know that

$$T(r,\ g)\leqq(1+o(1))T(r,\ f)$$

so that

$$o\{T(r)\}=o\{T(r,\ f)\}\,. \tag{5}$$

Hence

$$\begin{aligned}T(r,\ f)&<[2-\delta(0,\ f)-\Theta(\infty,\ f)]T(r,\ f)\\&\quad+[1-\delta(0,\ g)]T(r,\ g)+o\{T(r)\}\quad(r\notin E)\\&\leqq[3-\delta(0,\ f)-\Theta(\infty,\ f)-\delta(0,\ g)]T(r,\ f)+o\{T(r)\},\quad(r\notin E).\end{aligned} \tag{6}$$

Since

$$3-\delta(0,\ f)-\Theta(\infty,\ f)-\delta(0,\ g)<\frac{3}{2}-\delta(0,\ g)<1\,,$$

by (5) and (6) we deduce that

$$T(r,\ f)=o\{T(r,\ f)\},\qquad (r\notin E)$$

which is a contradiction.

It shows that if h is a constant function, h must be equal to μ/λ. Hence we obtain from (1)

$$\frac{f}{\mu}=\frac{g}{\lambda}.$$

CASE 2. $h\not\equiv$constant, let

$$f_1=\frac{f}{\mu},\qquad f_2=\frac{\lambda}{\mu}h,\qquad f_3=-\frac{g}{\mu}h\,.$$

then from (3)

$$\sum_{j=1}^{3}f_j\equiv 1\,. \tag{7}$$

Suppose f_j ($j=1, 2, 3$) are linearly independent, it is easy to see from Lemma 1 that

$$T(r, f)<N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+N(r, f)+N(r, D)$$
$$-\sum_{j=1}^{3} N(r, f_j)+o\{T(r)\}, \qquad (r\notin E) \tag{8}$$

where

$$D=\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}.$$

From (7) we get

$$D=\begin{vmatrix} f_1 & f_2 & 1 \\ f_1' & f_2' & 0 \\ f_1'' & f_2'' & 0 \end{vmatrix}=\begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix}.$$

Hence

$$N(r, D)\leqq N(r, f)+2\bar{N}(r, f)+o\{T(r)\}.$$

Thus

$$N(r, f)+N(r, D)-\sum_{j=1}^{3} N(r, f_j)\leqq 2\bar{N}(r, f)+o\{T(r)\}. \tag{9}$$

Then, from (8) we obtain

$$T(r, f)<N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+2\bar{N}(r, f)+o\{T(r)\}. \qquad (r\notin E) \tag{10}$$

Next, according to Lemma 3 we know that $g_1=-f/\lambda h$, $g_2=\mu/\lambda h$, $g_3=g/\lambda$ are also linearly independent. Similary, we can get

$$T(r, g)<N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+2\bar{N}(r, g)+o\{T(r)\}, \qquad (r\notin E). \tag{11}$$

By (10) and (11) we have

$$T(r, f)+T(r, g)<2\left[N\left(r, \frac{1}{f}\right)+\bar{N}(r, f)\right]+2\left[N\left(r, \frac{1}{g}\right)+\bar{N}(r, g)\right]$$
$$+o\{T(r)\} \qquad (r\notin E)$$
$$\leqq 2[2-\delta(0, f)-\Theta(\infty, f)]T(r, f)+o\{T(r)\}$$
$$+2[2-\delta(0, g)-\Theta(\infty, g)]T(r, g), \qquad (r\notin E) \tag{12}$$

but

$$2[2-\delta(0, f)-\Theta(\infty, f)]<1,$$

and

$$2[2-\delta(0,\ g)-\Theta(\infty,\ g)]<1\ .$$

Hence from (12) we deduce that

$$T(r)=o\{T(r)\}, \qquad (r\notin E).$$

This is a contradiction.

It shows that f_j $(j=1,\ 2,\ 3)$ are linearly dependent, i.e., there exist three constants $(c_1,\ c_2,\ c_3)\neq(0,\ 0,\ 0)$ such that

$$c_1f_1+c_2f_2+c_3f_3=0\ . \tag{13}$$

If $c_1=0$, since $h\not\equiv$constant and $h\not\equiv$constant, from (13) we get

$$g=\frac{c_2}{c_3}\lambda\ ,$$

contradicting given condition $T(r,\ \lambda)=o\{T(r,\ g)\}$. Hence $c_1\neq0$. Then, combining (7) and (13) we have

$$\left(1-\frac{c_2}{c_1}\right)\frac{\lambda}{\mu}h+\left(\frac{c_3}{c_1}-1\right)\frac{g}{\mu}h=1\ , \tag{14}$$

We assert that $1-(c_2/c_1)=0$. Otherwise, then $1-(c_2/c_1)\neq0$. If $(c_3/c_1)-1\neq0$, we get by Lemma 2

$$\begin{aligned}T(r,\ g)&<N\left(r,\ \frac{1}{g}\right)+\bar{N}(r,\ g)+o\{T(r,\ g)\} &&(r\notin E)\\ &\leqq[2-\delta(0,\ g)-\Theta(\infty,\ g)]T(r,\ g)+o\{T(r,\ g)\} &&(r\notin E)\\ &\leqq\frac{1}{2}T(r,\ g)+o\{T(r,\ g)\}, &&(r\notin E)\ .\end{aligned}$$

It is impossible. If $(c_3/c_1)-1=0$, then

$$h=\frac{c_1}{c_1-c_2}\cdot\frac{\mu}{\lambda}\ ,$$

from (1) we get

$$\frac{f}{\mu w}-\frac{g}{\lambda w}\cdot\frac{c_1}{c_1-c_2}=1\ ,$$

where $w=1-(c_1/c_1-c_2)$. Here we may assume $w\neq0$, because, if $w=0$, then we have $f/\mu\equiv g/\lambda$. By Lemma 2 we have

$$\begin{aligned}T(r,\ f)&<N\left(r,\ \frac{1}{f}\right)+N\left(r,\ \frac{1}{g}\right)+\bar{N}(r,\ f)+o\{T(r,\ f)\} &&(r\notin E)\\ &\leqq[3-\delta(0,\ f)-\delta(0,\ g)-\Theta(\infty,\ f)]T(r,\ f)+o\{T(r,\ f)\}, &&(r\notin E)\end{aligned}$$

but

$$3-\delta(0, f)-\delta(0, g)-\Theta(\infty, f)<1 .$$

It is also impossible. Thus $1-(c_2/c_1)=0$. i.e.,

$$\frac{g}{\mu}h=\frac{c_1}{c_3-c_1}. \tag{15}$$

Substituting this into (1) we obtain

$$\frac{f}{\mu}=\frac{c_3}{c_3-c_1}-\frac{\lambda}{\mu}h . \tag{16}$$

It is easy to see that $c_3=0$ by Lemma 2. Hence from (15) and (16) we get, respectively;

$$f=-\lambda h, \qquad g=-\frac{\mu}{h},$$

i.e.,

$$f\cdot g=\mu\cdot\lambda .$$

This complete the proof of Theorem 1.

The proof of Theorem 2. First, by Theorem 1 we get

$$\frac{f}{\varphi_1}=\frac{g}{h_1} \tag{17}$$

or

$$f\cdot g=\varphi_1\cdot h_1 , \tag{18}$$

and

$$\frac{f}{\varphi_2}=\frac{g}{h_2} \tag{19}$$

or

$$f\cdot g=\varphi_2\cdot h_2 , \tag{20}$$

from (17) and (19) we have

$$\frac{f}{\varphi_1}=\frac{g}{h_1} \quad \text{and} \quad \frac{\varphi_1}{h_1}=\frac{\varphi_2}{h_2},$$

from (18) and (20) we have

$$f\cdot g=\varphi_1\cdot h_1 \quad \text{and} \quad \varphi_1\cdot h_1=\varphi_2\cdot h_2 .$$

On the other hand, from (17) and (20) or (18) and (19) we obtain, respectively;

$$T(r, f)=o\{T(r, f)\}, \qquad T(r, g)=o\{T(r, g)\},$$

which is a contradiction. This completes the proof of Theorem 2.

The proof of Corollary. Let

$$f=\mu_1 e^{\alpha}+\mu_2, \qquad g=\lambda_1 e^{\beta}+\lambda_2,$$

where α and β are two entire functions, μ_i and λ_i $(i=1, 2)$ are meromorphic functions, satisfying $\mu_1\not\equiv 0$, $\mu_2\not\equiv$const., $\lambda_1\not\equiv 0$ and $\lambda_2\not\equiv$const.,

$$T(r, \mu_i)=o\{T(r, e^{\alpha})\}, \qquad T(r, \lambda_i)=o\{T(r, e^{\beta})\}. \quad (i=1, 2)$$

Again let

$$f^*=\frac{f-\mu_2}{\mu_1}, \qquad g^*=\frac{g-\lambda_2}{\lambda_1},$$

then

$$f^*=e^{\alpha}, \qquad g^*=e^{\beta}.$$

Obviouly,

$$E(\infty, f^*)=E(\infty, g^*)$$

and

$$\delta(0, f^*)+\Theta(\infty, f^*)=2>\frac{3}{2},$$

$$\delta(0, g^*)+\Theta(\infty, g^*)=2>\frac{3}{2}.$$

From $E(c_i, f)=E(c_i, g)$ $(i=1, 2)$ we have

$$E(c_i-\mu_2, \mu_1 f^*)=E(c_i-\lambda_2, \lambda_1 g^*). \qquad (i=1, 2).$$

By Theorem 2 we have

(i)
$$\frac{\mu_1 f^*}{c_1-\mu_2}=\frac{\lambda_1 g^*}{c_1-\lambda_2} \tag{21}$$

and

$$c_1-\mu_2/c_1-\lambda_2=c_2-\mu_2/c_2-\lambda_2, \tag{22}$$

or

(ii)
$$\mu_1 f^* \lambda_1 g^*=(c_1-\mu_2)\cdot(c_1-\lambda_2) \tag{23}$$

and

$$(c_1-\mu_2)\cdot(c_1-\lambda_2)=(c_2-\mu_2)\cdot(c_2-\lambda_2). \tag{24}$$

If (i) holds, then from (21) and (22) we obtain

$$\mu_2=\lambda_2$$

and

$$\frac{f-\mu_2}{\mu_1}=\frac{g-\lambda_2}{\lambda_1}\cdot\frac{\lambda_1}{\mu_1},$$

i.e., $f\equiv g$.

If (ii) holds, then from (23) and (24) we obtain

$$(\mu_2+\lambda_2)=(c_1+c_2)$$

and

$$(f-\mu_2)\cdot(g-\lambda_2)=(c_1-\mu_2)\cdot(c_1-\lambda_2).$$

Hence

$$\mu_1 e^{\alpha}\cdot\lambda_1 e^{\beta}=(\lambda_2-c_2)(c_1-\lambda_2).$$

Thus

$$f=\frac{(c_2-\lambda_2)(c_1-\lambda_2)}{\lambda_1 e^{\beta}}+(c_1+c_2-\lambda_2).$$

Obviously, only letting

$$\lambda_1=\frac{h}{1-\lambda}, \qquad e^{\beta}=e^{\phi}, \qquad \lambda_2=\frac{c_1-c_2\lambda}{1-\lambda},$$

we can deduce the conjecture of C. C. Yang.

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