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KODAI MATH. J.
16 (1993), 244-252

SURFACES OF FINITE TYPE WITH CONSTANT MEAN CURVATURE

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Abstract

Which surfaces in the Euclidean space E^3 with constant mean curvature are of finite type? We show that a 3-type surface has non constant mean curvature. Moreover, among surfaces of revolution with constant mean curvature the only ones which are of finite type are: the plane, the sphere, the catenoid and the circular cylinder.

1. Introduction.

Finite type submanifolds were introduced by B.-Y. Chen in [1]. These can be regarded as a generalization of minimal submanifolds. From the class of finite type submanifolds the 2-type are those that attracted the interest and it is a striking fact that almost all the results concerned with 2-type spherical submanifolds. In [6] it was proved that every 2-type hypersurface M of S^{n+1} has non-zero constant mean curvature in S^{n+1} and constant scalar curvature. On the other hand, it was proved ([3], [4]) that every 3-type spherical hypersurface has non-constant mean curvature. As far as we know nothing is known about 3-type hypersurfaces of a Euclidean space E^{n+1}, with constant mean curvature.

It is well known that the minimal surfaces, the ordinary spheres and the circular cylinders in the Euclidean space E^3, are at most of finite 2-type. Moreover all these have constant mean curvature.

As it is known there is an abundance of surfaces of constant mean curvature in the Euclidean space E^3. Among them are certain of the surfaces of revolution, called Delaunay surfaces. Moreover, Wente [9] demonstrated the existence of an immersed torus of constant mean curvature and Kapouleas [7] has shown that there also exist compact immersed surfaces with constant mean curvature of every genus $g \geqq 3$.

In this article, we ask the following geometric question:

"Which surfaces in E^3 with constant mean curvature are of finite type?"

After some preliminaries we prove the following two theorems, which answer partially the question.

Received October 26, 1992.

THEOREM 1. *A 3-type surface (not necessarily compact) in the Euclidean space E^3 has non constant mean curvature.*

THEOREM 2. *Let M be a surface in the Euclidean space E^3, which is a surface of revolution with constant mean curvature. Assume M is of finite type; then M is an open piece of an ordinary sphere, a plane, a catenoid or a circular cylinder.*

2. Preliminaries.

Let M be a hypersurface of a Euclidean space E^{n+1}. It is well known that the position vector field x and the mean curvature vector field $\vec{H}$ of M satisfy

$$\Delta x = -n\vec{H}, \tag{2.1}$$

where Δ stands for the Laplace operator of the induced metric (with sign conventions such that $\Delta = -d^2/dx^2$ on the real line E).

The hypersurface M is said to be of *k-type* if its position vector field can be written as

$$x = x_0 + x_1 + x_2 + \cdots + x_k,$$

where x_0 is a constant vector, $x_1, \cdots, x_k$ are non-constant maps satisfying $\Delta x_i = \lambda_i x_i$, $i=1, \cdots, k$ and all eigenvalues $\{\lambda_1, \cdots, \lambda_k\}$ are mutually different. In particular, if one of $\{\lambda_1, \cdots, \lambda_k\}$ is zero, then M is said to be of *null k-type*.

It is obvious that for a surface of k-type we have

$$\Delta^k x - \sigma_1 \Delta^{k-1} x + \cdots + (-1)^k \sigma_k (x - x_0) = 0 \tag{2.2}$$

$$\Delta^k \vec{H} - \sigma_1 \Delta^{k-1} \vec{H} + \cdots + (-1)^k \sigma_k \vec{H} = 0, \tag{2.3}$$

where σ_i is the i-th elementary symmetric function of $\lambda_1, \cdots, \lambda_k$.

From now on we refer to some preliminaries useful for the proof of theorems.

We choose an origin $O \in E^{n+1}$, denote by x the position vector of M, and set $|x| = r$ for the corresponding distance function. Let N be the unit normal vector field of M. The support function p of M is defined as $p = \langle x, N \rangle$. We decompose the position vector x of M in a component normal to M, and a component x_T tangent to M:

$$x = x_T + pN. \tag{2.4}$$

It is obvious that

$$x_T = \frac{1}{2} \text{grad } r^2. \tag{2.5}$$

Let ∇ be the induced connection on M and A be the Weingarten map. Differentiating (2.4) in the direction of a tangent vector X and using the formulae of Gauss and Weingarten we obtain (see for example [5]) the following formulae

$$\nabla_X x_T = X + pAX \tag{2.6}$$

and

$$Ax_T = -\operatorname{grad} p\,. \tag{2.7}$$

Moreover, by easy computations one finds

$$\Delta r^2 = -2n - 2npH \tag{2.8}$$

and

$$\Delta p = n\langle x_T, \operatorname{grad} H\rangle + nH + pS\,, \tag{2.9}$$

where H and S stand for the mean curvature and the square of the length of the second fundamental form.

Assuming that M has constant mean curvature and is of 3-type, and computing $\Delta\vec{H}$ and $\Delta^2\vec{H}$ one finds

$$\Delta\vec{H} = HSN \tag{2.10}$$

$$\Delta^2\vec{H} = 2HA\operatorname{grad} S + H(\Delta S + S^2)N\,. \tag{2.11}$$

Moreover, since M is of 3-type, we have

$$\Delta^2\vec{H} = (\lambda_1+\lambda_2+\lambda_3)\Delta\vec{H} - (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\vec{H} - \frac{\lambda_1\lambda_2\lambda_3}{n}\vec{x}\,. \tag{2.12}$$

Taking into account (2.10) and comparing the tangential and normal components of $\Delta^2\vec{H}$ in (2.11) and (2.12) we obtain the following useful equations

$$A\operatorname{grad} S = -\frac{\lambda_1\lambda_2\lambda_3}{2nH}x_T \tag{2.13}$$

and

$$\Delta S + S^2 = (\lambda_1+\lambda_2+\lambda_3)S - (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) - \frac{\lambda_1\lambda_2\lambda_3}{nH}p\,. \tag{2.14}$$

Remark 2.1. Note that the constant H is non-zero. In fact, if $H=0$, then M must be of 1-type as implies from (2.1). Moreover, in equations (2.13) and (2.14), the vector field x_T and the support function p correspond to a suitable origin $O\in E^{n+1}$, so that $x = x_1 + x_2 + x_3$.

At this point we mention a well known result (see for example [8], Theorem 3.1.3), which plays a conspicuous part in the proof of the Theorem 1.

LEMMA 2.1. *Let M be a surface in E^3 with constant mean curvature H. Let*

K be the Gaussian curvature, and Δ the Laplacian with respect to the induced metric on M. Then, in a neighborhood of a non-umbilic point, K satisfies the following equation

$$\Delta \log(H^2-K)=-4K. \tag{2.15}$$

Remark 2.2. Our Δ has opposite sign of that in [8].

Now, let $x(s, \theta)=(f(s)\cos\theta, f(s)\sin\theta, g(s))$, $s\in I$, $0<\theta<2\pi$, be the position vector of a regular surface of revolution M in the Euclidean space E^3, where the smooth curve $(f(s), g(s))$ is the profile curve C parametrized by the arc length and I is a real interval. Since M is a regular surface we may assume that $f(s)>0$, everywhere on I. The first fundamental form of M is $ds^2+f^2(s)d\theta^2$ and so its Laplacian operator Δ is given by

$$\Delta=-\frac{f'(s)}{f(s)}\frac{\partial}{\partial s}-\frac{\partial^2}{\partial s^2}-\frac{1}{f^2(s)}\frac{\partial^2}{\partial\theta^2},$$

where $f'(s)$ denotes the derivative of $f(s)$.

As it is known the curvature function $k(s)$ of C and the principal curvatures $k_1(s, \theta)$, $k_2(s, \theta)$ of M are given by

$$k(s)=f'(s)g''(s)-f''(s)g'(s), \quad k_1(s, \theta)=k(s) \quad \text{and} \quad k_2(s, \theta)=\frac{g'(s)}{f(s)},$$

respectively. So, for the mean curvature $H(s, \theta)$ of M we have

$$2H(s, \theta)=k(s)+\frac{g'(s)}{f(s)}. \tag{2.16}$$

Since, $(f'(s))^2+(g'(s))^2=1$, by using Frenet's formula we, easily, obtain

$$f''(s)=-k(s)g'(s) \quad \text{and} \quad g''(s)=k(s)f'(s). \tag{2.17}$$

At this point we prove the following lemma which is useful in the proof of Theorem 2.

LEMMA 2.2. *Let M be a regular surface of revolution with profile curve C as above. If M has constant mean curvature c, then*

$$k(s)=c+\frac{c_1}{f^2(s)}, \tag{2.18}$$

where c_1 is a constant. Moreover, we have

$$g'(s)=cf(s)-\frac{c_1}{f(s)}. \tag{2.19}$$

Proof. Since $H(s, \theta)=c$, we have from (2.16)

$$g'(s)=2cf(s)-k(s)f(s). \tag{2.20}$$

Differentiating (2.20) and using the second of (2.17) we obtain

$$\frac{d}{ds}((k(s)-c)f^2(s))=0$$

or

$$(k(s)-c)f^2(s)=c_1,$$

where c_1 is a constant of integration. Now from (2.20) by using (2.18) we obtain (2.19).

We are ready to prove the theorems.

3. Proof of theorem 1.

Let M be a 3-type surface in E^3. If M had constant mean curvature H, then H is non-zero (see Remark 2.1). From now on, we assume that M has non-zero mean curvature H. For brevity's sake we set $\lambda=-(\lambda_1\lambda_2\lambda_3/4H)$. We distinguish the following two cases:

CASE I. $\lambda=0$; that is M is of null 3-type. In that case (2.13) becomes $A\,\mathrm{grad}S=0$. From this we conclude that S is constant and so the Gaussian curvature K of M must be constant. In fact, if $\mathrm{grad}S$ is non-zero in an open set $U\subset M$, then the principal curvatures on U are the constants 0 and $2H$; that is S is a constant. Since surfaces in E^3 with non-zero constant mean curvature and constant Gaussian curvature are either of 1-type or of 2-type, because these are spheres or right circular cylinders (see [2]), we conclude that this case is impossible.

CASE II. $\lambda\neq 0$. In that case (2.13) and (2.14) become

$$A\,\mathrm{grad}S=\lambda x_T \tag{3.1}$$

and

$$\Delta S+S^2=(\lambda_1+\lambda_2+\lambda_3)S-(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)+2\lambda p. \tag{3.2}$$

Because of (3.1) and since A satisfies its own characteristic equation we obtain

$$2\lambda Hx_T-\lambda Ax_T=K\,\mathrm{grad}S, \tag{3.3}$$

from which, by using (2.5) and (2.7) one finds

$$K\,\mathrm{grad}S=\mathrm{grad}(\lambda p+\lambda Hr^2). \tag{3.4}$$

Taking the divergence of both sides of (3.4) and bearing in mind the Gauss equation $2K=4H^2-S$ we get

$$(4H^2-S)\Delta S+|\mathrm{grad} S|^2=2\lambda\Delta p+2\lambda H\Delta r^2 .$$

The last equation, by using (2.8) and (2.9) becomes

$$(4H^2-S)\Delta S+|\mathrm{grad} S|^2=-4\lambda H+2\lambda pS-8\lambda H^2 p . \tag{3.5}$$

Moreover, from (2.15) by a suitable manipulation we obtain

$$(S-2H^2)\Delta S+|\mathrm{grad} S|^2=2(S-4H^2)(S-2H^2)^2 . \tag{3.6}$$

The subtraction of (3.5) from (3.6) yields

$$(S-3H^2)\Delta S=(S-4H^2)(S-2H^2)^2+2\lambda H-\lambda pS+4\lambda H^2 p ,$$

from which by using (3.2) we obtain

$$\begin{aligned} 2S^3-(\lambda_1+\lambda_2+\lambda_3+11H^2)S^2+\{(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)+3H^2(\lambda_1+\lambda_2+\lambda_3)+20H^4\}S \\ -3\lambda pS+10\lambda H^2 p-3H^2(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)+2\lambda H-16H^6=0 . \end{aligned} \tag{3.7}$$

From (3.7) we find

$$\alpha\, \mathrm{grad} S+\beta\, \mathrm{grad} p=0 , \tag{3.8}$$

where we set

$$\begin{aligned} \alpha=6S^2-2S(\lambda_1+\lambda_2+\lambda_3+11H^2)+(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) \\ +3H^2(\lambda_1+\lambda_2+\lambda_3)+20H^4-3\lambda p \end{aligned} \tag{3.9}$$

and

$$\beta=10\lambda H^2-3\lambda S . \tag{3.10}$$

If β vanishes identically on an open subset $U\subset M$, then S should be constant on U, a contradiction (as in Case I). Thus, we may suppose that β is non-zero on M. Now, from (3.8) by using (2.7), (3.1) and (3.3) one finds

$$Ax_T=\frac{\alpha\lambda+\beta K}{2\beta H}x_T .$$

If x_T is identically zero on an open subset $V\subset M$, then V should be a sphere and thus of 1-type, a contradiction. This shows that x_T is in a principal direction with corresponding principal curvature $(\alpha\lambda+\beta K)/2\beta H$. Therefore, the other principal curvature is $2H-(\alpha\lambda+\beta K)/2\beta H$ and thus we have for the Gaussian curvature

$$K=\frac{\alpha\lambda+\beta K}{2\beta H}\left(2H-\frac{\alpha\lambda+\beta K}{2\beta H}\right)$$

or

$$\beta^2 K^2+2\alpha\beta\lambda K+\alpha^2\lambda^2-4\alpha\beta\lambda H^2=0 . \tag{3.11}$$

Solving (3.7) with respect to p and setting in (3.9) we find, from (3.9) and (3.10), α and β as functions of S. Substituting α and β in (3.11), and bearing in mind the Gauss equation $2K=4H^2-S$, we get the following polynomial equation for S

$$\sum_{i=0}^{6} \mu_i S^i=0,$$

where $\mu_6=(1089/4)\lambda^4$ is non-zero. This shows that S is constant, a contradiction, and the proof is completed.

4. Proof of theorem 2.

Suppose that M is a regular surface of revolution with constant mean curvature c as in paragraph 2. Assuming that M is of finite k-type ($k\geqq 1$), we conclude that its position vector $x(s, \theta)=(f(s)\cos\theta, f(s)\sin\theta, g(s))$ satisfies the following equation (equation 2.2)

$$\text{(4.1)} \quad \Delta^k x(s, \theta)-\sigma_1\Delta^{k-1}x(s, \theta)+\cdots+(-1)^{k-1}\sigma_{k-1}\Delta x(s, \theta)+(-1)^k\sigma_k(x(s, \theta)-x_0) =0$$

where $\sigma_i\,(i=1, \cdots, k)$ are some constants. From (4.1) we obtain

$$\Delta^k g(s)-\sigma_1\Delta^{k-1}g(s)+\cdots+(-1)^{k-1}\sigma_{k-1}\Delta g(s)+(-1)^k\sigma_k(g(s)-g_0)=0$$

or

$$\text{(4.2)} \quad \Delta^{k+1}g(s)-\sigma_1\Delta^k g(s)+\cdots+(-1)^{k-1}\sigma_{k-1}\Delta^2 g(s)+(-1)^k\sigma_k\Delta g(s)=0.$$

By computing $\Delta g(s)$ and $\Delta^2 g(s)$ one finds

$$\text{(4.3)} \quad \begin{aligned} \Delta g(s)&=-2cf'(s) \\ \Delta^2 g(s)&=-4cf'(s)\frac{P_2(f(s))}{f^4(s)}, \end{aligned}$$

where $P_2(t)=c_1^2+c^2t^4$ is a polynomial of degree 4. In these computations we take into account the relation (2.19) and $(f'(s))^2+(g'(s))^2=1$.

By a straightforward computation we prove the following:

$$\text{(4.4)} \quad \Delta^{m+1}g(s)=-4cf'(s)\frac{P_{m+1}(f(s))}{f(s)^{4m}}, \qquad m\geqq 0$$

where $P_1(t)=1/2$ and $P_{m+1}(t)$ is a polynomial of degree $4m$ with constant coefficients. Moreover, denoting by $\mathrm{const}(P_{m+1}(t))$ the constant term of $P_{m+1}(t)$ one finds

$$\mathrm{const}\,(P_{m+1}(t))=(4m-3)(4m-2)c_1^2\,\mathrm{const}\,(P_m(t)), \qquad m\geqq 1$$

from which we obtain

$$\text{const}\,(P_{m+1}(t))=5.6.9.10\cdots(4m-3)(4m-2)c_1^{2m}, \qquad m\geqq 2. \tag{4.5}$$

Setting (4.3) and (4.4) in (4.2) we find

$$4cf'(s)Q(f(s))=0, \tag{4.6}$$

where

$$Q(t)=P_{k+1}(t)-\sigma_1 P_k(t)t^4+\cdots+(-1)^{k-1}\sigma_{k-1}P_2(t)t^{4k-4}+(-1)^k\frac{\sigma_k}{2}t^{4k}.$$

It is obvious that

$$\text{const}\,(Q(t))=\text{const}\,(P_{k+1}(t)). \tag{4.7}$$

We distinguish the following two cases.

CASE I. Assume that $cf'(s)=0$ on I. Then either $c=0$ or $f(s)$ is constant on I. If $c=0$, then M is a minimal surface and since it is a surface of revolution we conclude that M is an open piece of a plane or a catenoid. If $f(s)$ is constant on I, then M is an open piece of a circular cylinder.

CASE II. Assume that $cf'(s)\neq 0$ on an open interval $J\subset I$. Then $f(s)$ is non constant on J and from (4.6) we have that $Q(f(s))=0$ on J, which means that the polynomial $Q(t)$ has infinite zeros. So, the constant term of $Q(t)$ and thus of $P_{m+1}(t)$ must be zero. Thus, $c_1=0$. Hence, from (2.18) we obtain $k_1(s, \theta)=k_2(s, \theta)=c$; that is all points of M, for $s\in J$, are umbilics. So, M must be a sphere since $c\neq 0$.

This completes the proof of Theorem 2.

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