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ON MONOMIALS AND HAYMAN'S PROBLEM

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1. Introduction and main results

Let $f(z)$ be a meromorphic function in the plane. We shall, for brevity, write / insted of $f(z)$. It is assumed that the reader is familiar with the notations of Nevanlinna theory (see, for example [1]). Throughout this paper we denote by $S(r, /)$, as usual, any function satisfying

$$S(r, f)=o(T(r, /))$$

as $r\to\infty$, possibly outside a set of r value of finite linear measure and $N_{1)}(r, f)$ and $N_{2)}(r, /)$ count only the simple and multiple poles of / respectively.

L. R. Sons ([5]) has considered the monomial of form

$$\psi=f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k} \tag{1}$$

where $n_0, n_1, \cdots, n_k$ are non-negative integers. The following result is proved.

THEOREM A. (i) *Iff is a transcendental meromorphicfunction in the plane with*

$$N_{1)}\left(r,\frac{1}{f}\right)=S(r, /)$$

and ψ *has the form* (1) *where* $n_0\geqq 1$, $n_k\geqq 1$, $n_i\geqq 0$ *for* $i\neq\theta$, k *and if*

$$2^k\left(2n_0+\underset{i=0}{23}(1+i)n_i\right)<(2^k+2n_0-1)\left(\underset{i=0}{23}(1+i)n_i\right) \tag{2}$$

then $\delta(c, \psi)<1$ *for* $c\neq 0, \infty$.

(ii) *// / is a transcendental meromorphic function in the plane and* ψ *has the form* (1) *where* $n_0\geqq 2$, $n_k\geqq 1$, $n_i\geqq 0$ *for* $i\neq 0$, k, *and if*

$$2^k\left(n_0+\sum_{i=0}^{k}(1+i)n_i\right)<(2^k+n_0-1)\left(\sum_{i=0}^{k}(1+i)n_i\right) \tag{3}$$

then $\delta(c, \psi)<1$ *for* $c\neq 0, \infty$.

The assumption of Theorem A can be weakened. For $n_0\geqq 2$ N. Steinmetz ([7]) proved the following theorem:

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THEOREM B. *Let f be a transcendental meromorphic function in the plane and* ψ *has the form* (1). *If* $n_0 \geqq 2$, $n_1 + \cdots + n_k \geqq 1$, *then*

$$\limsup_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{\psi-c}\right)}{T(r, \psi)} > 0$$

for $c \neq 0, \infty$.

In this paper we use a modified version of Steinmetz's proof to consider the case of $n_0=1$ and prove condition (2) is not necessary. The result is the following:

THEOREM 1. *Let f be a transcendental meromorphic function in the plane with*

$$N_{1)}\left(r, \frac{1}{f}\right) = S(r, f) \tag{4}$$

and let

$$\psi = f(f')^{n_1}(f'')^{n_2} \cdots (f^{(k)})^{n_k} \tag{5}$$

where $n_1, n_2, \cdots, n_k$ *are non-negative integers. If* $n_1 \geqq 1$ *then*

$$\limsup_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{\psi-c}\right)}{T(r, \psi)} > 0$$

for $c \neq 0, \infty$.

Obviously, Theorem 1 improves Sons's result.

Let f be a transcendental meromorphic function in the plane. W. K. Hayman ([2]) and E. Mues ([4]) proved respectively if $n \geqq 3$ and $n=2$ then $f^n f'$ assumes all values except possibly zero infinitely often. The case $n=1$ is still open (W. K. Hayman [3], Problem 1.19), but our Theorem 1 enables us to obtain the following theorem:

THEOREM 2. *Let f be a transcendental meromorphic function in the plane with* $N_{1)}(r, 1/f) = S(r, f)$. *Then* ff' *assumes all values except possibly zero infinitely often.*

2. Preliminary results and lemmas

For the proof of theorem we introduce some results on algebroid functions (cf. [8]).

The solution $w=w(z)$ of the functional equation

$$a_n(z)w^n + \cdots + a_0(z) = 0 \tag{6}$$

is called an algebroid function, where $a_n(z), \cdots, a_0(z)$ are meromorphic functions, n is a positive integer.

LEMMA 1 ([8]). *// $a_n(z)\not\equiv 0$, then equation* (6) *has at least one solution.*

Obviously, meromorphic functions are algeoroid.

A polynomial in w and their derivatives of the form

$$Q[w]=\sum_{j=1}^{l} a_j(z)w^{l_0(j)}(w')^{l_1^{(j)}}\cdots(w^{(k_j)})^{l_{k_j}^{(j)}} \tag{7}$$

is called a differential polynomial in w, where $a_j(z)$ ($j=1, \cdots, /$) are meromorphic functions satisfying

$$T(r,\mathrm{fl},)=\mathrm{S}(\mathrm{r},w), \quad \jmath=1, \cdots, l. \tag{8}$$

If $Q[w]$ has only one term, it is called a (differential) monomial in w. We denote $(d/dz)Q[w]$as $Q'[w]$.

If (8) is replaced with $m(r, a_j)=S(r,w)$, then $Q[w]$ is called a quasi-differential polynomial in w. The following lemma on quasi-differential polynomials is essentially due to He Yu-Zhan and Xiao Xiu-Zhi ([8, 9])

LEMMA 2. *Let w be a nonconstant algebroid function, $Q_1[w]$ and $Q_2[w]$ be quasi-differentıapolynomials in w and n be a positive integer. If*

$$w^n Q_1[w]=Q_2[w]$$

and $n\geqq\gamma_{Q_2}$ then $m(r, Q_1[w])=S(r, w)$, where γ_{Q_2} is the degree of $Q_2[w]$.

LEMMA 3. *Let w be an algebroid function, $Q[w]$ be a differential polynomial in w, and n be a positive integer. If*

$$w^n Q[w]=d \quad \text{and} \quad d\neq 0 \text{ is Const}, \tag{9}$$

then $w\equiv$Const.

Proof. Obviously, $Q[w]\not\equiv 0$. Suppose $u\not\equiv$Const, then Lemma 2 yields $m(r, Q[w])=S(r, w)$.

The poles of w are not any poles of $Q[w]$ by (9). Combining (7) and (8), we get

$$N(r, Q[w])=S(r, w).$$

Thus

$$T(r, Q[w])=S(r, w)$$

and

$$nT(r, w)=T\left(r, \frac{1}{Q[w]}\right)+O(1)$$

$$=T(r, Q[w])+O(1)=S(r, w).$$

This is impossible. Thus Lemma 3 is proved.

3. Proof of Theorem 1

Suppose that there is some $c \neq 0, \infty$, such that

$$\limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{\phi - c}\right)}{T(r, \phi)} = 0 .$$

Since $T(r, \phi) = O(T(r, /))$, we get

$$\bar{N}\left(r, \frac{1}{\phi - c}\right) = S(r, f) .$$

Without loss of the generality, we may assume that $c = 1$. Set

$$Q[f'] = (f')^{n_1} \cdots (f^{(k)})^{n_k}$$

and

$$F = \phi - 1 = fQ[f'] - 1. \tag{10}$$

Then

$$\bar{N}\left(r, \frac{1}{F}\right) = S(r, f) . \tag{11}$$

Obviously, $F \not\equiv 0$. By (10) we obtain

$$fQ'[f'] + f'Q[f'] = fQ[f'] \frac{F'}{F} - \frac{F'}{F} .$$

That is,

$$f a(z) = -\frac{F'}{F} , \tag{12}$$

where

$$a(z) = Q'[f'] + \frac{f'}{f} Q[f'] - Q[f'] \frac{F'}{F} \tag{13}$$

is a quasi-differential polynomial in /, since $m(r, f/f) = S(r, /)$ and $m(r, F'/F) = S(r, /)$.

If $a(z) \equiv 0$, then $F \equiv \text{Const}$. Further $f \equiv \text{Const}$ by Lemma 3 and (10). Hence $a(z) \not\equiv 0$.

From (12) and Lemma 2 we obtain

$$m(r, a) = S(r, f) . \tag{14}$$

Now we note that $a(z)$ can have poles only at the poles or zeros of / or the zeros of F by (13). Since $n_1 \geqq 1$ and

$$\frac{f'}{f} Q[f'] = \frac{(f')^{n_1+1}}{f} (f'')^{n_2} \cdots (f^{(k)})^{n_k}$$

it is easily seen from (13) that the multiple zeros of / are not any poles of $a(z)$. On the other hand, by (12) the poles of / are not any poles of $a(z)$.

Thus

$$N(r, a) \leqq N_{1)}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{F}\right).$$

Together with above inequalities (4), (11) and (14), we get

$$\mathrm{T(r,}\ a) = S(r, /). \tag{15}$$

Dividing equation (12) by $a(z)$, we deduce

$$m(r, f) \leqq m\left(r, \frac{1}{a}\right) + m\left(r, \frac{F'}{F}\right) + O(1)$$

$$\leqq T(r, a) + S(r, f),$$

so that

$$m(r, f) = S(r, f). \tag{16}$$

It is easily seen from (12) that $a(z)$ has a zero of multiplicity at least $q-1$ at any pole of / with order $q(\geqq 2)$. Thus, we have

$$N_{(2}(r, f) \leqq 2N\left(r, \frac{1}{a}\right) \leqq 2T(r, a) + O(1),$$

so that

$$N_{(2}(r, f) = S(r, /). \tag{17}$$

Thus / must have infinite simple poles.

Now we multiply (12) by $fQ[f']$ and (13) by / respectively and subtract. This gives

$$a(z)Q[f']f^2 + Q'[f']f + Q[f']f' - a(z)f = 0. \tag{18}$$

Let z_0 be a simple pole of /, then $a(z_0) \neq 0, \infty$ by (12). We may write $f(z)$ and $a(z)$ near z_0 in the form

$$f(z) = \frac{\bar{d}_1}{z\ \ z_0} + \bar{d}_0 + O(z - z_0)$$

and

$$a(z) = a(z_0) + a'(z_0)(z - z_0) + O((z - z_0)^2),$$

where $\bar{d}_1 \neq 0$ and $\bar{d}_0$ depend on z_0. Combining these with (18) we see that the coefficients $\bar{d}_1$ and $\bar{d}_0$ have the form

$$\bar{d}_1 = \frac{\ }{a(z_0)}, \qquad \bar{d}_0 = -\frac{(\Gamma+1)^2}{\Gamma+2}\frac{a'(z_0)}{a^2(z_0)}$$

where $\Gamma = 2n_1 + - + (k+1)n_k$. Thus if let

$$d_1(z) = \frac{\Gamma+1}{a(z)}, \qquad d_0 = -\frac{(\Gamma+1)^2}{\Gamma+2}\frac{a'(z)}{a^2(z)}$$

then $d_1(z_0) = \bar{d}_1$, $d_0(z_0) = \bar{d}_0$ and

$$f(z)=\frac{d_1(z_0)}{z-z_0}+d_0(z_0)+O(z-z_0)$$

for any simple pole z_0 of /. Furthermore $d_1(z)$ and $d_0(z)$ are meromorphic and

$$T(r,\ \text{rfO}+T(r,\ d_0)=S(r,\ f)$$

by (15). Here by using (16), (17) and Steinmetz's Lemma 2 (see [6, P 156]), we know that there exist the meromorphic functions $b_0(z)$, $b_1(z)$, $b_2(z)(\not\equiv 0)$ satisfying

$$\mathrm{T}(\mathrm{r},\ b_i)=S(r,\ f), \qquad (i=0,\ 1,\ 2) \tag{19}$$

such that

$$f'=b_0(z)+b_1(z)f+b_2(z)f^2. \tag{20}$$

That is, / satisfies Riccati equation

$$w'=b_0(z)+b_1(z)w+b_2(z)w^2. \tag{21}$$

Using (20) over and over again we deduce that

$$f^{(j)}=j!\ b_2^j(z)f^{j+1}+ \quad , \qquad j=1,\ 2,\ \cdots$$

are polynomials in /. Thus we may write

$$Q[f']=P(z,\ f)$$

and

$$\begin{aligned} F&=fQ[f']-1\\ &=fP(z,\ f)-1, \end{aligned} \tag{22}$$

where $P(z,\ /)$ is a polynomial in / and the coefficients $\{\alpha_i\}$ are meromorphic functions satisfying

$$T(r,\ \alpha_i)=S(r,\ f), \qquad \iota=0,\ 1,\ 2,\ \cdots \tag{23}$$

by (19).

Now we consider the function of z, w

$$G(z,\ w)=wP(z,\ w)-1. \tag{24}$$

This is a polynomial in w and satisfies the identity

$$G(z,\ f(z))\equiv F$$

by (22). We will prove that the solution $w=w(z)$ of the functional equation

$$G(z,\ w)=0 \tag{25}$$

satisfies Riccati equation (21) and so that

$$w(z)Q[w'(z)]-1\equiv G(z,\ w(z))\equiv 0. \tag{26}$$

We rewrite (12) in the form

$$a(z)fF+F'\equiv 0$$

it follows that $H(z,f)\equiv 0$, where

$$H(z, w)=a(z)wG(z, w)+G'_z(z, w)+G'_w(z,w)(b_0(z)+b_1(z)w+b_2(z)w^2)$$

is a polynomial in w and the coefficients $\{\beta_i\}$ are meromorphic functions satisfying

$$T(r, \beta_i)=S(r, f)$$

by (15), (19), (23) and (24). Hence $H(z,f)\equiv 0$ implies $H(z, w)=0$ for arbitrary complex z and w. That is,

$$a(z)wG(z, w)+G'_z(z, w)+G'_w(z, w)(b_0(z)+b_1(z)w+b_2(z)w^2)=0 \qquad (27)$$

for arbitrary complex z and w.

Let $w=w(z)$ be a solution of (25). Then there is a unique positive integer λ such that

$$G(z, w)=(w-w(z))^{\lambda}G^*(z,w), \qquad G^*(z, w(z))\not\equiv 0. \qquad (28)$$

The equations (27) and (28) yield

$$(w-w(z))^{\lambda}(a(z)wG^*(z, w)+G_z^{*\prime}(z,w)+G_w^{*\prime}(z, w)(b_0(z)+b_1(z)w+b_2(z)w^2)$$
$$-\lambda(w-w(z))^{\lambda-1}(w'(z)-(b_0(z)+b_1(z)w+b_2(z)w^2))G^*(z, w)\equiv 0.$$

Dividing by $(w-w(z))^{\lambda-1}$ and letting $w\to w(z)$ we get the desired result that

$$w'(z)\equiv b_0(z)+b_1(z)w(z)+b_2(z)w^2(z).$$

By (26) the functional equation (25) has not any constant solution. On the other hand, by using Lemma 3 to (26) we know that the functional equation (25) has only constant solution. These imply that the functional equation (25) has not any solution. It contradicts Lemma 1. Theorem 1 is proved.

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