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NORMAL OPERATORS CONSTRUCTED FROM GENERALIZED HARMONIC MEASURES ON OPEN RIEMANN SURFACES

Dedicated to Professor Nobuyuki Suita on his 60th birthday

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Introduction.

Let R be an open Riemann surface and V be a union of a finite number of regular subregions in R with disjoint closures. We assume that $R-\bar{V}$ is connected. Denote by $C^{\omega}(\partial V)$ the space of real analytic functions on ∂V and by $H(R-V)$ the space of harmonic functions on $R-V$. A linear operator L from $C^{\omega}(\partial V)$ to $H(R-V)$ is called a normal operator if L satisfies the following conditions:

$$Lf|_{\partial V}=f,$$

$$\min_{\partial V} f \leqq Lf \leqq \max_{\partial V} f,$$

$$\int_{\partial V} {}^*dLf=0.$$

The notion of normal operators was introduced by L. Sario [13]. He constructed two normal operators L_0 and L_1. Here we are specially concerned with L_1-operator. If R is a compact bordered surface with smooth boundary, $L_1 f$ is characterized by the following additional properties:

$$L_1 f=\text{constant on } \beta_j,$$

$$\int_{\beta_j} {}^*dL_1 f=0,$$

where β_j are the boundary components of R. For a general open Riemann surface R, $L_1 f$ is defined as $\lim_{n\to\infty} L_1^{R_n} f$, where $\{R_n\}$ is a canonical exhaustion of R and $L_1^{R_n}$ is the L_1-operator from $C^{\omega}(\partial V)$ to $H(R_n-V)$.

Let $\Gamma_h(R)$ be the Hilbert space of real square integrable harmonic differentials on R and $\Gamma_{hse}(R)$ be the space of semiexact differentials in $\Gamma_h(R)$. Let us denote by $\Gamma_{hm}(R)$ the orthogonal complement of ${}^*\Gamma_{hse}(R)$ in $L_h(R)$. Then $L_1 f$

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is characterized by the following properties:

There exist a harmonic function u_{hm} on R with $du_{hm} \in \Gamma_{hm}(R)$ and a Dirichlet potential p on R such that

$$L_1 f = u_{hm} + p$$

on $R-V$ and

$$\int_c {}^*dL_1 f = 0$$

for all dividing cycles $c = \partial\Omega$ with $\Omega \subset R-V$.

In [7], we introduced $\Gamma_{\widehat{hm}}$ the space generated by the differentials of generalized harmonic measures and Γ_{hwe} the space of harmonic differentials which have vanishing periods along almost all weakly dividing cycles.

In the present paper we construct a normal operator $\hat{L}_1$, which is a generalization of L_1-operator. In contrast with $L_1 f$, $\hat{L}_1 f$ is characterized by the following properties:

There exist a harmonic function $u_{\widehat{hm}}$ on R with $du_{\widehat{hm}} \in \Gamma_{\widehat{hm}}(R)$ and a Dirichlet potential p on R such that

$$\hat{L}_1 f = u_{\widehat{hm}} + p$$

on $R-V$ and

$$\int_{\hat{c}} {}^*d\hat{L}_1 f = 0$$

for almost all weakly dividing cycles $\hat{c} = \partial G$ with $G \subset R-V$.

Roughly speaking $\hat{L}_1 f$ takes a constant value on each connected component of the Royden harmonic boundary of R and ${}^*d\hat{L}_1 f$ has vanishing period along cycles dividing the components of the Royden harmonic boundary.

First, we shall define a finite partition (P) of the Royden harmonic boundary and define the subspaces $(P)\Gamma_{\widehat{hm}}(R)$ and $(P)\Gamma_{hwe}(R)$ of $\Gamma_h(R)$. Further we shall define periods of a differential along components of the harmonic boundary.

Next, we construct $(P)\hat{L}_1$-operator and $\hat{L}_1$-operator. We also study an extremal property of $\hat{L}_1$-operator.

Finally, we shall introduce a modulus function obtained from $\hat{L}_1$-operator and give an example related to the topic.

1. Preliminaries.

Let R be an open Riemann surface and $\Gamma = \Gamma(R)$ the Hilbert space of real square integrable differentials on R (cf. [2]). For $\omega_1, \omega_2 \in \Gamma(R)$, $(\omega_1, \omega_2)_R = \int_R \omega_1 \wedge {}^*\omega_2$ denotes the inner product of ω_1, ω_2. where ${}^*\omega$ is the conjugate differential of ω and $\|\omega\|_R$ denotes the norm of ω on R.

We use the notation $|\omega|$ for the density $\sqrt{a^2+b^2}|dz|$ if $\omega = adx + bdy$ locally. For the sake of convenience we recall some definitions of subspaces of Γ used below. Let Γ_e be the space of exact differentials in Γ and Γ_{e0} be the closure of differentials of C^1-functions with compact supports. Let Γ_h the space of harmonic differentials in Γ, Γ_{hse} be the space of semiexact differentials in Γ_h and $\Gamma_{he}=\Gamma_h\cap\Gamma_e$. We denote by Γ_{hm} the orthogonal complement of $*\Gamma_{hse}$ in Γ_h, where $*\Gamma_x$ is the class of differentials conjugate to those in Γ_x. Then the following orthogonal decompositions are well known:

$$\Gamma=\Gamma_h+\Gamma_{e0}+*\Gamma_{e0},$$

$$\Gamma_e=\Gamma_{he}+\Gamma_{e0}.$$

Let $D(R)$ be the class of real continuous Dirichlet functions on R and $BD(R)$ be the class of bounded functions in $D(R)$ (cf. [3], [14]). Let $HD(R)$ (resp. $HBD(R)$) be the class of harmonic functions in $D(R)$ (resp. $BD(R)$) and $D_0(R)$ (resp. $BD_0(R)$) be the class of potentials in $D(R)$ (resp. $BD(R)$). Since $dD_0=\{df\,;\ f\in D_0\}\subset\Gamma_{e0}$, we have $(\sigma, dp)_R=0$ for any $\sigma\in\Gamma_h(R)$ and $p\in D_0(R)$. The class $BD(R)$ forms an algebra and the class $D(R)$ has the following lattice property; if $f, g\in D(R)$ then $f\cup g=\max(f, g)$ and $f\cap g=\min(f, g)$ belong to $D(R)$.

Let R^* be the Royden compactification of R and Δ the (Royden) harmonic boundary of R. Every function f in $D(R)$ can be extended continuously to R^*. Since the extension of f is unique, we may use the same notation f for the extension.

We know that $BD(R)$ enjoys the following Urysohn's property. That is, for any two non-empty disjoint compact sets K_1, K_2 in R^* and two real values a_1, a_2, there is a function f in $BD(R)$ such that $f=a_i$ on $K_i (i=1, 2)$ and $\min(a_1, a_2)\leqq f\leqq\max(a_1, a_2)$.

We use the following lemmas ([14]) in the sequal.

LEMMA 1.1. *Let $\{f_n\}$ be a sequence of functions in $BD_0(R)$ and f a bounded function on R. If $\|df_n\|_R$ is uniformly bounded and $\{f_n\}$ converges to f uniformly on every compact subset of R then $f\in BD_0(R)$.*

LEMMA 1.2. *A BD-function (resp. D-function) f on R belongs to $BD_0(R)$ (resp. $D_0(R)$) if and only if $f=0$ on Δ.*

LEMMA 1.3. *Any BD-function (resp. D-function) f on R can be uniquely decomposed into the form $f=u+p$, where $u\in HBD(R)$ (resp. $HD(R)$) and $p\in BD_0(R)$ (resp. $D_0(R)$) (the Royden decomposition).*

LEMMA 1.4. *Every HD-function on R has μ-integrable boundary value on Δ, where μ is the harmonic measure of Δ with respect to a point $z_0\in R$.*

2. Generalized harmonic measures.

DEFINITION. A harmonic function u on R is called a generalized harmonic measure if the greatest harmonic minorant $u\wedge(1-u)$ of u and $1-u$ vanishes identically on R ([5]).

LEMMA 2.1 ([7]). *Suppose that u is a nonconstant generalized harmonic measure with finite Dirichlet integral on R. For each $0<r<1$, set $G_r=\{p\in R;\ u(p)>r\}$. Then*

$$(du, \omega)_R=-\int_{\partial G_r}{}^*\omega$$

for any $\omega\in\Gamma_h(R)$ with $\int_{\partial G_r}|\omega|<\infty$.

We note that $\int_{\partial G_r}|\omega|<\infty$ for almost all r $(0<r<1)$, where each relative boundary of an open set is oriented so that the open set lies on the lefthand side of the boundary (cf. [1], [9]).

DEFINITION. We say that an exact differential du on R belongs to the class $\Gamma_{\widehat{hm}}(R)$ if there exists a sequence of functions $\{u_n\}$, each u_n being a real linear combination of generalized harmonic measures with finite Dirichlet integral and $\|du_n-du\|_R\to 0\,(n\to\infty)$.

Then clearly $\Gamma_{\widehat{hm}}(R)$ is a closed subspace of $\Gamma_h(R)$.

3. Partitions of the harmonic boundary.

DEFINITION. We say that $(P)=(P: \delta_1, \cdots, \delta_N)$ is a finite partition of the harmonic boundary Δ if $\delta_1, \cdots, \delta_N$ are mutually disjoint nonempty compact subsets of Δ and $\Delta=\delta_1\cup\cdots\cup\delta_N$.

DEFINITION. An exact differential du in $\Gamma_{\widehat{hm}}(R)$ belongs to the class $(P)\Gamma_{\widehat{hm}}(R)$ if u takes a constant value on each part $\delta_j\,(1\leqq j\leqq N)$ of the partition (P) of Δ.

PROPOSITION 3.1. *The class $(P)\Gamma_{\widehat{hm}}(R)$ is a closed subspace in $\Gamma_{\widehat{hm}}(R)$.*

Proof. Clearly $(P)\Gamma_{\widehat{hm}}(R)\subset\Gamma_{\widehat{hm}}(R)$. Suppose that $du_n\in(P)\Gamma_{\widehat{hm}}(R)$, $du\in\Gamma_{\widehat{hm}}(R)$ and $\|du_n-du\|_R\to 0$. We may assume that there is a point $z_0\in R$ such that $u_n(z_0)=u(z_0)=0$ and $\{u_n\}$ converges to u uniformly on every compact subset of R. Let $u_n=c_n^{(j)}$ on $\delta_j\,(1\leqq j\leqq N)$.

First, we prove that $\{u_n\}$ is uniformly bounded. Suppose that $\{u_n\}$ is not uniformly bounded. We may assume that $c_n^{(1)}\leqq 0$ and $c_n^{(2)}\to\infty$. Let M be an arbitraly positive number. Then for sufficiently large number n, $0\cup(u_n\cap M)=0$ on δ_1, $=M$ on δ_2 and converges to $0\cup(u\cap M)$ uniformly on every compact subset of R. Let h be an HBD-function on R such that $h=1$ on δ_2 and $h=0$ on $\Delta-\delta_2$. Then $h(0\cup(u_n\cap M)-M)$ converges to $h(0\cup(u\cap M)-M)$ uniformly on every compact subset of R. Further, for sufficiently large number n,

$h(0\cup(u_n\cap M)-M)\in BD_0(R)$ and

$$\|d(h(0\cup(u_n\cap M)-M))\|_R\leqq\|d(hu_n)\|_R+2M\|dh\|_R$$
$$\leqq 3(M\|dh\|_R+\|du\|_R).$$

By Lemma 1.1, $h(0\cup(u\cap M)-M)\in BD_0(R)$ and $M=u\cap M\leqq u$ on δ_2. While, HD-function u is μ-integrable on Δ and $\mu(\delta_2)>0$, where μ is the harmonic measure with respect to z_0. This is a contradiction. Hence, $\{u_n\}$ must be uniformly bounded and $u\in HBD(R)$.

Since $\{c_n^{(j)}\}$ is uniformly bounded, we may assume that there are constants $c^{(1)}, \cdots, c^{(N)}$ such that $c_n^{(j)}\to c^{(j)}$ $(n\to\infty)$ for each j. For each δ_j, let g be an HBD-function on R such that $g=1$ on δ_j and $g=0$ on $\Delta-\delta_j$. Then $g(u_n-c_n^{(j)})\in BD_0(R)$. By the similar argument above, we conclude that $g(u-c^{(j)})\in BD_0(R)$. Hence $u=c^{(j)}$ on δ_j. ∎

4. Weakly dividing cycles.

We say that c is a curve on R if c is an image of a homeomorphic mapping from an open interval or the unit circle into R. Let $\{c_k\}$ be a set of (at most countable number of) oriented piecewise analytic curves clustering nowhere in R.

Let $(P)=(P: \delta_1, \cdots, \delta_N)$ be a finite partition of the harmonic boundary Δ. We say that a formal sum $c=\sum c_k$ is a *(P)-weakly dividing cycle* in R if there exists an open set G such that

(1) $c=\sum c_k$ coincides with the relative boundary ∂G of G,

(2) $\overline{\partial G}\cap\Delta=\emptyset$,

(3) for each δ_j, it holds either $\delta_j\subset\overline{G}\cap\Delta$ or $\delta_j\subset\Delta-\overline{G}$,

where the closure is taken in R^*.

In (1), ∂G is oriented so that G lies on the left hand side of ∂G. So, if G is the complement of a curve γ in R, then ∂G is the sum of two oriented curves γ^+ and γ^- which have the same image as γ and are oriented reversely. We write (1) simply $c=\partial G$. While, in (2) ∂G is the topological relative boundary of G in R.

We say that c is a *weakly dividing cycle* if (1) and (2) hold ([7]).

We say that a property holds for *almost every curve* or *almost all curves* in a family of curves if the subfamily of exceptional curves has infinite extremal length (cf. [11]).

DEFINITION. We say that a differential ω belongs to the class $(P)\Gamma_{hwe}(R)$ (resp. $\Gamma_{hwe}(R)$) if $\omega\in\Gamma_h(R)$ and $\int_c\omega=0$ for almost all (P)-weakly dividing cycles (resp. weakly dividing cycles) c.

We note that if $\omega\in\Gamma(R)$, then $\int_c|\omega|<\infty$ for almost every weakly dividing cycle and (P)-weakly dividing cycle c.

We know that the class $\Gamma_{hwe}(R)$ is a closed subspace in $\Gamma_h(R)$, and that the orthogonal decomposition

$$\Gamma_h(R)=\Gamma_{\widehat{hm}}(R)+{}^*\Gamma_{hwe}(R)$$

holds ([7]). By the similar argument, we can prove that $(P)\Gamma_{hwe}(R)$ is a closed subspace of $\Gamma_h(R)$ and the following

PROPOSITION 4.1. $\Gamma_h(R)=(P)\Gamma_{\widehat{hm}}(R)+{}^*(P)\Gamma_{hwe}(R)$.

5. Periods along the harmonic boundary.

Let V be a union of a finite number of regular subregions of R with disjoint closures. By a regular region we mean one which is relatively compact and bounded by a finite number of disjoint analytic curves. Suppose that $R-\bar{V}$ is connected. Let Γ_x be a subspace of Γ_h. We say that $\sigma\in\Gamma_x(R-V)$ if σ is a harmonic differential on a neighborhood of $(R-\bar{V})\cup\partial V$ and $\sigma\in\Gamma_x(R-\bar{V})$.

Let $(\delta, \Delta-\delta)$ be a partition of Δ. For $\sigma\in\Gamma_h(R-V)$, we shall define the period of σ along δ.

LEMMA 5.1. *Let G be a subregion of R with piecewise analytic boundary. Let σ be a harmonic differential on a neighborhood of $G\cup\partial G$ such that $\in\Gamma_h(G)$ and $\int_{\partial G}|\sigma|<\infty$. If $\bar{G}\cap\Delta=\emptyset$, then $\int_{\partial G}\sigma=0$.*

Proof. Let $\hat{G}$ be the double of G along ∂G. If $\hat{G}$ is compact, then the statement clearly holds. Hence, we assume that $\hat{G}$ is noncompact. Since $\hat{G}$ has no Green's function, there exists an exhaustion $\{\Omega_n\}$ of $\hat{G}$ such that Ω_n is symmetric with respect to ∂G and

$$\lim_{n\to\infty}\int_{\partial\Omega_n\cap G}|\sigma|=0.$$

Since

$$\int_{\partial\Omega_n\cap G}\sigma+\int_{\partial G\cap\Omega_n}\sigma=0,$$

we have $\int_{\partial G}\sigma=0$. ■

DEFINITION. Let $(\delta, \Delta-\delta)$ be a partition of Δ and $v_\delta=v_\delta^{R-V}$ be an *HBD*-function on $R-V$ such that $v_\delta=1$ on δ and $v_\delta=0$ on $(\Delta-\delta)\cup\partial V$. We call v_δ a generalized harmonic measure of δ on $R-V$

Set $G_r=\{p\in R;\ v_\delta(p)>r\}$ for $0<r<1$. Then ∂G_r is a weakly dividing cycle such that $\bar{G}_r \cap \Delta = \delta$ and $G_r \subset R-V$. By Lemma 2.1, we have the following

LEMMA 5.2. *Let* $\sigma \in \Gamma_h(R-V)$. *Then*

$$\int_{\partial G_r} \sigma = (dv_\delta, {}^*\sigma)_{R-V}$$

for almost all $0<r<1$.

DEFINITION. For $\sigma \in \Gamma_h(R-V)$, we define the period of σ along δ as

$$\int_\delta \sigma = -\int_{\partial G_r} \sigma = -(dv_\delta, {}^*\sigma)_{R-V},$$

where r is the value for which Lemma 5.2 holds.

PROPOSITION 5.3. *Let* $(\delta, \Delta-\delta)$ *be a partition of* Δ *and* $c=\partial G$ *be a weakly dividing cycle such that* $\bar{G} \cap \Delta = \delta$ *and* $G \subset R-V$. *Let* $\sigma \in \Gamma_h(R-V)$ *with* $\int_c |\sigma| < \infty$. *Then for almost all* $0<r<1$,

$$\int_{\partial G_r} \sigma = \int_c \sigma .$$

Proof. There is a *BD*-function w such that $w=1$ on δ, $w=0$ on $R-G$ and harmonic on G. Then the harmonic part of the Royden decomposition of w on $R-\bar{V}$ is the generalized harmonic measure v_δ of δ on $R-V$. By Lemma 5.1, for almost all $0<r<1$,

$$\int_c \sigma = \int_{\partial\{w>r\}} \sigma = \int_{\partial G_r} \sigma . \quad \blacksquare$$

THEOREM 5.4. *Let* $(P)=(P\colon \delta_1, \cdots, \delta_N)$ *be a partition of* Δ *and* $\sigma \in \Gamma_h(R)$. *Then* $\sigma \in (P)\Gamma_{hwe}(R)$ *if and only if* $\int_{\delta_j} \sigma = 0$ *for all* j.

Proof. Let $v_j = v_{\delta_j}^{R-V}$ be the generalized harmonic measure of δ_j on $R-V$, that is, $v_j \in HBD(R-V)$ such that $v_j=1$ on δ_j and $v_j=0$ on $(\Delta-\delta_j) \cup \partial V$. We extend v_j on R so that $v_j=0$ on V. Let $v_j = w_j + p_j$ be the Royden decomposition of v_j on R. Then w_j is a generalized harmonic measure on R such that $w_j=1$ on δ_j and 0 on $\Delta-\delta_j$. Since $(dp_j, \sigma)_R=0$ for $\sigma \in \Gamma_h(R)$, we have

$$(dw_j, \sigma)_R = (dv_j, \sigma)_{R-V} .$$

Hence, $(P)\Gamma_{\widehat{hm}}(R)$ being generated by $\{dw_j\}$ $(1 \leqq j \leqq N)$ proves the assertion. $\blacksquare$

6. $(P)\hat{L}_1$-operator.

Let V be a union of a finite number of relatively compact regular subregions of R with disjoint closures. We assume that $R-\bar{V}$ is connected.

THEOREM 6.1. *Let $f\in C^{\omega}(\partial V)$ and $(P)=(P: \delta_1, \cdots, \delta_N)$ be a partition of the harmonic boundary Δ. There exists a unique function $u\in HBD(R-V)$ satisfying the follwoing conditions:*

(1) $$u|_{\partial V}=f,$$

(2) $$u=constant \text{ on } \delta_j \ (1\leqq j\leqq N),$$

(3) $$\int_{\delta_j} {}^*du=0 \ (1\leqq j\leqq N).$$

Proof. (Uniqueness) Suppose that $u_1, u_2\in HBD(R-V)$ satisfying (1), (2), (3). Then $u_1-u_2=0$ on ∂V and u_1-u_2 is constant on each δ_j. Let $v_j=v_{\delta_j}^{R-V}$ is a generalized harmonic measure of δ_j on $R-V$. Then (3) implies

$$\int_{\delta_j} {}^*d(u_1-u_2)=(dv_j, d(u_1-u_2))_{R-V}=0 \quad (1\leqq j\leqq N).$$

Since u_1-u_2 is a linear combination of $\{v_j\}$, we conclude that $u_1-u_2\equiv 0$ on $R-V$.

(Existence) The matrix whose (i, j)-element is defined by

$$\int_{\delta_i} {}^*dv_j=(dv_i, dv_j)_{R-V}$$

is symmetric and positive definite. In fact, for real variables $x_1, \cdots, x_N$,

$$\Big\| \sum_{j=1}^{N} x_j dv_j \Big\|_{R-V}^2 = \sum_{i,j=1}^{N} x_i x_j (dv_i, dv_j)_{R-V} \geqq 0$$

and the equality holds if and only if $\sum_{j=1}^{N} x_j dv_j\equiv 0$, i.e. $x_j=0$ for all j.

Let $Hf\in HBD(R-V)$ such that $Hf=f$ on ∂V and 0 on Δ. Consider the function $u=Hf+\sum_{j=1}^{N} c_j v_j$ where c_j are real constants. Then

$$\sum_{j=1}^{N} c_j \int_{\delta_i} {}^*dv_j=\int_{\delta_i} {}^*du-\int_{\delta_i} {}^*dHf.$$

There exist $c_1, \cdots, c_N$ such that $\int_{\delta_i} {}^*du=0 \,(1\leqq i\leqq N)$. Therefore, there exists u satisfying (1), (2) and (3). ■

We denote the function u in Theorem 6.1 by $(P)\hat{L}_1 f$.

THEOREM 6.2. *The operator $(P)\hat{L}_1$ from $C^{\omega}(\partial V)$ to $HBD(R-V)$ is a normal operator. That is, $(P)\hat{L}_1$ is a linear operator satisfying the following conditions:*

(1) $$(P)\hat{L}_1 f|_{\partial V}=f,$$

(2) $$\min_{\partial V} f\leqq (P)\hat{L}_1 f\leqq \max_{\partial V} f,$$

(3) $$\int_{\partial V} {}^*d((P)\hat{L}_1 f)=0.$$

Proof. It is easy to see that $(P)\hat{L}_1$ is a linear operator and satisfies (1).

We prove (2). Let $(P)\hat{L}_1 f=c_j$ on δ_j. It is clear that $(P)\hat{L}_1 1=1$. Therefore, it is sufficient to see that if $f\geqq 0$ then all c_j are non-negative. Suppose that there exists a $c_j<0$. We may assume that c_j is the minimum value of $(P)\hat{L}_1 f$ on Δ. Let $\delta=\{p\in\Delta;\ (P)\hat{L}_1 f=c_j\}$. Then δ is a union of some parts of the partition (P). For $\varepsilon>0$, let $G_\varepsilon=\{p\in R;\ (P)\hat{L}_1 f(p)<c_j+\varepsilon\}$. Then, for almost all sufficiently small $\varepsilon>0$, ∂G_ε is a weakly dividing cycle such that $\bar{G}_\varepsilon\cap\Delta=\delta$, and $G_\varepsilon\subset R-V$ and

$$\int_\delta {}^*d((P)\hat{L}_1 f)=-\int_{\partial G_\varepsilon} {}^*d((P)\hat{L}_1 f)<0\,.$$

Thus there exists a part δ_k of (P) such that $\int_{\delta_k} {}^*d((P)\hat{L}_1 f)<0$. This contradicts the property (3) in Theorem 6.1.

Finally we prove (3). Let $v_j=v_{\delta_j}^{R-V}$ be the generalized harmonic measure of δ_j on $R-V$. By the following Lemma, we have

$$\begin{aligned}\int_{\partial V} {}^*d((P)\hat{L}_1 f)&=-\Big(d\Big(1-\sum_{j=1}^N v_j\Big),\ d((P)\hat{L}_1 f)\Big)_{R-V}\\&=\Big(d\Big(\sum_{j=1}^N v_j\Big),\ d((P)\hat{L}_1 f)\Big)_{R-V}\\&=\sum_{j=1}^N\int_{\delta_j} {}^*d((P)\hat{L}_1 f)=0\,.\quad\blacksquare\end{aligned}$$

LEMMA 6.3 ([7]). *Suppose that $v\in HD(R-V)$ and $v=0$ on Δ. Then*

$$(dv,\ \omega)_{R-V}=-\int_{\partial V} v^*\omega$$

for any $\omega\in\Gamma_h(R-V)$.

PROPOSITION 6.4. *For every $f\in C^\omega(\partial V)$,*

$$\|d((P)\hat{L}_1 f)\|_{R-V}^2=-\int_{\partial V} f^*d((P)\hat{L}_1 f)\,.$$

Proof. We recall that $(P)\hat{L}_1 f=Hf+\sum_{j=1}^N c_j v_j$ in the proof of Theorem 6.1. Then by Lemma 6.3,

$$\begin{aligned}\|d((P)\hat{L}_1 f)\|_{R-V}^2&=(dHf,\ d((P)\hat{L}_1 f))_{R-V}+\sum_{j=1}^N c_j(dv_j,\ d((P)\hat{L}_1 f))_{R-V}\\&=-\int_{\partial V} Hf^*d((P)\hat{L}_1 f)+\sum_{j=1}^N c_j\int_{\delta_j} {}^*d((P)\hat{L}_1 f)\\&=-\int_{\partial V} f^*d((P)\hat{L}_1 f)\,.\quad\blacksquare\end{aligned}$$

7. Refinement of partions.

DEFINITION. Let $(P)=(P: \delta_1, \cdots, \delta_N)$ and $(P')=(P': \delta'_1, \cdots, \delta'_M)$ are partitions of Δ. We say that (P') is a refinement of (P) if each δ'_j is a subset of some δ_i.

LEMMA 7.1. *If (P') is a refinement of (P) then*

$$\|d((P)\hat{L}_1 f)\|_{R-V} \geqq \|d((P')\hat{L}_1 f)\|_{R-V}$$

for every $f \in C^\omega(\partial V)$.

Proof. Let $u=(P)\hat{L}_1 f$ and $u'=(P')\hat{L}_1 f$. Let $v'_j=v_{\delta'_j}^{R-V}$ be the generalized harmonic measure of δ'_j on $R-V$. Then $u-u'=\sum_{j=1}^M c'_j v'_j$, for some $c'_j (1 \leqq j \leqq M)$ and

$$(d(u-u'), du')_{R-V}=\sum_{j=1}^M c'_j \int_{\delta'_j} *du'=0 .$$

Hence

$$(du, du')_{R-V}=\|du'\|_{R-V}^2$$

and

$$0 \leqq \|du-du'\|_{R-V}^2=\|du\|_{R-V}^2-\|du'\|_{R-V}^2 .$$

Thus

$$\|du\|_{R-V} \geqq \|du'\|_{R-V} . \quad \blacksquare$$

8. $\hat{L}_1$-operator.

DEFINITION. We define a constant

$$\kappa_{R-V}=\kappa_{R-V}(f)=\inf\{\|d((P)\hat{L}_1 f)\|_{R-V}\} ,$$

where the infimum is taken over all finite partitions (P) of Δ.

Since $\|dL_0 f\|_{R-V} \leqq \|dv\|_{R-V}$ for any $v \in HBD(R-V)$ with $v|_{\partial V}=f$, it follows that $\kappa_{R-V}>0$ for every non-constant function f (see [12], [14], [15] for L_0-operator).

PROPOSITION 8.1. *There exist a sequence of partitions $\{(P_n)\}$ of Δ and $u \in HBD(R-V)$ such that*

(1) $\quad (P_{n+1})$ *is a refinement of* (P_n) $(n=1, 2, \cdots)$,

(2) $\quad \|du\|_{R-V}=\kappa_{R-V}$,

(3) $\quad u|_{\partial V}=f$,

(4) $\quad \|d((P_n)\hat{L}_1 f)-du\|_{R-V} \longrightarrow 0 \quad (n \to \infty)$.

Proof. There is a sequence of partitions (P_n) such that $\|d(P_n)\hat{L}_1 f\|_{R-V}\to\kappa_{R-V}(n\to\infty)$. By Lemma 7.1, we may assume that (P_{n+1}) is a refinement of (P_n) $(n=1, 2, \cdots)$. Let $u_n=(P_n)\hat{L}_1 f$. By the same argument as in Lemma 7.1, for $n<m$,

$$\|du_n-du_m\|_{R-V}^2=\|du_n\|_{R-V}^2-\|du_m\|_{R-V}^2 .$$

Hence, there exists a $u\in HBD(R-V)$ such that $u|_{\partial V}=f$, $\|du_n-du\|_{R-V}\to 0$ $(n\to\infty)$ and $\|du\|_{R-V}=\kappa_{R-V}$. ■

We note that u does not depend on the choice of a sequence of partitions in Proposition 8.1. In fact, suppose that $\{(P'_n)\}$ is another sequence of partitions such that $\|d(P'_n)\hat{L}_1 f\|_{R-V}\to\kappa_{R-V}$ $(n\to\infty)$ and (P'_{n+1}) is a refinement of (P'_n) $(n=1, 2, \cdots)$. Let $u'_n=(P'_n)\hat{L}_1 f$ and $u'=\lim_{n\to\infty}u'_n$. There is a sequence of partitions $\{(P''_n)\}$ such that (P''_{n+1}) is a refinement of (P''_n) and (P''_n) is a refinement of both (P_n) and (P'_n) $(n=1, 2, \cdots)$. Let $u''=\lim_{n\to\infty}u''_n=\lim_{n\to\infty}(P''_n)\hat{L}_1 f$. By the same argument as in Lemma 7.1, $(d(u''_n-u_n), du''_n)_{R-V}=0$. Since $\|du''_n-du''\|_{R-V}\to 0$ and $\|du_n-du\|_{R-V}\to 0$

$$\begin{aligned}(du'', du)_{R-V}&=\lim_{n\to\infty}(du''_n, du_n)_{R-V}\\&=\lim_{n\to\infty}\|du''_n\|_{R-V}^2=\|du''\|_{R-V}^2 .\end{aligned}$$

Hence

$$0\leqq\|du''-du\|_{R-V}^2=\|du\|_{R-V}^2-\|du''\|_{R-V}^2=0 .$$

Thus, $u=u''$. Similarly, $u=u'$.

For any $w\in HD(R)$, there exists a unique HD-function $I_{R-V}(w)$ on $R-V$ such that $I_{R-V}(w)=w$ on Δ and $I_{R-V}(w)=0$ on ∂V. We call $I_{R-V}(w)$ the *inextremisation* of w to $R-V$. It is clear that I_{R-V} is a linear operator.

LEMMA 8.2 ([7]). *If $u\in HD(R)$ with $du\in\Gamma_{\widehat{hm}}(R)$ then $dI_{R-V}(u)\in\Gamma_{\widehat{hm}}(R-V)$.*

THEOREM 8.3. *For every $f\in C^\omega(\partial V)$, there exists a unique function $u\in HBD(R-V)$ satisfying the following conditions·*

(1) $$u|_{\partial V}=f ,$$

(2) *there exist a harmonic function $u_{\widehat{hm}}$ on R with $du_{\widehat{hm}}\in\Gamma_{\widehat{hm}}(R)$ and a Dirichlet potential p on R such that*

$$u=u_{\widehat{hm}}+p$$

on $R-V$ and

(3) $$\int_\delta {}^*du=0$$

for any partition $(\delta, \Delta-\delta)$ of Δ consisting of two parts.

Proof. (Existence) We use the notation $u_n=(P_n)\hat{L}_1 f$ and u in Proposition 8.1 and its proof. We have already proved (1).

We prove (2). Let $Hf\in HBD(R-V)$ such that $Hf=f$ on ∂V and $Hf=0$ on Δ. Since $u_n-Hf=0$ on ∂V, $d(u_n-Hf)\in\Gamma_{\widehat{hm}}(R-V)$. Hence $d(u-Hf)\in\Gamma_{\widehat{hm}}(R-V)$. We set $u_n-Hf=0$ and $u-Hf=0$ on V so that u_n-Hf, $u-Hf\in BD(R)$. The Royden decomposition gives $u_n-Hf=w_n+q_n$, $u-Hf=w+q$, where w_n, $w\in HBD(R)$ and q_n, $q\in BD_0(R)$. Since $dw_n\in\Gamma_{\widehat{hm}}(R)$ and $\|dw_n-dw\|_R\to 0$, we have $dw\in\Gamma_{\widehat{hm}}(R)$. We can extend Hf to a BD_0-function on R so that $u=w+(q+Hf)$ on $R-V$. Denoting w by $u_{\widehat{hm}}$ and $q+Hf$ by p gives (2).

Finally, we shall prove (3). Let $v_\delta=v_\delta^{R-V}$ be a generalized harmonic measure of δ on $R-V$. By the note following Proposition 8.1, we may assume that each (P_n) is a refinement of $(\delta, \Delta-\delta)$. Then

$$\int_\delta {}^*du=(du_\delta, du)_{R-V}=\lim_{n\to\infty}(dv_\delta, du_n)_{R-V}=0.$$

(Uniqueness) Let $u=u_{\widehat{hm}}+p$ and $u'=u'_{\widehat{hm}}+p'$ satisfy (1), (2) and (3). Then $u-u'=I_{R-V}(u_{\widehat{hm}}-u'_{\widehat{hm}})$. By Lemma 8.2, $d(u-u')\in\Gamma_{\widehat{hm}}(R-V)$. There is a sequence $\{w_n\}$ of HBD-functions on $R-V$ each w_n being a linear combination of generalized harmonic measures with finite Dirichlet integral on $R-V$, $w_n|_{\partial V}=0$ and $\|dw_n-d(u-u')\|_{R-V}\to 0$. While, by (3), we have $(dw_n, d(u-u'))_{R-V}=0$. Hence $du-du'\equiv 0$. ∎

We denote the function u by $\hat{L}_1 f$. Then $\|d\hat{L}_1 f\|_{R-V}=\kappa_{R-V}(f)$.

THEOREM 8.4. *The operator $\hat{L}_1$ from $C^\omega(\partial V)$ to $HBD(R-V)$ is a normal operator. That is, $\hat{L}_1$ is a linear operator satisfying the following conditions:*

(1) $$\hat{L}_1 f|_{\partial V}=f,$$

(2) $$\min_{\partial V} f\leqq \hat{L}_1 f\leqq \max_{\partial V} f,$$

(3) $$\int_{\partial V} {}^*d\hat{L}_1 f=0.$$

Proof. We use the notation $(P_n)\hat{L}_1$ in Proposition 8.1. By Theorem 6.2, $(P_n)\hat{L}_1$ are normal operators. By Proposition 8.1, $(P_n)\hat{L}_1 f$ converges to $\hat{L}_1 f$ uniformly on every compact subset of $R-V$. Hence (1), (2) and (3) hold. ∎

9. An extremal property.

For every $v\in HBD(R-V)$ there exists a unique HBD-function $E(v)$ on R such that $E(v)=v$ on Δ. We call $E(v)$ the *extremisation* of v (see [7], [8]). It is clear that E is a linear operator and satisfies the following

LEMMA 9.1. *Let* $v \in HBD(R-V)$ *and* $v=w+p$ *on* $R-V$, *where* $w \in HBD(R)$ *and* $p \in BD_0(R)$. *Then* $E(v)=w$. *Moreover, if* $v=0$ *on* ∂V *then* $I_{R-V}(E(v))=v$ *on* $R-V$.

THEOREM 9.2. *Let* $f \in C^{\omega}(\partial V)$. *The function* $\hat{L}_1 f$ *minimizes* $\|dv\|_{R-V}$ *in* $v \in HBD(R-V)$ *such that* $v|_{\partial V}=f$ *and* $dE(v) \in \Gamma_{\widehat{hm}}(R)$.

Proof. Let $v \in HBD(R-V)$ such that $v|_{\partial V}=f$ and $dE(v) \in \Gamma_{\widehat{hm}}(R)$. Since $dE(v)-dE(\hat{L}_1 f) \in \Gamma_{\widehat{hm}}(R)$, $dI_{R-V}(E(v)-E(\hat{L}_1 f))=d(v-\hat{L}_1 f) \in \Gamma_{\widehat{hm}}(R-V)$ by Lemma 8.2. Hence there is a sequence $\{w_n\}$ of HBD-functions on $R-V$ such that each w_n is a linear combination of generalized harmonic measures with finite Dirichlet integral, equals 0 on ∂V and $\|dw_n-(dv-d\hat{L}_1 f)\|_{R-V} \to 0$. Since $(dw_n, d\hat{L}_1 f)_{R-V}=0$, $(dv, d\hat{L}_1 f)_{R-V}=\|d\hat{L}_1 f\|^2_{R-V}$. Hence, $\|dv\|_{R-V} \geqq \|d\hat{L}_1 f\|_{R-V}$. ∎

10. Regular operators.

An operator L from $C^{\omega}(\partial V)$ to $HBD(R-V)$ is called a *regular operator* if

(1) $$Lf|_{\partial V}=f,$$

(2) $$(dLf, dLg)_{R-V}=-\int_{\partial V} f^* dLg$$

for any $f, g \in C^{\omega}(\partial V)$ ([17]).

THEOREM 10.1. *Let* (P) *be a finite partition of* Δ. *Then* $(P)\hat{L}_1$ *and* $\hat{L}_1$ *are regular operators.*

Proof. It is sufficient to prove that $(P)\hat{L}_1$ satisfies (2). Let $Hf \in HBD(R-V)$ such that $Hf=f$ on ∂V and $Hf=0$ on Δ. Then $d((P)\hat{L}_1 f-Hf) \in \Gamma_{\widehat{hm}}(R-V)$. Hence

$$(d((P)\hat{L}_1 f-Hf),\quad d((P)\hat{L}_1 g))_{R-V}=0.$$

Thus

$$(d((P)\hat{L}_1 f), d((P)\hat{L}_1 g))_{R-V}=(dHf, d((P)\hat{L}_1 g))_{R-V}$$
$$=-\int_{\partial V} f^* d((P)\hat{L}_1 g). \quad ∎$$

11. Modulus functions.

Let V_0 and V_1 be two relatively compact regular subregions of R with disjoint closures. We assume that $R-\bar{V}_0 \cup \bar{V}_1$ is connected. Let $f=0$ on ∂V_0 and $f=1$ on ∂V_1. Then $\int_{\partial V_0} {}^* d\hat{L}_1 f=\|d\hat{L}_1 f\|^2_{R-V_0 \cup V_1}>0$. Set $\hat{q}_1=(2\pi/\int_{\partial V_0} {}^* d\hat{L}_1 f)\hat{L}_1 f$.

THEOREM 11.1. *There exists a unique HBD-function* $\hat{q}_1$ *on* $R-V_0 \cup V_1$ *such that*

(1) $\hat{q}_1|_{\partial V_0}=0,$

(2) $\hat{q}_1|_{\partial V_1}=\hat{k}_1=constant,$

(3) $\hat{L}_1(\hat{q}_1|_{\partial V_0\cup\partial V_1})=\hat{q}_1 \quad on\ R-V_0\cup V_1,$

(4) $\int_{\partial V_0}{}^*d\hat{q}_1=2\pi.$

We call $\hat{q}_1$ $\hat{L}_1$-*modulus function* on $R-V_0\cup V_1$ with respect to ∂V_0 and ∂V_1. The constant $e^{\hat{k}_1}$ is called the $\hat{L}_1$-*modulus* of $R-V_0\cup V_1$ with respect to ∂V_0 and ∂V_1.

We denote usual L_1-modulus function for L_1-operator by q_1 (see [15]). That is, q_1 satisfies (1), (2), (4) of Theorem 11.1 and $L_1(q_1|_{\partial V_0\cup\partial V_1})=q_1$ on $R-V_0\cup V_1$. If $q_1|_{\partial V_1}=k_1$, e^{k_1} is called the L_1-modulus of $R-V_0\cup V_1$ with respect to ∂V_0 and ∂V_1.

12. An example.

Now, we consider a two sheeted branched covering surface of the unit disk. Let D be the unit disk and $\{a_n\}$, $\{b_n\}$ be sequences of positive numbers such that $0<a_0<b_0<a_1<b_1<\cdots<a_n<b_n<\cdots$ and $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=1$. Consider the region obtained from D by deleting the closed intervals $[a_n, b_n]$ ($n=0, 1, \cdots$). Join two such copies, one being D_0 and another being D_1, crosswise along $[a_n, b_n]$ ($n=0, 1, \cdots$), so as to obtain a 2-sheeted branched covering surface R of D. Denote by π the projection from R onto D. In [6], we show that the number of components of the harmonic boundary Δ of R is at most 2. Moreover, if intervals $[a_n, b_n]$ are sufficiently small then Δ consists of two components and if gaps (b_n, a_{n+1}) are sufficiently small then Δ is connected. (See [6, p. 639] for precise estimations.)

Let U be the sufficiently small disk with center 0 in D so that $\pi^{-1}(U)$ consists of two disjoint disk V_0 in D_0 and V_1 in D_1.

Denote by ∂D_k the boundary of D_k corresponding to $\{|z|=1\}-\{1\}$ $(k=0, 1)$. Note that every HBD-function u on $R-V_0\cup V_1$ is uniquely determined by the boundary values on $\partial V_0\cup\partial V_1\cup\partial D_0\cup\partial D_1$. Moreover, if $du\in\Gamma_{\widehat{hm}}(R-V_0\cup V_1)$ then u is constant on ∂V_0, ∂V_1, ∂D_0 and ∂D_1 respectively ([6]).

Let τ be the nontrivial covering transformation of R. Let ϕ be the anti-conformal automorphism of R which preserves the sheets D_0, D_1 and is identical with the mapping $z\mapsto\bar{z}$ on each sheet.

Let $f=0$ on ∂V_0 and 1 on ∂V_1. Since R has one Stoïlow ideal boundary component, $L_1=(I)\hat{L}_1$ for the identity partition $(I)=(\Delta)$. Let $L_1f=k$ on Δ. Since $(L_1f)\circ\tau=L_1(1-f)=1-L_1f$, $k=1-k$. Hence $k=1/2$.

Further, $(L_1f)\circ\phi\circ\tau=1-L_1f$. Hence $L_1f=1/2$ on $\cup_n[a_n, b_n]$.

If Δ is connected, then $L_1f=\hat{L}_1f$. If Δ is not connected, then $L_1f\neq\hat{L}_1f$.

Proof. We shall prove the latter half. Contraly to the assertion, suppose that $L_1f=\hat{L}_1f$. Then $\hat{L}_1f=1/2$ on Δ, hence on $\partial D_0\cup\partial D_1$, and $\hat{L}_1f=1/2$ on $\cup_n[a_n, b_n]$.

If Δ is not connected then there exists $v\in HBD(R-V_0\cup V_1)$ such that $v=0$ on $\partial V_0\cup\partial V_1\cup\partial D_1$ and $v=1$ on ∂D_0. Then $dv\in\Gamma_{\widehat{hm}}(R-V_0\cup V_1)$. Therefore, $(dv, d\hat{L}_1f)_{R-V}=0$. While,

$$
\begin{aligned}
(dv, d\hat{L}_1f)_{R-V} &= \left(dv, d\left(\hat{L}_1f-\frac{1}{2}\right)\right)_{R-V} \\
&= -\int_{\partial(V_0\cup V_1)}\left(\hat{L}_1f-\frac{1}{2}\right)^*dv \\
&= \frac{1}{2}\left(\int_{\partial V_0}{}^*dv-\int_{\partial V_1}{}^*dv\right)>0.
\end{aligned}
$$

For, $v|_{D_0}-v|_{D_1}$ is considered as an HBD-function on $D-U-\cup_n[a_n, b_n]$ whose boundary values equals 0 on $(\cup_n[a_n, b_n])\cup\partial U$ and equals 1 on $\partial D-\{1\}$. This is a contradiction. ■

In the latter case, by Lemma 7.1 and its proof, we have $\|dL_1f\|_{R-V_0\cup V_1}>\|d\hat{L}_1f\|_{R-V_0\cup V_1}$. Therefore, $\hat{k}_1>k_1$.

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