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CHARACTERISTIC CLASSES OF ORIENTED 6-DIMENSIONAL SUBMANIFOLDS IN THE OCTONIANS

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§ 1. Introduction.

Let (M^6, c) be an oriented 6-dimensional submanifold in the 8-dimensional Euclidean space R^8 with the immersion c. In this paper, we shall identify R^8 with the octonians (or Cayley algebra) O in the natural way. By making use of the algebraic properties of the octonians, we can define an almost complex structure on (M^6, c). We may observe that this almost complex structure J is orthogonal with respect to the induced metric $\langle , \rangle$. Hence $M^6=(M^6, J, \langle , \rangle)$ is an almost Hermitian manifold ([B], [C], [G]). R. Bryant ([B]) established the structure equations of (M^6, c) from the standpoint of (O, Spin (7)) geometry. These equations play an important role in this paper.

C. T. C. Wall [W] has proved the following

THEOREM A. *Let M^6 be a 6-dimensional closed, simply-connected spinor manifold with torsion free homology. Then we have*

(1) *There exists an immersion from M^6 into $\boldsymbol{R}^8$ if and only if $p_1(M^6)+X^2=0$ holds for some $X\in 2H^2(M^6; \boldsymbol{Z})$, where $p_1(M^6)$ is the 1-st Pontrjagin class of M^6, In particular,*

(2) *There exists an embedding from M^6 into $\boldsymbol{R}^8$ if and only if $p_1(M^6)=0$.*

The purpose of this paper is to show some results related to the above Theorem A by making use of the properties of the induced almost Hermitian structure on (M^6, c). Namely, we shall prove the following

THEOREM B. *Let $M^6=(M^6, J, \langle , \rangle)$ be a 6-dimensional almost Hermitian submanifold immersed in the octonians O. Then, we have the following relations*

(1) $c_1(T^{1,0})=-c_1(\boldsymbol{\nu}^{1,0})=-e(\boldsymbol{\nu})$,

(2) $c_2(T^{1,0})=c_1(T^{1,0})^2$,

(3) $p_1(TM^6)+c_1(T^{1,0})^2=0$,

where $p_1(TM^6)$ is the 1-st Pontrjagin class of the tangent bundle TM^6 of M^6, $c_i(T^{1,0})$ is the i-th Chern class of the bundle $T^{1,0}=\{v\in TM^6\otimes \boldsymbol{C} \mid Jv=\sqrt{-1}v\}$, $e(\boldsymbol{\nu})$ is the Euler class of the normal bundle $\boldsymbol{\nu}$ and $c_1(\boldsymbol{\nu}^{1,0})$ is the 1-st Chern class of the bundle $\boldsymbol{\nu}^{1,0}=\{v\in \boldsymbol{\nu}\otimes \boldsymbol{C} \mid Jv=\sqrt{-1}v\}$, respectively.

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COROLLARY 1. *Let $M^6=(M^6, J, \langle , \rangle)$ be a 6-dimensional almost Hermitian submanifold immersed in the octonians O with flat normal connection. Then, we have*

$$c_1(T^{1,0})=c_1(\nu^{1,0})=e(\nu)=0, \quad and \quad c_2(T^{1,0})=p_1(TM^6)=0.$$

COROLLARY 2. *Let $M^6=(M^6, J, \langle , \rangle)$ be a 6-dimensional almost Hermitian submanifold in the octonians O which is embedded as a closed subset in O. Then, we have*

$$c_1(T^{1,0})=c_1(\nu^{1,0})=e(\nu)=0, \quad and \quad c_2(T^{1,0})=p_1(TM^6)=0.$$

Remark 1. E. Calabi ([C]) proved that an oriented 6-dimensional hypersurface in purely imaginary octonians $Im\,\boldsymbol{O}\cong\boldsymbol{R}^7$ is an almost Hermitian manifold and its 1-st Chern class vanishes. Corollary 1 is a generalization of this result.

Remark 2. Corollary 2 improves slightly the necessary part of (2) of Theorem A in our situation.

Remark 3. If (M^6, ι) satisfies the assumption in Corollary 1 or 2, then it is a spin manifold (see [L-M, Remark 1.8, p. 82]).

In this paper, we adopt the same notational convention as in [B], [H2] and all the manifolds are assumed to be connected and of class C^∞ unless otherwise stated. The author would like to express his hearty thanks to Professor Sekigawa for his valuable suggestions and to the refree for his valuable comments.

§ 2. Preliminaries.

We shall recall the following formulation of the Spinor group Spin(7) ([H-L]). Let $S^6=\{u\in Im\,\boldsymbol{O}\,|\,\langle u, u\rangle=1\}$ where $Im\,\boldsymbol{O}$ is the purely imaginary octonians. Then, for any $u\in S^6$, we have $u=-\bar{u}$ and $u^2=-uu=-\langle u, u\rangle=-1$. So, we may use $u\in S^6$ to define a map $J_u: \boldsymbol{O}\to\boldsymbol{O}$ such that $J_u(x)=xu$ for any $x\in\boldsymbol{O}$. Each J_u is an orthogonal complex structure on O. It is known that Spin(7) is isomorphic to the subgroup of $SO(8)$ generated by the set $\{J_u\,|\,u\in S^6\}$. Also Spin(7) is isomorphic to the group $\{g\in SO(8)\,|\,g(uv)=g(u)\chi(g)(v)$ for any $u, v\in\boldsymbol{O}\}$, where I is the map from $SO(8)$ to itself defined by $\chi(g)(v)=g(g^{-1}(1)v)$ for any $v\in\boldsymbol{O}$. Then we may observe that $\chi|_{\mathrm{Spin}(7)}$: Spin(7)$\to SO(7)$ is a double covering map and satisfies the following equivariance $g(u)\times g(v)=\chi(g)(u\times v)$ for any $g\in$Spin(7), where X is the *vector cross product* defined by $u\times v=(\bar{v}u-\bar{u}v)/2$. Now, we shall recall the structure equations of an oriented 6-dimensional submanifold in (O, Spin(7)). It is known that the octonians O is considerd as the algebra $\boldsymbol{H}\oplus\boldsymbol{H}$ where H is the quaternions. We put a basis of $\boldsymbol{C}\otimes_{\boldsymbol{R}}\boldsymbol{O}$ by; N, $E_1=iN$, $E_2=jN$, $E_3=kN$, $\bar{N}$, $\bar{E}_1=i\bar{N}$, $\bar{E}_2=j\bar{N}$, $\bar{E}_3=k\bar{N}$ where $\varepsilon=(0, 1)\in\boldsymbol{H}\oplus\boldsymbol{H}$, $N=(1-\sqrt{-1}\,\varepsilon)/2$, $\bar{N}=(1+\sqrt{-1}\,\varepsilon)/2\in\boldsymbol{C}\otimes_{\boldsymbol{R}}\boldsymbol{O}$ and $\{l, i, j, k\}$ is the canonical basis of H. We call this basis the standard one of $\boldsymbol{C}\otimes_{\boldsymbol{R}}\boldsymbol{O}$ and a basis $(n, f, \bar{n}, \bar{f})$ of $C\otimes_R O$ is said to be *admissible*, if $(n, f, \bar{n}, \bar{f})=(N, E, \bar{N}, \bar{E})g$ for some $g\in \mathrm{Spin}(7)\subset M_{8\times 8}(C)$. We shall identify Spin(7) with the admissible basis. Here, we may note that the Grassmannian manifold $G_2(O)$ of the oriented 2-planes in O is isomorphic to the homogeneous space $\mathrm{Spin}(7)/U(3)$. So, we can set

$$\mathcal{F}_\iota(M^6)=\{(p\ (n, f, \bar{n}, \bar{f}))\mid -2\sqrt{-1}\, n\wedge\bar{n}=T_p^{\perp}M^6 \text{ for any } p\in M^6\}.$$

Then $x: \mathcal{F}_\iota(M^6)\to M^6$ is a principal $U(3)$-bundle over M^6. The induced almost complex structure is defined by:

$$\iota_*(JX)=(\iota_*X)(\eta\times\xi) \tag{2.1}$$

for $X\in T_pM^6$, where ξ, η are an orthonormal pair of the normal space and $n=1/2(\xi-\sqrt{-1}\eta)$ (for details, see [B], [H1]). By making use of the properties of Spin(7), we may observe that this almost complex structure is an invariant of Spin(7) in the following sence Let M^6 be an oriented 6-dimensional manifold and $\iota, \iota': M^6\to O$ be isometric immersions. If there exists $g\in\mathrm{Spin}(7)$ such that $\iota'=g\circ\iota$ (up to parallel displacement) then $J=J'$ where J and J' are the almost complex structures on M^6 induced by the immersions ι and ι', respectively. Also, we can easily see that $T^{1,0}=\mathrm{span}_C\{f_1, f_2, f_3\}$ where $T^{1,0}$ is the subbundle of the complexified tangent bundle $TM^6\otimes C$ whose fibre is $\sqrt{-1}$-eigenspace of the almost complex structure J. Then we have the following structure equations:

$$d\iota=f\omega+\bar{f}\bar{\omega}, \tag{2.2}$$

$$df=-n{}^t\bar{\mathfrak{h}}+f\kappa-\bar{n}{}^t\bar{\theta}+\bar{f}[\theta], \qquad \text{(Gauss formula)} \tag{2.3}$$

$$dn=n(\sqrt{-1}\rho)+f\mathfrak{h}+\bar{f}\bar{\theta}, \qquad \text{(Weingarten formula)} \tag{2.4}$$

$$d(\sqrt{-1}\rho)={}^t\bar{\mathfrak{h}}\wedge\mathfrak{h}+{}^t\theta\wedge\bar{\theta}, \qquad \text{(Ricci equation)} \tag{2.5}$$

$$\begin{cases} d\mathfrak{h}=-\mathfrak{h}\wedge(\sqrt{-1}\rho)-\kappa\wedge\mathfrak{h}-[\bar{\theta}]\wedge\bar{\theta}, \\ d\theta=-\kappa\wedge\theta+\theta\wedge(\sqrt{-1}\rho)-[\theta]\wedge\bar{\mathfrak{h}}, \end{cases} \qquad \text{(Codazzi equation)} \tag{2.6}$$

$$d\kappa=\mathfrak{h}\wedge{}^t\bar{\mathfrak{h}}-\kappa\wedge\kappa+\theta\wedge{}^t\bar{\theta}-[\bar{\theta}]\wedge[\theta] \qquad \text{(Gauss equation)} \tag{2.7}$$

where ρ: R-valued 1-form, $\mathfrak{h}$, θ: $M_{3\times 1}(C)$-valued 1-forms, and κ: $M_{3\times 3}(C)$ valued 1-form on $\mathcal{F}_\iota(M^6)$ which satisfy $\kappa+{}^t\bar{\kappa}=0$ and $tr\kappa+\sqrt{-1}\rho=0$. Here, $[\theta]$ is defined by

$$[\theta]=\begin{pmatrix} 0 & \theta^3 & -\theta^2 \\ -\theta^3 & 0 & \theta^1 \\ \theta^2 & -\theta^1 & 0 \end{pmatrix}$$

where $\theta={}^t(\theta^1, \theta^2, \theta^3)$. By (2.2) and (2.3), the second fundamental form Π is given by

$$\text{(2.8)} \qquad \mathrm{II}=-2Re\{({}^t\bar{\mathfrak{h}}\circ\omega+{}^t\theta\circ\bar{\omega})n\}\,.$$

Applying the Cartan's lemma, we may conclude that there exists 3x3 matrices of functions A, B, C on $\mathcal{F}_\iota(M^6)$ (with complex values) satisfying

$$\text{(2.9)} \qquad A={}^tA\,, \qquad C={}^tC\,, \qquad \begin{pmatrix}\mathfrak{h}\\ \theta\end{pmatrix}=\begin{pmatrix}\bar{B} & \bar{A}\\ {}^tB & \bar{C}\end{pmatrix}\begin{pmatrix}\omega\\ \bar{\omega}\end{pmatrix}.$$

Hence, we have the following canonical splittings:

$$\text{(2.10)} \quad \mathrm{II}^{2,0}=(-{}^t\omega\circ A\omega)n\,, \quad \mathrm{II}^{1,1}=(-{}^t\bar{\omega}\circ{}^tB\omega-{}^t\omega\circ B\bar{\omega})n\,, \quad \mathrm{II}^{0,2}=(-{}^t\bar{\omega}\circ\bar{C}\bar{\omega})n\,.$$

§ 3. Proofs of Theorem B and Corollaries 1, 2.

First, we shall define the Hermitian connections on $T^{1,0}$ and $\nu^{1,0}$. Let X be a section of the bundle $T^{1,0}$. Then, we can write $X=f\alpha$, where a is $M_{3\times 1}(\boldsymbol{C})$-Valued function on $\mathcal{F}_\iota(M^6)$. We define the operator on $T^{1,0}$ such that $\tilde{\nabla}(f\alpha)=f(d\alpha+\kappa\alpha)$. Then we have

LEMMA 3.1. *The operator tf defined above is a connection on $T^{1,0}$ and satisfies the following conditions,*

(1) $\tilde{\nabla}$ *is complex, that is* $\tilde{\nabla}J=0$,

(2) $\tilde{\nabla}$ *preserves the Hermitian metric, that is*

$$d\langle X, \bar{Y}\rangle=\langle\tilde{\nabla}X, \bar{Y}\rangle+\langle X, \overline{\tilde{\nabla}Y}\rangle\,,$$

where X, Y are sections of $T^{1,0}$ and $\bar{Y}$ is the conjugation of Y.

Proof. Let $f'=(f_1', f_2', f_3')$ be another frame field on M^6, where f_i' is a section of $T^{1,0}$, then there exists $U(3)$-valued function A on $\mathcal{F}_\iota(M^6)$ such that $f'=fA$. By direct calculation, we have

$$\kappa'=A^{-1}dA+A^{-1}\kappa A\,.$$

Hence, $\tilde{\nabla}$ is well-defined. For any section X of $T^{1,0}$, we have

$$\begin{aligned}(\tilde{\nabla}J)X&=(\tilde{\nabla}J)(f\alpha)=\tilde{\nabla}(J(f\alpha))-J(\tilde{\nabla}(f\alpha))\\ &=\sqrt{-1}\tilde{\nabla}(f\alpha)-J(f(d\alpha+\kappa\alpha))=\sqrt{-1}(f(d\alpha+\kappa\alpha))-J(f(d\alpha+\kappa\alpha))=0\,.\end{aligned}$$

Hence, we have (1).

$$\begin{aligned}&\langle\tilde{\nabla}X, \bar{Y}\rangle+\langle X, \overline{\tilde{\nabla}Y}\rangle\\ &=\langle f(d\alpha+\kappa\alpha), \bar{f}\bar{\beta}\rangle+\langle f\alpha, \bar{f}(d\bar{\beta}+\bar{\kappa}\bar{\beta})\rangle\end{aligned}$$

$$=\frac{1}{2}\{{}^t(d\alpha)\bar\beta+{}^t(\kappa\alpha)\bar\beta\}+\frac{}{2}\{{}^t(\alpha)d\bar\beta+{}^t(\alpha)\bar\kappa\bar\beta\}$$

$$=\frac{1}{2}\{{}^t(d\alpha)\bar\beta+{}^t(\alpha)d\bar\beta\}=d\langle X,\bar Y\rangle\,.$$

Hence we have (2). □

Similarly, let v be a section of $\nu^{1,0}$. Then we can write $v—n\zeta$ where ζ is the C-valued function. We define the operator $\nabla^\perp$ as follows: $\nabla^\perp v=n(d\zeta+\zeta\sqrt{-1}\rho)$.

LEMMA 3.2. *The operator $\nabla^\perp$ defined above is a connection on $\nu^{1,0}$ and satisfy the following conditions,*

(1) *$\nabla^\perp$ is complex, that is* $7V=0$,

(2) *$\nabla^\perp$ preserves the Hermitian metric, that is*

$$d\langle u,\bar v\rangle=\langle\nabla^\perp u,\bar v\rangle+\langle u,\overline{\nabla^\perp v}\rangle$$

where u, v are sections of $\nu^{1,0}$ and v is the conjugation of v.

Proof. Same as that of Lemma 3.1. D

We are now in a position to prove Theorem B. By Lemma 3.1, we see that the 1-st Chern class of $T^{1,0}$ is given by

$$c_1(T^{1,0})=-(2\pi\sqrt{-1})^{-1}[tr\Omega]\in H^2_{DR}(M^6)\,, \tag{3.1}$$

where $\Omega=d\kappa+\kappa\wedge\kappa$ is the curvature form of $\tilde\nabla$. By (2.7), we get

$$\Omega=\mathfrak{h}\wedge{}^t\bar{\mathfrak{h}}+\theta\wedge{}^t\bar\theta-[\bar\theta]\wedge[\theta]\,, \tag{3.2}$$

By (2.5), (3.1) and (3.2), we get

$$c_1(T^{1,0})=-[(2\pi\sqrt{-1})^{-1}({}^t\mathfrak{h}\wedge\bar{\mathfrak{h}}-{}^t\theta\wedge\bar\theta)]=[(2\pi\sqrt{-1})^{-1}(d\sqrt{-1}\rho)]\,. \tag{3.3}$$

On the other hand, by Lemma 3.2 and (2.5), we have

$$c_1(\nu^{1,0})=-[(2\pi\sqrt{-1})^{-1}(d\sqrt{-1}\rho)]\,. \tag{3.4}$$

Since the codimension is two, we see that

$$c_1(\nu^{1,0})=e((\nu^{1,0})_R)=e(\nu)\,. \tag{3.5}$$

By (3.3), (3.4) and (3.5), we have (1) of Theorem B. Next we shall prove (2) of Theorem B. Since the restriction $T\boldsymbol{O}|_{\iota(M^6)}$ is the pull back of $T\boldsymbol{O}$ to M^6 under the immersion ι, by the functoriality of the total Pontrjagin class

$$P(T\boldsymbol{O}|_{\iota(M^6)})=\iota^*(P(T\boldsymbol{O}))=1\,.$$

On the other hand, by (1), we get

$$\begin{aligned} P(T\boldsymbol{O}\,|\,_{\iota(M^6)}) &= c(T\boldsymbol{O}\,|\,_{\iota(M^6)}\otimes \boldsymbol{C}) = c((\iota_*(TM^6)\oplus\nu)\otimes\boldsymbol{C}) \\ &= c((\iota_*(TM^6)\otimes C)\oplus(\nu\otimes\boldsymbol{C})) = c(T^{1,0}\oplus T^{0,1}\oplus\nu^{1,0}\oplus\nu^{0,1}) \\ &= (1+c_1(T^{1,0})+c_2(T^{1,0})+c_3(T^{1,0}))(1-c_1(T^{1,0}) \\ &\quad +c_2(T^{1,0})-c_3(T^{1,0}))(1+c_1(\nu^{1,0}))(1-c_1(\nu^{1,0})) \\ &= 1+2c_2(T^{1,0})-2c_1(T^{1,0})^2. \end{aligned}$$

Hence we have (2). From (2), we have the equality (3). D

We see that Corollary 1 follows from Theorem B. The following Proposition 3.3 will then complete the proof of Corollary 2.

PROPOSITION 3.3 ([M-S; p. 120]). *Let M^n be an oriented, n-dimensional manifold which is embedded as a dosed subset in $(n+k)$-dimensional Euclidean space $\boldsymbol{R}^{n+k}$. Then we have $e(\nu)=0$ where $e(\nu)$ is the Euler class of the normal bundle ν.*

§ 4. Applications.

In this section, we shall give some applications of the main Theorem B and Corollaries 1, 2, and some examples.

Let M^6 be a 6-dimensional compact irreducible Riemannian 3-symmeteric space, i.e., M^6 is one the following spaces:

(1) $SU(3)/T^2$, (2) $SU(4)/S(U(1)\times U(3))$,
(3) $SO(5)/U(1)\times SO(3)$, (4) $SO(5)/U(2)$,
(5) $Sp(2)/U(1)\times Sp(1)$, (6) $Sp(2)/U(2)$,
(7) $SO(6)/U(3)$, (8) $G_2/SU(3)=S^6$.

We note that the spaces (2), (4), (5) and (7) are diffeomorphic to $\boldsymbol{P}^3(\boldsymbol{C})$, and, (3) is diffeomorpic to $G_2(\boldsymbol{R}^5)$. *T.* Koda [K] has calculated the characteristic classes of compact irreducible Riemannian 3-symmetric spaces. From his results and Corollary 2, we have

THEOREM 4.1. *Let M^6 be a 6-dimensional compact irreducible Riemannian 3-symmetric space. If M^6 can be embedded in $\boldsymbol{R}^8$, then it is* (1) *or* (8). *In fact, $SU(3)/T^2$ can be embedded in $S^7\subset\boldsymbol{O}$ as a Cartan hypersurface.*

Next, we shall calculate characteristic classes of three examples.

Example 1. Let $\iota: S^2\to\boldsymbol{R}^3$ be the totally umbilical embedding and $\iota\times id: S^2\times\boldsymbol{R}^4\to\boldsymbol{R}^3\oplus\boldsymbol{R}^4=Im\,\boldsymbol{O}\subset\boldsymbol{O}$ be the product embedding. By Corollary 1, its 1-st Chern class and 1-st Pontrjagin class vanish.

Example 2. (Example of non-zero 1-st Chern class with zero 1-st Pontrjagin class). Let $\iota: S^2(1/3)\to S^4(1)$ be the Veronese surface which is defined by;

$$\iota(x, y, z)=\left(\frac{xy}{\sqrt{3}}, \frac{xz}{\sqrt{3}}, \frac{yz}{\sqrt{3}}, \frac{x^2-y^2}{2\sqrt{3}}, \frac{x^2+y^2-2z^2}{6}\right),$$

where $x^2+y^2+z^2=3$. We fix $p\in S^4(1)\setminus\iota(S^2(1/3))$ and denote by π_p the stereographic projection; π_p: $S^4\setminus\{p\}\to\boldsymbol{R}^4$. We shall consider the following product immersion

$$\pi_p\ \iota\times id:\ S^2\times\boldsymbol{R}^4\longrightarrow\boldsymbol{H}\oplus\boldsymbol{H}=\boldsymbol{O}.$$

Since $H^4_{DR}(S^2\times\boldsymbol{R}^4)=0$, the 1-st Pontrjagin class of $T(S^2\times\boldsymbol{R}^4)$ vanishes. Next, we shall prove that the 1-st Chern class does not vanish. We note that the induced almost complex structure satisfies the following:

$$J(T_{\tilde{p}}S^2)=T_{\tilde{p}}S^2,\qquad J(T_q\boldsymbol{R}^4)=T_q\boldsymbol{R}^4,$$

for any $(\tilde{p}, q)\in S^2\times\boldsymbol{R}^4$. Hence, we may compute the following

$$\int_{\pi_p\circ\iota(S^2)}c_1(T^{1,0})=\int_{\pi_p\circ\iota(S^2)}\frac{1}{2\pi}(d\rho)=-\frac{1}{2\pi}\int_{\pi_p\circ\iota(S^2)}K^{\perp}\sigma_0$$

where $K^{\perp}$, σ_0 are the normal curvature, volume element of $\pi_p\circ\iota(S^2)$, respectively. Since $K^{\perp}\sigma_0$ is a conformal invaiant, we see that $\pi_p^*(K^{\perp}\sigma_0)=(2/3)\sigma'$, where 2/3 is the normal curvature of the Veronese surface. Therefore, we get

$$-\frac{1}{2\pi}\int_{\pi_p\circ\iota(S^2)}K^{\perp}\sigma_0=-\frac{1}{2\pi}\int_{\iota(S^2)}\frac{2}{3}\sigma'=-\frac{1}{6\pi}\int_{S^2}\sigma=-\frac{1}{6\pi}4\pi\times3=-2,$$

where σ', σ are the volume element of $P^2(\boldsymbol{R})$, $S^2(1/3)$, respectively. Hence, we have $c_1(T^{1,0}(S^2\times\boldsymbol{R}^4))\neq0$.

Remark. $\pi_p\circ\iota$ is an immersion but not an embedding.

Example 3. Let $\tilde{\iota}=\iota\times\iota\times\iota: S^2\times S^2\times S^2\to S^8\subset\boldsymbol{R}^9=\boldsymbol{R}^3\oplus\boldsymbol{R}^3\oplus\boldsymbol{R}^3$ be the product embedding where ι is the totally umbilical embedding. We fix $p\in S^8\setminus\iota(S^2\times S^2\times S^2)$ and let π_p: $S^8\setminus\{p\}\to\boldsymbol{R}^8$ be the stereographic projection. Then $\pi_p\circ\tilde{\iota}: S^2\times S^2\times S^2\to\boldsymbol{R}^8$ is an embedding. So the 1-st Chern class of $T^{1,0}(S^2\times S^2\times S^2)$ and the 1-st Pontrjagin class of $T(S^2\times S^2\times S^2)$ vanish. On the other hand, if we identify S^2 with the complex projective space $P^1(\boldsymbol{C})$, then we have $c_1(T^{1,0}(P^1(\boldsymbol{C})\times P^1(\boldsymbol{C})\times P^1(\boldsymbol{C})))\neq0$. Therefore, the induced almost complex structure is different from the product complex structure.

Lastly, we shall give some curvature condition that the immersion has a self intersection. We shall recall the following

PROPOSITION 4.2 ([B]). *Let $M^6=(M^6, J, \langle\ ,\ \rangle)$ be a 6-dimensional almost Hermitian submanifold immersed in the octonians O. Then, its almost Hermitian structure is semi-Kähler, that is, $d\Omega^2=0$ where $\Omega=(\sqrt{-1}/2)\omega\wedge\bar{\omega}=(\sqrt{-1}/2)\sum_{i=1}^3\omega^i\wedge\bar{\omega}^i$ is the Kähler form of M^6.*

PROPOSITION 4.3. *Let $M^6=(M^6, J, \langle , \rangle)$ be a compact 6-dimensional almost Hermitian submanifold immersed in the octonians O. Then we have*

$$\int_{M^6} c_1(T^{1,0})\wedge\Omega^2=-\frac{2}{\pi}\int_{M^6}(|\mathrm{II}^{2,0}|^2-|\mathrm{II}^{0,2}|^2)\sigma$$

where σ is the volume element of M^6, $|\mathrm{II}^{2,0}|^2:=trA\bar{A}$ and $|\mathrm{II}^{0,2}|^2:=trC\bar{C}$.

Proof. By (2.5), (2.9), (2.10) and (3.3), we get

$$c_1(T^{1,0})\wedge\Omega^2=-\frac{1}{2\pi}d\rho\wedge\Omega^2$$

$$=\frac{\sqrt{-1}}{4\pi}(trA\bar{A}-trC\bar{C})\omega^1\wedge\bar{\omega}^1\wedge\omega^2\wedge\bar{\omega}^2\wedge\omega^3\wedge\bar{\omega}^3=-\frac{2}{\pi}(tr\bar{A}A-tr\bar{C}C)\sigma$$

From this, we get the desired results. □

THEOREM 4.4. *Let $M^6=(M^6, J, \langle , \rangle)$ be a compact 6-dimensional almost Hermitian submanifold immersed in the octonians O. If $\int_{M^6}|\mathrm{II}^{2,0}|^2\sigma\neq\int_{M^6}|\mathrm{II}^{0,2}|^2\sigma$, then the immersion has a self intersection.*

Proof. If the immersion is an embedding, by Proposition 4.2 and Corollary 2, we see that $c_1(T^{1,0})\wedge\Omega^2$ is closed. By Proposition 4.3 and Stokes' theorem, we get the desired result. □

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