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HARMONIC MAPPINGS, MINIMAL AND TOTALLY GEODESIC IMMERSIONS OF COMPACT RIEMANNIAN HOMOGENEOUS SPACES INTO GRASSMANN MANIFOLDS

Dedicated to Professor Hisao Nakagawa on his sixtieth birthday

BY OSAMU IKAWA

§1. Introduction.

Let M and N be two compact connected Riemannian manifolds. A smooth mapping $F: M \to N$ is called harmonic if it is an extremal of the energy. Moreover, if harmonic mapping $F: M \to N$ is an isometric immersion, then F is a minimal immersion. An isometric immersion $F: M \to N$ is called totally geodesic if F carries every geodesic of M to a geodesic of N. A totally geodesic immersion is especially minimal. The existence and construction of minimal immersions and harmonic mappings are interesting and important problems in various situations. In the previous paper [1], we construct harmonic mappings and minimal immersions from compact Riemannian homogeneous spaces into Grassmann manifolds. In this paper, we study different construction of harmonic mappings, minimal and totally geodesic immersions of compact Riemannian homogeneous spaces into Grassmann manifolds (see Theorem A and B).

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§2. A construction of harmonic mappings and minimal immersions of compact Riemannian homogeneous spaces into Grassmann manifolds.

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and K be a closed subgroup of G with Lie algebra $\mathfrak{k}$. Take a bi-invariant Riemannian metric $\langle , \rangle$ on G and denote also by $\langle , \rangle$ the induced $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{m}=\mathfrak{k}^{\perp}$. Thus $M=(M^n, \langle , \rangle)=G/K$ is a compact Riemannian homogeneous space. The subspace $\mathfrak{m}$ of $\mathfrak{g}$ is naturally identified with the tangent space $T_o(M)$ of M at $o=\pi(e)$, where $\pi: G \to M$ is a natural projection.

Take a nontrivial real spherical representation (ρ, V) of (G, K), that is,

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(ρ, V) is a nontrivial real irreducible representation of G, and there exists a nonzero vector $v_0 \in V$ such that

$$\rho(k)v_0 = v_0 \quad \text{for each} \quad k \in K.$$

Take a G-invariant inner product $\langle , \rangle$ on V. Put

$$(2.1) \quad \begin{cases} V_0 = \boldsymbol{R}v_0, \\ V_1 = \rho(\mathfrak{m})v_0, \\ V_2 = \text{the orthogonal projection of } \operatorname{span}\{\rho(X)\rho(Y)v_0 \,;\, X, Y \in \mathfrak{m}\} \\ \qquad \text{to } (V_0 + V_1)^{\perp}, \\ \qquad \cdots \\ V_k = \text{the orthogonal projection of } \operatorname{span}\{\rho(X_1)\cdots\rho(X_k)v_0 \,;\, X_1, \cdots, X_k \in \mathfrak{m}\} \\ \qquad \text{to } (V_0 + \cdots + V_{k-1})^{\perp}, \\ \qquad \cdots, \end{cases}$$

where we denote the differential representation of ρ of G by the same symbol ρ. Since ρ is irreducible, there exists an integer m such that

$$V = \sum_{i=0}^{m} V_i \quad \text{(the orthogonal direct sum of } K\text{-invariant subspaces),}$$

$$V_i \neq \{0\} \qquad \text{for } 0 \leqq i \leqq m.$$

Since ρ is nontrivial, we get $m \geqq 1$. Put $S_m = \{0, \cdots, m\}$. For subsets $P(\neq \emptyset)$, $Q(\neq \emptyset)$ with $S_m = P \cup Q$ (disjoint union), put $V_P = \sum_{p \in P} V_p$, $V_Q = \sum_{q \in Q} V_q$, $a = \dim V_P$, $b = \dim V_Q$. Then $V + V_P + V_Q$ (orthogonal direct sum of K-invariant subspaces). Put

$$(2.2) \qquad F\colon M = G/K \longrightarrow G_{a,b}(\boldsymbol{R}) = SO(a+b)/S(O(a) \times O(b))\,;$$

$$gK \longmapsto \rho(g)V_P = \rho(g)S(O(a) \times O(b)).$$

We call $G_{a,b}(\boldsymbol{R})$ the Grassmann manifold consisting of all a-dimensional subspaces in V.

We explain that F is $\boldsymbol{R}$-full. Let V'_P and V'_Q be subspaces of V_P and V_Q, respectively. Put $a' = \dim V'_P$ and $b' = \dim V'_Q$. Then $SO(a'+b')$ is considered as a closed subgroup of $SO(a+b)$ in a natural manner. So $G_{a',b'}(\boldsymbol{R})$ is a totally geodesic submanifold of $G_{a,b}(\boldsymbol{R})$. The mapping F is said to be $\boldsymbol{R}$-full when the image $F(M)$ is not contained in these totally geodesic submanifolds $G_{a',b'}(\boldsymbol{R})$ with $a' + b' < a + b$. From the irreducibility of (ρ, V), the mapping F defined in (2.2) is clearly $\boldsymbol{R}$-full.

We prove the following theorem.

THEOREM A. *F is a nonconstant $\boldsymbol{R}$-full equivariant harmonic mapping. If the isotropy action of K is irreducible, then F is a minimal immersion. In particular, if we put $P=\{0\}$, $Q=\{1, \cdots, m\}$, then F is a minimal immersion of M into a projective space.*

In order to prove this, we prepare a few lemmas. First, we note that $\rho(\mathfrak{m})V_k \subset V_0+\cdots+V_{k+1}$ for $k=0, \cdots, m$, where we put $V_{m+1}=\{0\}$.

LEMMA 2.1.

$$\rho(\mathfrak{m})V_k \subset V_{k-1}+V_k+V_{k+1} \text{ for } k=0, 1, \cdots, m, \text{ where we put } V_{-1}=\{0\}.$$

Proof. We prove this by induction on k. It is clear when $k=0$. We assume that this lemma holds until k. For $0\leqq i\leqq k-1$, by the hypothesis of induction, we get $\langle\rho(\mathfrak{m})V_{k+1}, V_i\rangle=\langle V_{k+1}, \rho(\mathfrak{m})V_i\rangle=\{0\}$. Q. E. D.

We denote an orthonormal basis of $\mathfrak{m}$ and $\mathfrak{k}$ by $\{E_i\}_{1\leq i\leq n}$ and $\{E_{n+j}\}_{1\leq j\leq l}$, respectively. We remark that the Casimir operator

$$C=\sum_{i=1}^{n+l}\rho(E_i)^2$$

of ρ is a scalar operator because C is a G-invariant symmetric transformation and ρ is irreducible. For $v\in V$, we denote the V_k-component of v by v_{V_k}.

LEMMA 2.2. $\sum_{i=1}^{n}\rho(E_i)(\rho(E_i)v_k)_{V_{k+1}}\in V_k+V_{k+1}$ *for each* $v_k\in V_k$

Proof. We have

$$V_k\ni Cv_k=\sum_{i=1}^{n+l}\rho(E_i)(\rho(E_i)v_k)_{V_k}+\sum_{i=1}^{n+l}\rho(E_i)(\rho(E_i)v_k)_{V_{k-1}}+\sum_{i=1}^{n+l}\rho(E_i)(\rho(E_i)v_k)_{V_{k+1}}.$$

Hence we have by Lemma 2.1

$$\sum_{i=1}^{n+l}\rho(E_i)(\rho(E_i)v_k)_{V_{k+1}}\in V_k+V_{k+1}.$$

Since V_k is K-invariant, we get the conclusion. Q. E. D.

The Lie algebra $\mathfrak{u}$ of $SO(a+b)$ acts on V, naturally. Put $\mathfrak{l}=\mathrm{Lie}(S(O(a)\times O(b)))$ and $\mathfrak{p}=\{T\in\mathfrak{u};\ TV_P\subset V_Q,\ TV_Q\subset V_P\}$. Then $\mathfrak{u}=\mathfrak{l}+\mathfrak{p}$ is the canonical decomposition of $\mathfrak{u}$. For $T\in\mathfrak{u}$, we denote the $\mathfrak{p}$(resp. $\mathfrak{l}$)-component of T by $T_{\mathfrak{p}}$(resp. $T_{\mathfrak{l}}$).

LEMMA 2.3. $\sum_{i=1}^{n}(\rho(E_i)_{\mathfrak{l}}\rho(E_i)_{\mathfrak{p}}+\rho(E_i)_{\mathfrak{p}}\rho(E_i)_{\mathfrak{l}})=0$.

Proof. For each $v_k\in V_k$, we have

$$\sum_{i=1}^{n}\rho(E_i)^2v_k=Cv_k-\sum_{j=1}^{l}\rho(E_{n+j})^2v_k\in V_k.$$

Since

$$\sum_{i=1}^{n}\rho(E_i)^2=\sum_{i=1}^{n}(\rho(E_i)_{\mathfrak{k}}\rho(E_i)_{\mathfrak{p}}+\rho(E_i)_{\mathfrak{p}}\rho(E_i)_{\mathfrak{k}})+\sum_{i=1}^{n}(\rho(E_i)_{\mathfrak{k}}\rho(E_i)_{\mathfrak{k}}+\rho(E_i)_{\mathfrak{p}}\rho(E_i)_{\mathfrak{p}}),$$

we get the conclusion. Q. E. D.

Proof of Theorem A. Let $H\in\mathfrak{p}$ denote the tension of F at o. Then by homogeneity F is harmonic if and only if $H=0$. By (4.1) in [1], we have $H=\sum_{i=1}^{n}[\rho(E_i)_{\mathfrak{k}},\ \rho(E_i)_{\mathfrak{p}}]$. From Lemma 2.3, we have

$$H=2\sum_{i=1}^{n}\rho(E_i)_{\mathfrak{k}}\rho(E_i)_{\mathfrak{p}}=-2\sum_{i=1}^{n}\rho(E_i)_{\mathfrak{p}}\rho(E_i)_{\mathfrak{k}}.$$

If $0,\ 1\in P$ or $0,\ 1\in Q$, then we have $Hv_0=0$ by $\rho(E_i)_{\mathfrak{p}}v_0=0$. If $0\in P,\ 1\in Q$ or $0\in Q,\ 1\in P$, then we have $Hv_0=0$ by $\rho(E_i)_{\mathfrak{k}}v_0=0$. Hence we have $H|V_0=0$. We assume that $H|(V_0+\cdots+V_j)=0$. We will prove $H|V_{j+1}=0$. Clearly, we have $HV_{j+1}\subset\sum_{i=0}^{j+3}V_i$. By the hypothesis, we have

$$0=\left\langle H\sum_{i=0}^{j}V_i,\ V_{j+1}\right\rangle=-\left\langle\sum_{i=0}^{j}V_i,\ HV_{j+1}\right\rangle.$$

Hence we have $HV_{j+1}\subset V_{j+1}+V_{j+2}+V_{j+3}$. We define two maps $\chi_P,\ \chi_Q:S_m\to\{0,\ 1\}$ as follows:

$$\chi_P(k)=\begin{cases}1 & (k\in P)\\ 0 & (k\in Q),\end{cases}\qquad \chi_Q(k)=\begin{cases}1 & (k\in Q)\\ 0 & (k\in P).\end{cases}$$

For each $v_{j+1}\in V_{j+1}$, we have by Lemma 2.1 and the hypothesis of induction

$$\begin{aligned}
&\Big(\sum_{i=1}^{n}\rho(E_i)_{\mathfrak{k}}\rho(E_i)_{\mathfrak{p}}\Big)v_{j+1}\\
&=\chi_P(j+1)\sum_{k=j}^{j+2}\chi_Q(k)\sum_{i=1}^{n}\sum_{l=k-1}^{k+1}\chi_Q(l)(\rho(E_i)(\rho(E_i)v_{j+1})_{V_k})_{V_l}\\
&\quad+\chi_Q(j+1)\sum_{k=j}^{j+2}\chi_P(k)\sum_{i=1}^{n}\sum_{l=k-1}^{k+1}\chi_P(l)(\rho(E_i)(\rho(E_i)v_{j+1})_{V_k})_{V_l}\\
&=\chi_P(j+1)\chi_Q(j+2)\sum_{i=1}^{n}(\rho(E_i)(\rho(E_i)v_{j+1})_{V_{j+2}})_{V_{j+2}}\\
&\quad+\chi_P(j+1)\chi_Q(j+2)\chi_Q(j+3)\sum_{i=1}^{n}(\rho(E_i)(\rho(E_i)v_{j+1})_{V_{j+2}})_{V_{j+3}}\\
&\quad+\chi_Q(j+1)\chi_P(j+2)\sum_{i=1}^{n}(\rho(E_i)(\rho(E_i)v_{j+1})_{V_{j+2}})_{V_{j+2}}\\
&\quad+\chi_Q(j+1)\chi_P(j+2)\chi_P(j+3)\sum_{i=1}^{n}(\rho(E_i)(\rho(E_i)v_{j+1})_{V_{j+2}})_{V_{j+3}},
\end{aligned}$$

$$\left(\sum_{i=1}^{n} \rho(E_i)_{\mathfrak{p}} \rho(E_i)_{\mathfrak{k}}\right) v_{j+1}$$

$$= \chi_P(j+1) \sum_{k=j}^{j+2} \chi_P(k) \sum_{i=1}^{n} \sum_{l=k-1}^{k+1} \chi_Q(l) (\rho(E_i)(\rho(E_i) v_{j+1})_{V_k})_{V_l}$$

$$+ \chi_Q(j+1) \sum_{k=j}^{j+2} \chi_Q(k) \sum_{i=1}^{n} \sum_{l=k-1}^{k+1} \chi_P(l) (\rho(E_i)(\rho(E_i) v_{j+1})_{V_k})_{V_l}$$

$$= \chi_P(j+1) \chi_Q(j+2) \sum_{i=1}^{n} (\rho(E_i)(\rho(E_i) v_{j+1})_{V_{j+1}})_{V_{j+2}}$$

$$+ \chi_P(j+1) \chi_P(j+2) \chi_Q(j+3) \sum_{i=1}^{n} (\rho(E_i)(\rho(E_i) v_{j+1})_{V_{j+2}})_{V_{j+3}}$$

$$+ \chi_Q(j+1) \chi_P(j+2) \sum_{i=1}^{n} (\rho(E_i)(\rho(E_i) v_{j+1})_{V_{j+1}})_{V_{j+2}}$$

$$+ \chi_Q(j+1) \chi_Q(j+2) \chi_P(j+3) \sum_{i=1}^{n} (\rho(E_i)(\rho(E_i) v_{j+1})_{V_{j+2}})_{V_{j+3}}.$$

Hence, if $j+1,\ j+2 \in P$ or $j+1,\ j+2 \in Q$, then we have $(\sum_{i=1}^{n} \rho(E_i)_{\mathfrak{k}} \rho(E_i)_{\mathfrak{p}}) v_{j+1} = 0$. If $j+1 \in P$, $j+2 \in Q$ or $j+1 \in Q$, $j+2 \in P$, then we have by Lemma 2.2

$$\left(\sum_{i=1}^{n} \rho(E_i)_{\mathfrak{p}} \rho(E_i)_{\mathfrak{k}}\right) v_{j+1} = \sum_{i=1}^{n} (\rho(E_i)(\rho(E_i) v_{j+1})_{V_{j+1}})_{V_{j+2}} \in V_{j+2},$$

$$\left(\sum_{i=1}^{n} \rho(E_i)_{\mathfrak{k}} \rho(E_i)_{\mathfrak{p}}\right) v_{j+1} = \sum_{i=1}^{n} (\rho(E_i)(\rho(E_i) v_{j+1})_{V_{j+2}})_{V_{j+2}} = 0.$$

Hence we have $H|V_{j+1}=0$. Therefore F is a nonconstant harmonic mapping.

We show that F is an isometric immersion if the isotropy action of K is irreducible. We define a symmetric linear transformation A of $T_o(M)$ by

$$\langle X, AY \rangle = (F_* X, F_* Y) \qquad \text{for} \quad X, Y \in T_o(M),$$

where $(\ ,\ )$ denote a $SO(a+b)$-invariant Riemannian metric on $G_{a,b}(\boldsymbol{R})$. Since A is a K-homomorphism, A is a scalar operator by the irreducibility of (G, K). The scalar is clearly nonnegative. So if F were not an isometric (more precisely, homothetic) immersion, then $F_*=0$. This means that V_P and V_Q are G-invariant (see (2.1) in [1]). This is a contradiction. Hence F is an isometric immersion. Q. E. D.

Remark 2.4. Put

$$F:\ M=G/K \longrightarrow S^{N-1} = \{v \in V\ ;\ \|v\| = \|v_0\|\} (\subset V = \boldsymbol{R}^N)\ ;\ gK \longmapsto \rho(g) v_0.$$

Then we can prove that F is a harmonic mapping into a sphere in the same way of the proof of this Theorem (see [2] and [3], Proposition 8.1, p. 21). ∎

Remark 2.5. Let (ρ, V) be a complex (resp. quaternion) spherical representation of (G, K). Put

$$V_K=\{v\in V\,;\ \rho(k)v=v \text{ for each } k\in K\}(\neq\{0\}).$$

If there exists a nonzero vector $v_0\in V_K$ such that

$$\langle\rho(g)v_0, v_0\rangle\in \boldsymbol{R} \qquad \text{for each } g\in G, \tag{2.3}$$

then we can construct a harmonic mapping from M into a complex (resp. quaternion) Grassmann manifold in the same way of Theorem A. Condition (2.3) means

PROPOSITION 2.6. *A complex (resp. quaternoin) spherical representation* (ρ, V) *is satisfied with* (2.3) *if and only if there exists a real spherical representation* (τ, W) *of* (G, K) *such that*

$$(\rho, V)=(\tau, W)^{C}(\textit{resp. } (\tau, W)^{H}), \tag{2.4}$$

where $(\tau, W)^C$ *(resp.* $(\tau, W)^H$*) denote the complex (resp. quaternion) representation of* G *obtained by extension of the coefficient field of* (τ, W) *to* $\boldsymbol{C}$ *(resp.* $\boldsymbol{H}$*).*

Proof. Clearly (2.4) implies (2.3). Conversely we assume (2.3). If we put

$$W=\boldsymbol{R}\text{-linear span of } \{\rho(g)v_0\,;\ g\in G\},$$

then (2.4) is concluded. Q. E. D.

If (G, K) is a compact symmetric pair (see § 3 for definition) of rank one, then every complex (or quaternion) spherical representation (ρ, V) of (G, K) is satisfied with (2.3) (see [3], p. 25, Cor. 8.2 and [1], § 3, Lemma 3.3 (3)).

We prepare a few lemmas for use later (§ 3).

LEMMA 2.7. $\rho(X_1)\cdots\rho(X_k)v_0\equiv\rho(X_{\tau(1)})\cdots\rho(X_{\tau(k)})v_0 \pmod{V_0+\cdots+V_{k-1}}$ *for* $X_1,\cdots,X_k\in\mathfrak{m}$, $\tau\in\mathfrak{S}_k$, *where we denote the symmetric group of degree* k *by* $\mathfrak{S}_k$.

Proof. We have

$$\begin{aligned}&\rho(X_1)\cdots\rho(X_i)\rho(X_{i+1})\cdots\rho(X_k)v_0\\ &\quad=\rho(X_1)\cdots\rho(X_{i-1})\rho(X_{i+1})\rho(X_i)\rho(X_{i+2})\cdots\rho(X_k)v_0\\ &\qquad+\rho(X_1)\cdots\rho(X_{i-1})\rho([X_i, X_{i+1}])\rho(X_{i+2})\cdots\rho(X_k)v_0.\end{aligned}$$

Hence we get the conclusion Q. E. D.

LEMMA 2.8.

V_k *=the orthogonal projection of span*$\{\rho(X)^k v_0\,;\ X\in\mathfrak{m}\}$ *to* $(V_0+\cdots+V_{k-1})^{\perp}$.

Proof. We prove this by induction on k. It is clear when $k=0$. We assume that this lemma holds for k. From this, we get

$$V_{k+1}=\text{the orthogonal projection of } \mathrm{span}\{\rho(X)\rho(Y)^k v_0\,;\ X,\ Y\in\mathfrak{m}\}$$
$$\text{to } (V_0+\cdots+V_k)^{\perp}.$$

From Lemma 2.7, we have

$$\rho(X+lY)^{k+1}v_0\equiv\sum_{s=0}^{k+1}\binom{k+1}{s}l^s\rho(X)^{k+1-s}\rho(Y)^s v_0 \pmod{V_0+\cdots+V_k}$$

for $l=1, 2, \cdots, k+2$. By the formula of Van der Monde, we have

$$\det\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2^0 & 2^1 & \cdots & 2^{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ (k+2)^0 & (k+2)^1 & \cdots & (k+2)^{k+1}\end{bmatrix}=\prod_{1\leq i<j\leq k+2}(j-i)\neq 0\,.$$

Hence the vector $\rho(X)\rho(Y)^k v_0$ is a linear combination of $\rho(X+Y)^{k+1}v_0, \cdots,$ $\rho(X+(k+2)Y)^{k+1}v_0 \pmod{V_0+\cdots+V_k}$. Q. E. D.

§ 3. A construction of totally geodesic immersions of compact irreducible Riemannian symmetric spaces into Grassmann manifolds.

Let (G, K) be a compact irreducible symmetric pair, that is, G is a compact connected Lie group with Lie algebra $\mathfrak{g}$, K is a closed subgroup of G with Lie algebra $\mathfrak{k}$, and there exists an involutive automorphism θ of G such that K lies between the identity component $(K_\theta)_0$ of K_θ and $K_\theta=\{g\in G\,;\ \theta(g)=g\}$. And the adjoint action of K on $\mathfrak{m}$ is irreducible.

An $\mathrm{Ad}(G)$ and θ-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ naturally induces a G-invariant Riemannian metric on $M=G/K$. M is a compact Riemanniau symmetric space with respect to the G-invariant Riemannian metric. Since θ is an involutive automorphism, we have a canonical orthogonal decomposition of $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{m}\,.$$

Put F as in § 2 with $P=\{\text{even}\}$, $Q=\{\text{odd}\}$, then we have the following theorem.

THEOREM B. *F is a totally geodesic immersion.*

In order to prove this, we prepar the following lemma.

LEMMA 3.1.

$\rho(\mathfrak{m})V_k\subset V_{k-1}+V_{k+1}$ for $k=0, 1, \cdots, m$, where we put $V_{-1}=V_{m+1}=\{0\}$.

Proof. We prove this by induction on k. It is clear when $k=0$. We assume that this lemma holds until k. From Lemma 2.1, it is sufficient to prove $\langle\rho(\mathfrak{m})V_{k+1}, V_{k+1}\rangle=\{0\}$. When k is even, put $k=2l$. For $X\in\mathfrak{m}$, by the hypothesis of induction, we get

$$\rho(X)v_0\in V_1,$$
$$\rho(X)^2v_0\in V_0+V_2,$$
$$\cdots$$
$$\rho(X)^{2l}v_0\in V_0+V_2+\cdots+V_{2l},$$
$$\rho(X)^{2l+1}v_0\in V_1+V_3+\cdots+V_{2l+1}.$$

For $Y\in\mathfrak{m}$, we get

$$\rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2l+1}}=\rho(Y)\rho(X)^{2l+1}v_0-\sum_{s=0}^{l-1}\rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2s+1}}.$$

By the hypothesis of induction, we get

$$\sum_{s=0}^{l-1}\rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2s+1}}\in V_0+V_2+\cdots+V_{2l}.$$

For each $Z\in\mathfrak{m}$, we have

$$\begin{aligned}&\langle\rho(Y)\rho(X)^{2l+1}v_0,\ \rho(Z)^{2l+1}v_0\rangle\\&=\langle\rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2l+1}}+\sum_{s=0}^{l-1}\rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2s+1}},\ \sum_{t=0}^{l}(\rho(Z)^{2l+1}v_0)_{V_{2t+1}}\rangle\\&=\langle\rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2l+1}},\ \sum_{t=0}^{l}(\rho(Z)^{2l+1}v_0)_{V_{2t+1}}\rangle\\&=\langle\rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2l+1}},\ (\rho(z)^{2l+1}v_0)_{V_{2l+1}}\rangle.\end{aligned}$$

From Lemma 2.8, it is sufficient to prove

$$\langle\rho(Y)\rho(X)^{2l+1}v_0,\ \rho(Z)^{2l+1}v_0\rangle=0 \qquad \text{for each } X, Y, Z\in\mathfrak{m}.$$

For $X_1, \cdots, X_{2l+2}, Y_1, \cdots, Y_{2l+1}\in\mathfrak{m}$, $\sigma\in\mathfrak{S}_{2l+2}$, by the hypothesis of induction and $[\mathfrak{m}, \mathfrak{m}]\subset\mathfrak{k}$, we have

$$\begin{aligned}&\langle\rho(X_1)\cdots\rho(X_{2l+2})v_0,\ \rho(Y_1)\cdots\rho(Y_{2l+1})v_0\rangle\\&\quad=\langle\rho(X_{\sigma(1)})\cdots\rho(X_{\sigma(2l+2)})v_0,\ \rho(Y_1)\cdots\rho(Y_{2l+1})v_0\rangle.\end{aligned}$$

Hence we have

$$\begin{aligned}\langle\rho(W)^{2l+2}v_0,\ \rho(Z)^{2l+l}v_0\rangle&=\langle\rho(W)^{2l+1}\rho(Z)v_0,\ \rho(Z)^{2l}\rho(W)v_0\rangle\\&=\langle\rho(W)^{2l}\rho(Z)^2v_0,\ \rho(Z)^{2l-1}\rho(W)^2v_0\rangle\\&\cdots\\&=\langle\rho(W)^{l+1}\rho(Z)^{l+1}v_0,\ \rho(Z)^l\rho(W)^{l+1}v_0\rangle\end{aligned}$$

$$=\langle\rho(Z)^{l+1}\rho(W)^{l+1}v_0,\ \rho(Z)^l\rho(W)^{l+1}v_0\rangle$$
$$=0 \quad \text{for each } W,\ Z\in\mathfrak{m}.$$

Hence we have

$$0=\langle\rho(Y+mX)^{2l+2}v_0,\ \rho(Z)^{2l+1}v_0\rangle$$
$$=\sum_{i=0}^{2l+2}\binom{2l+2}{i}m^i\langle\rho(Y)^{2l+2-i}\rho(X)^i v_0,\ \rho(Z)^{2l+1}v_0\rangle$$
$$\text{for } X,\ Y,\ Z\in\mathfrak{m},\ m=1,\cdots,2l+3.$$

By the formula of Van der Monde, we have

$$\det\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2^0 & 2^1 & \cdots & 2^{2l+2} \\ \vdots & \vdots & \ddots & \vdots \\ (2l+3)^0 & (2l+3)^1 & \cdots & (2l+3)^{2l+2} \end{bmatrix}=\prod_{1\leq i<j\leq 2l+3}(j-i)\neq 0.$$

Hence we have $\langle\rho(Y)\rho(X)^{2l+1}v_0,\ \rho(Z)^{2l+1}v_0\rangle=0$. When k is odd, we can prove this in the same way. Q. E. D.

Proof of Theorem B. We have $\rho(\mathfrak{m})\subset\mathfrak{p}$ by Lemma 3.1. Hence F is a totally geodesic immersion. Q. E. D.

Remark 3.2. Let (ρ, V) be a complex (resp. quaternion) spherical representation satisfied with (2.3). Then we can construct a totally geodesic immersion of M into a complex (resp. quaternion) Grassmann manifold in the same way of Theorem B.

The next example is not contained in Theorem 3.1, [1].

Example. $(G, K)=(SU(n), SO(n))$ $(n\geqq 3)$.

Since G acts on $\boldsymbol{C}^n$ naturally, G acts on a complex vector space $W=(\sigma, W)$ $=S^2(\boldsymbol{C}^n)=\text{span}\{u\cdot v=1/2(u\otimes v+v\otimes u)\,;\ u,\ v\in\boldsymbol{C}^n\}$. Let $\{e_i\}_{1\leq i\leq n}$ denote the canonical basis of $\boldsymbol{C}^n$. Put $v_0=\sum_{i=1}^n e_i^2\in W$. Then we have $\sigma(k)v_0=v_0$ for each $k\in K$. Put $(\rho, V)=(\sigma, W)_{\boldsymbol{R}}$. Then (ρ, V) is a nontrivial real spherical representation of (G, K) (see Lemma 3.5). The canonical inner product on $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$ natually induces a G-invariant inner product on V. We define K-invariant subspaces V_k as in (2.1). Then we have

$$V_0=\boldsymbol{R}v_0,$$
$$V_1=\sum_{1\leq i<j\leq n}\boldsymbol{R}\sqrt{-1}e_i\cdot e_j+\left\{\sum_{i=1}^n x_i\sqrt{-1}e_i^2;\ x_i\in\boldsymbol{R}\ (1\leqq i\leqq n),\ \sum_{i=1}^n x_i=0\right\},$$
$$V_2=\sum_{1\leq i<j\leq n}\boldsymbol{R}e_i\cdot e_j+\left\{\sum_{i=1}^n x_i e_i^2;\ x_i\in\boldsymbol{R}\ (1\leqq i\leqq n),\ \sum_{i=1}^n x_i=0\right\},$$

$$V_3 = \mathbf{R}\sqrt{-1}v_0,$$

$$V = \sum_{i=0}^{3} V_i.$$

Put F as in (2.2), then F is a minimal immersion. Since $V_0 \cong V_3$, $V_1 \cong V_2$ (K-isomorphic), this example is not contained in Theorem 3.1, [1].

In order to prove the irreducibility of (ρ, V), we prepare a few lemmas.

LEMMA 3.3. *Let (σ, W) be a complex irreducible representation of a compact connected Lie group G. If there exists a weight λ of (σ, W) such that $-\lambda$ is not a weight of (σ, W), then $(\sigma, W)_{\mathbf{R}}$ is a real irreducible representation of G.*

Proof. If $(\sigma, W)_{\mathbf{R}}$ were not irreducible, then there exists a real representation (ρ, V) of G such that $(\sigma, W) = (\rho, V)^{\mathbf{C}}$ by Lemma 2.2, [1]. Let J denote the conjugation of W with respect to V. Then J is a conjugate G-linear mapping with $J^2 = 1$. Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$. Let v_λ be a nonzero weight vector of λ, that is,

$$\rho(H)v_\lambda = \sqrt{-1}\lambda(H)v_\lambda \qquad \text{for each} \quad H \in \mathfrak{t}.$$

Since J is conjugate G-linear, we have

$$\rho(H)Jv_\lambda = -\sqrt{-1}\lambda(H)Jv_\lambda \qquad \text{for each} \quad H \in \mathfrak{t}.$$

Since $-\lambda$ is not a weight, we have $Jv_\lambda = 0$. Since $J^2 = 0$, this is a contradiction. Q. E. D.

LEMMA 3.4. *(σ, W) is a complex irreducible representation of G.*

Proof. We first let E_{ij} denote the matrix, whose r-th row and s-th column are given by $\delta_{ir}\delta_{js}$, i.e., E_{ij} has a 1 in the i-th row and j-th column and zeros elsewhere.

It is sufficient to prove that the complexification $\mathfrak{sl}(n, \mathbf{C})$ of $\mathfrak{su}(n)$ acts on W irreducibly. Suppose $W_0 (\neq \{0\})$ is an $\mathfrak{sl}(n, \mathbf{C})$-invariant subspace of W. In order to prove $W_0 = W$, first we show $v_0 \in W_0$. Let $v = \sum_{1 \leq k \leq l \leq n} a_{kl} e_k \cdot e_l \in W_0$. Put $i = \min\{k ; a_{kl} \neq 0\}$, $j = \min\{l ; a_{il} \neq 0\}$. We may assume $(i, j) \neq (n, n)$. If $i = j (< n)$, then we have $W_0 \ni \sigma(E_{ni})^2 v = 2a_{ii}e_n^2$. If $i < j$, then we have $W_0 \ni \sigma(E_{ni})v = \sum_{j \leq l \leq n} a_{il} e_l \cdot e_n$. Hence, if $i < j = n$, then we have $W_0 \ni a_{in}e_n^2$. If $i < j < n$, then we have $W_0 \ni \sigma(E_{nj})\sigma(E_{ni})v = a_{ij}e_n^2$. Hence we get $e_n^2 \in W_0$. For $1 \leqq i \leqq n-1$, we have $W_0 \ni \sigma(E_{in})^2 e_n^2 = 2e_i^2$. Hence we have $v_0 \in W_0$. Since $W = \operatorname{span}\{\rho(G)v_0\}$, we have $W_0 = W$. Q. E. D.

LEMMA 3.5. *(ρ, V) is a real irreducible representation of G.*

Proof. Put

$$T=S(\underbrace{U(1)\times\cdots\times U(1)}_{n\text{-times}})$$

and

$$\mathfrak{t}=\left\{\sqrt{-1}\operatorname{diag}\{x_1, \cdots, x_n\}\ ;\ x_i\in \boldsymbol{R}\ (1\leqq i\leqq n),\ \sum_{i=1}^{n} x_i=0\right\}.$$

Then T is a maximal torus of G with Lie algebra $\mathfrak{t}$. For $H=\sqrt{-1}\operatorname{diag}\{x_1, \cdots, x_n\}\in\mathfrak{t}$, we have $\sigma(H)(e_i\cdot e_j)=\sqrt{-1}(x_i+x_j)e_i\cdot e_j$. Since $n\geqq 3$, this shows that (ρ, V) is irreducible by Lemma 3.3 and 3.4. Q.E.D.

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