K. KATO
KODAI MATH. J.
16 (1993), 1—31

IWASAWA THEORY AND p-ADIC HODGE THEORY

BY KAZUYA KATO

The aim of this paper is to formulate the Iwasawa main conjecture for varieties (or motives) over arbitrary number fields. See (4.9) for the statement of the conjecture, and (4.15) for "philosophical comments" on it. To formulate our conjecture, we need the p-adic Hodge theory developed by Tate [Ta_2], Fontaine and Messing [FM], and Faltings [Fa_1].

The classical Iwasawa theory relates special values of partial Riemann zeta functions to the Galois module structures of the ideal class groups of cyclotomic extensions of Q. In our conjecture, we replace Q by an arbtirary number field K, a cyclotomic extension of Q by a finite abelian extension L of K, and partial Riemann zeta functions by partial L-functions $L_S(M, \sigma\text{-part}, s)$ of a motive M over K for $\sigma \in \mathrm{Gal}(L/K)$. (Here S is a finite set of finite places of K including "bad places", and L_S means the L-function without the S-part. For the meaning of the σ-part, see (4.6).) Our conjecture relates special values of $L(M, \sigma$-part, s) to the $\mathrm{Gal}(L/K)$-module structures of the etale cohomology of $\mathrm{Spec}(O_{L,S})$ with coefficients in an etale sheaf coming from M, where $O_{L,S}$ is the ring of elements of L which are integral outside S.

In [BK], Bloch and the author formulated a conjecture on Tamagawa numbers of motives which generalizes the Birch Swinnerton-Dyer conjecture to general Hasse-Weil L-functions. The Iwasawa main conjecture in this paper is a natural generalization of the conjecture of [BK] (which is regarded as the case $L=K$ in the above description).

In our conjecture, we assume the variety is smooth proper over K but we put no other assumption on our variety. We do not assume the variety is of ordinary reduction at the prime number p in problem. We do not assume our motive is critical in the sense of Deligne [De_1]. However this does not mean that we can define p-adic L-functions without these assumptions. Our conjecture treats directly the special values of complex L-functions, but do not treat p-adic interpolations of special values.

In §1-§3, we review known results and conjectures on p-adic Hodge theory, K-theory, and the duality in Galois cohomology of number fields. We formulate the Iwasawa main conjecture in §4. In §5 we consider the case of Tate motives over Q, and in §6 we show that in case of the motive $Q(r)$ over Q with r even and positive, and L real abelian extensions of Q, our Iwasawa main conjecture

Received November 4, 1992.

coincides with the classical Iwasawa main conjecture. In §6 we use a relation (5.12) between p-adic Hodge theory and values of partial Riemann zeta functions proved in a forthcoming paper [Ka_1]. In §7 we show that our conjecture is a generalization of the conjecture in [BK] on Tamagawa number of motives.

After I completed this paper, I learned that Fontaine and Perrin-Riou found a similar approach to the arithmetic of values of L-functions. ([FP] I, II, III). They did not consider partial L-functions, but they found how to treat mixed motives (especially the hight pairing of mixed motives) though my study is limited to pure motives. The motivation of my study was the hope to extend Kolyvagin's Euler systems (which are related to partial L-functions) to motives, and so partial L-functions were essential to me.

I was inspired much by the nice atmosphere in the number theory seminar at Komaba organized by Professors K. Iwasawa, G. Fujisaki, and S. Nakajíma. I found the general conjecture during I was preparing a lecture in this seminar on related subjects. I am very thankful to participants of this seminar. I thank Prof. T. Saito for his advice on the proof of (4.17), and Prof. S. Bloch who introduced me to this fascinating field.

Notations: For a field k, $\bar{k}$ denotes an algebraic closure of k, and char (k) denotes the characteristic of k. As usual, Q (resp. Q_p, R, C) denotes the field of rational (resp. p-adic, real, complex) numbers.

§ 1. p-adic Hodge theory.

(1.1) We review some results concerning *p-adic* Hodge theory. In this section, let K be a complete discrete valuation field with perfect residue field k such that char$(K)=0$ and char$(k)=p>0$.

Fontaine defined a big ring B_{dR} over K endowed with an action of Gal$(\bar{K}/K)$. For the definition of B_{dR}, see [Fo]. B_{dR} is the field of fractions of a complete discrete valuation ring B_{dR}^+, K is contained in B_{dR}^+, the residue field of B_{dR}^+ is isomorphic over K to the *p-adic* completion of $\bar{K}$, and K coincides with the Gal$(\bar{K}/K)$-invariant part of B_{dR}.

(1.2) We review the de Rham conjecture of Fontaine [F0] proved by Fontaine and Messing [FM] under certain assumptions and by Faltings [Fa_1] in general.

Let X be a smooth proper scheme over K. Then, on one hand, we have the *p-adic* etale cohomology $H_{et}^m(\bar{X}, Q_p)$ $(\bar{X}=X\otimes_K\bar{K})$ with an action of Gal$(\bar{K}/K)$. On the other hand we have the de Rham cohomology group $H_{dR}^m(X/K)$ with the Hodge filtration. The de Rham conjecture (1.3) relates these two different *p-adic* cohomologies.

THEOREM (1.3) ([Fa_1] VIII). *For any* ra^O, *there exists a canonical isomorphism*

$$H_{et}^m(\bar{X}, Q_p)\otimes_{Q_p}B_{dR}\cong H_{dR}^m(X/K)\otimes_K B_{dR}$$

preserving the actions of B_{dR} *and* Gal$(\bar{K}/K)$, *and the filtrations. (Here the action* *of* $\sigma\in\mathrm{Gal}(\bar{K}/K)$ *on the left (resp. right) hand side is* $\sigma\otimes\sigma$ *(resp.* $\sigma\otimes id.$*). For* $n\in Z$, fil^n *of the left (resp. right) hand side is*

$$H^m_{et}(\bar{X}, Q_p)\otimes_{Q_p} fil^n B_{dR} \ (\text{resp. } \textstyle\sum_{i+j=n} fil^i H^m_{dR}(X/K)\otimes_K fil^j B_{dR}) .$$

Here the filtration on B_{dR} *is defined by the discrete valuation of* B_{dR}.

(1.4) We review de Rham representations in the sense of Fontaine.

Let V be a finite dimensional vector space over Q_p endowed with a continuous action of $\mathrm{Gal}(\bar{K}/K)$. Let

$$D_{dR}(V)=H^0(K, \ V\otimes_{Q_p}B_{dR})$$

($H^0(K,\ )$ means the fixed part under $\mathrm{Gal}(\bar{K}/K)$), which is endowed with the filtration $\{D^i_{dR}(V)\}_i$ coming from the filtration $V\otimes_{Q_p}fil^{\cdot}B_{dR}$. Then

$$\dim_K(D_{dR}(V))\leqq\dim_{Q_p}(V)$$

holds always, and the following two conditions (i) (ii) are equivalent.

(i) $\dim_K(D_{dR}(V))=\dim_{Q_p}(V)$.

(ii) The canonical map

$$D_{dR}(V)\otimes_K B_{dR}\longrightarrow V\otimes_{Q_p}B_{dR}$$

is bijective.

We say V is a de Rham representation of $\mathrm{Gal}(\bar{K}/K)$ if these equivalent conditions (i) (ii) are satisfied. If V is a de Rham representation, the bijection in (ii) gives an isomorphism of filtrations.

The theorem (1.3) says that for a smooth proper scheme X over K, $H^m_{et}(\bar{X},Q_p)$ is a de Rham representation of $\mathrm{Gal}(\bar{K}/K)$ and $D_{dR}(H^m_{et}(\bar{X},Q_p))$ is canonically isomorphic to $H^m_{dR}(X/K)$ as a filtered K-vector space.

(1.5) We review the exponential map of a de Rham representation defined in [BK]. Let V be a de Rham representation of $\mathrm{Gal}(\bar{K}/K)$, and let $H^i(K, V)$ be the continuous Galois cohomology. Then we have a canonical homomorphism

(1.5.1) $$\exp: D_{dR}(V)/D^0_{dR}(V)\longrightarrow H^1(K, V)$$

defined as follows. Recall that Fontaine defined a subring B_{crys} of B_{dR} containing the field of fractions of the p-Witt ring $W(k)$, endowed with the Frobenius operator $f: B_{crys}\to B_{crys}$ (cf. [Fo] [FM]), and defined the functor D_{crys} by

$$D_{crys}(V)=H^0(K, \ V\otimes_{Q_p}B_{crys})\subset D_{dR}(V).$$

The sequence

$$0\longrightarrow Q_p\xrightarrow{\alpha} B_{crys}\oplus B^+_{dR}\xrightarrow{\beta} B_{crys}\oplus B_{dR}\longrightarrow 0$$

is exact where $\alpha(x)=(x, x)$, $\beta(x, y)=((1-f)(x), x-y)$. By tensoring with V and by taking Galois cohomologies, we have an exact sequence

(1.5.2)
$$0 \longrightarrow H^0(K, V) \longrightarrow D_{crys}(V) \oplus D^0_{dR}(V) \xrightarrow{\ r\ } D_{crys}(V) \oplus D_{dR}(V) \xrightarrow{\ \delta\ } H^1(K, V),$$

where $\gamma(x, y)=((1-f)(x), x-y)$ The exponential map (1.5.1) is defined by the second component of δ.

(1.6) Let / be a prime number and let V be a finite dimensional Q_l-vector space endowed with a continuous action of $\mathrm{Gal}(\overline{K}/K)$. In the case $l=p$, assume that V is a de Rham representation of $\mathrm{Gal}(\overline{K}/K)$. Then, the "finite part" $H^1_f(K, 7)$ of $H^1(K, V)$ is defined as follows ([BK] § 3)). If $l \neq p$,

$$H^1_f(K,V)=\mathrm{Ker}(H^1(K, V) \longrightarrow H^1(K_{nr}, V))$$

where K_{nr} denotes a maximal unramified extension of K. If $l=p$, $H^1_f(K, V)$ is the image of the map δ in (1.5.2). If $l=p$, we have an exact sequence ([BK] (3.8.4))

(1.6.1)
$$0 \longrightarrow H^0(K, 7) \longrightarrow \mathrm{Ker}(1-f : D_{crys}(V))$$
$$\longrightarrow D_{dR}(V)/D^0_{dR}(V) \xrightarrow{\exp} H^1_f(K, V)$$
$$\longrightarrow \mathrm{Coker}(1-f : D_{crys}(V)) \longrightarrow 0.$$

Assume the residue field of K is finite. Then, for any i, the cup product

$$H^i(K, V) \times H^{2-i}(K, V^*(1)) \longrightarrow H^2(K, Q_l(1)) \xrightarrow[=]{\text{trace}} Q_l$$

gives a perfect duality of finite dimensional Q_l-vector spaces (Tate duality). Here $V^*=\mathrm{Hom}(V, Q_l)$ on which $\sigma \in \mathrm{Gal}(\overline{K}/K)$ acts by $h \to h \circ \sigma^{-1}$, and (1) means the Tate twist. In this pairing, if $l \neq p$ or if $l=\rho$ and 7 is de Rham, $H^1_f(K, 7)$ and $H^1_f(K,V^*(1))$ are the annihilators of each other ([BK] (3.8)).

§ 2. K-theory.

In this section, K is a number field, p is a fixed prime number, and M is a pure motive (in Q-coefficients) over K of weight $wt(M)$. We review standard conjectures concerning the "K-theory of M".

(2.1) We do not ask seriously what motives are, but it is better to fix a definition. A pure motive over K of weight $w \in Z$ is a finite family of 4-ples $\{(X_i, m_i, r_i, \varepsilon_i)\}_i$ where X_i are smooth proper schemes of pure dimension over K, $m_i, r_i \in Z$ with $w=m_i-2r_i$, and ε_i is an idempotent in the ring of algebraic cycles on $X_i \times_K X_i$ with Q-coefficients modulo rational equivalence which are regarded as algebraic correspondences from X_i to X_i. We denote the single family (X, m, r, Δ_X) by $H^m(X)(r)$(Δ_X denotes the diagonal, which is regarded as the identity correspondence). We interpret $\{(X_i, m_i, r_i, \varepsilon_i)\}_i$ as the direct sum of the direct summands of $H^{m_\iota}(X_\iota)(r_\iota)$ corresponding to ε_ι. (This is just a very non-smart modification of the original definition of the motive of Grothendieck.) We do not discuss morphisms of motives. For a motive $M=\{(X_\iota, m_\iota, r_\iota, \varepsilon_\iota)\}_\iota$, the Tate twist $M(r)$ for $r\in Z$ (resp. the dual M^*) is defined as $\{(X_\iota, m_\iota, n_i+r, \varepsilon_\iota)\}_\iota$ (resp. $\{(X_\iota, 2n_\iota-m_\iota, n_\iota-r_\iota, \varepsilon_\iota^*)\}_\iota$ where $n_\iota=\dim(X_\iota)$ and ε_ι^* is the transpose of ε_ι).

(2.2) We fix notations for various realizations of M.

Let $V_p(M)$ be the p-adic etale realization of M which is a Q_p-sheaf on $\mathrm{Spec}(K)_{et}$ Once we fix an algebraic closure $\bar{K}$ of K, $V_p(M)$ is identified with a finite dimensional Q_p-vector space endowed with a continuous action of $\mathrm{Gal}(\bar{K}/K)$. Let M_h be the Q-structure in the Hodge structure of M which we regard as a sheaf of Q-vector spaces on $\mathrm{Spec}(K\otimes_Q \boldsymbol{R})_{et}$. Finally let $D(M)$ be the de Rham realization of M, which is a K-vector space endowed with the Hodge filtration $\{D^i(M)\}_{i\in Z}$. These realizations are defined as follows. Assume $M=H^m(X)(r)$ with X, m, r as in (2.1). Then, $V_p(M)=H^m_{et}(\bar{X}, Q_p)(r)$ as a Gal $(\bar{K}/K)$-module, where $\bar{X}=X\otimes_K \bar{K}$. For $\alpha\in\mathrm{Spec}(K\otimes_Q\boldsymbol{R})$, the stalk $M_h(\{\bar{\alpha}\})$ of M_h at the algebraic closure $\bar{\alpha}$ of $\alpha\in\mathrm{Spec}(K\otimes_Q\boldsymbol{R})$ is $H^m_{cl}(X\otimes_K\bar{\alpha}, Q(2\pi i)^r)$ where $H^{\cdot}_{cl}$ is the classical cohomology, and $M_h(\{\alpha\})$ is the $\mathrm{Gal}(\bar{\alpha}/\alpha)$-invariant part of $M_h(\{\bar{\alpha}\})$ where $\mathrm{Gal}(\bar{\alpha}/\alpha)$ acts simultaneously on $\bar{\alpha}$ and on $Q(2\pi i)^r$. Finally $D(M)=H^m_{dR}(X/K)$ with the filtration $D^i(M)=fil^{i+r}H^m_{dR}(X/K)$ where fil' is the Hodge filtration. In general if $M=\{(X_\iota, m_\iota, r_\iota, \varepsilon_\iota)\}_\iota$, a realization of M is defined as the direct sum of the direct summands of the realizations of $H^{m_\iota}(X_\iota)(r_\iota)$ correspondin to ε_ι.

(2.3) We have a canonical map

(2.3.1) $$H^0(K\otimes_Q R,\ M_h)\otimes_Q\boldsymbol{R} \longrightarrow (D(M)/D^0(M))\otimes_Q\boldsymbol{R},$$

which is injective if $wt(M)\leqq -1$. This map (2.3.1) is induced from the isomorphism

$$H^0(K\otimes_Q\boldsymbol{C},\ M_h)\otimes_Q\boldsymbol{C}\cong D(M)\otimes_Q\boldsymbol{C}$$

which is compatible with the action of $\mathrm{Gal}(\boldsymbol{C}/\boldsymbol{R})$. ($\sigma\in\mathrm{Gal}(\boldsymbol{C}/\boldsymbol{R})$ acts by $\sigma\otimes\sigma$ on the left hand side and by $1\otimes\sigma$ on the right hand side. If M is the motive $H^m(X)(r)$ for X, m, r as in (2.1), this isomorphism is the classical isomorphism

$$H^m_{cl}(X\otimes_Q\boldsymbol{C},\ \boldsymbol{C})\cong H^m_{dR}((X\otimes_Q\boldsymbol{C})/\boldsymbol{C}).$$

(2.4) Here we fix some notations concerning etale cohomology. For a scheme Y of finite type over O_K and for a smooth Z_p-sheaf F on Y_{et}, let $H^i(Y, F)=\varprojlim_n H^i_{et}(Y, F/p^nF)$. Then if p is invertible on Y, each $H^i_{et}(Y, F/p^nF)$ is a finite group and $H^i(Y, F)$ is a finitely generated Z_p-module. For such Y and for a smooth Q_p-sheaf F on Y_{et}, let $H^i(Y, F)=H^i(Y, F')\otimes_{Z_p}Q_p$, where F' is a smooth Z_p-sheaf such that $F=F'\otimes_{Z_p}Q_p$, which is independent of the choice of F'. For a scheme Y of finite type over K and for a smooth Z_p (resp. Q_p)-sheaf F on Y_{et} which comes from some smooth Z_p (resp. Q_p)-sheaf on a scheme Y' of finite type over O_K such that $Y=Y'\otimes_{O_K}K$, let

$$H^i_{lim}(Y, F)=\varinjlim_{U}(H^i(Y'\times_{O_K}U,\ F'))$$

where U ranges over all non-empty open subsets of Spec (O_K). Then $H^i_{lim}(Y, F)$ depends only on Y and F, and is independent of the choices of Y' and F'.

(2.5) We shall need K-theoretic Q-vector spaces denoted by $H^i(K, M)$ $(i\in Z)$ and a subspace $H^1_f(K, M)$ of $H^1(K, M)$. For $i\geqq 2$ or for $i<0$, define $H^i(K, M)=0$. We define $H^1(K, M)$ as follows.

First assume $M=H^m(X)(r)$ with X, m, r as in (2.1). Let $K_*(X)$ be Quillen's K-group. Then

$$K_n(X)\otimes Q=\bigoplus_{i\in Z}(K_n(X)\otimes Q)^{(i)}$$

where $(K_n(X)\otimes Q)^{(i)}$ is the part of $K_n(X)\otimes Q$ on which the Adams operators ψ^k act by k^i for any fe^1 ([Sei]). If tttf(M)=£-1 (i.e. if $m\neq 2r-1$), define

$$H^1(K,\ M)=(K_{2r-m-1}(X)\otimes Q)^{(r)}.$$

If $wt(M)=-1$ (i.e. if $m=2r-1$), define $H^1(K,\ M)$ to be the part of $(K_0(X)\otimes Q)^{(r)}=CH^r(X)\otimes Q$ (CH^r denotes the Chow group of algebraic cycles of codimension r) consisting of elements which are homologically equivalent to zero.

In general $H^1(K, M)$ is defined from the case $M=H^m(X)(r)$ by taking the direct summands and the direct sum.

Next we define $H^1_f(K, M)$. Consider the chern class mapping

$$H^1(K,\ M)\longrightarrow H^1_{lim}(K,\ V_p(M)) \tag{2.5.1}$$

which is induced when $M=H^m(X)(r)$, from the chern class map

$$K_{2r-m-1}(X)\longrightarrow H^{m+1}_{lim}(X,\ Q_p(r))$$

(Soule [So_1]) and from the map

$$\mathrm{Ker}\{H^{m+1}_{lim}(X, Q_p(r))\longrightarrow H^{m+1}_{et}(\bar{X},\ Q_p(r))\}\longrightarrow H^1_{lim}(K, H^m_{et}(\bar{X}, \mathcal{Q}_p(r)))$$

coming from Leray's spectral sequence

$$E_2^{i,j}=H^i_{lim}(K,\ H^j_{et}(\bar{X},\ Q_p(r)))\Longrightarrow H^{i+j}_{lim}(X,\ \ Q_p(r)).$$

Let $H^1_f(K, V_p(M))$ be the subspace of $H^1(K, V_p(M))$ consisting of elements whose images in $H^1(K_v,\ V_p(M))$ belong to $H^1_f(K_v,\ V_p(M))$ for all finite places v of K (cf. (1.6)). Let $H^1_f(K, M)\subset H^1(K, M)$ be the inverse image of $H^1_f(K, V_p(M))\subset H^1(K,\ V_p(M))$.

Finally we define $H^0(K, M)$ as follows. It is enough to consider the case $M=H^m(X)(r)$. If $m\neq 2r$ (i.e. if $wt(M)\neq 0$), define $H^0(K,\ M)=0$. If $m=2r$, let

$$H^0(K,\ M)=(CH^r(X)\otimes Q)/(\mathrm{hom.}\sim 0)$$

where (hom.$\sim$0) means the part homologically equivalent to zero.

CONJECTURE (2.6). (1) *$H^1_f(K, M)$ and $H^0(K, M)$ are finite dimensional Q-vector spaces.*

(2) $H^1_f(K, M)\otimes_Q Q_p \overset{\cong}{\to} H^1_f(K, V_p(M))$ *(cf.* [*BK*] (5.3))

(3) *(Tate conjecture.)* $H^0(K, M)\otimes_Q Q_p \overset{\cong}{\to} H^0(K, V_p(M))$.

(4) *// $M=H^m(X)(r)$ and $\mathfrak{X}$ is a proper flat regular scheme over O_K such that $X=\mathfrak{X}\otimes_{O_K}K$, $H^1_f(K, M)\subset K_{2r-m-1}(X)\otimes Q$ coincides with the intersection of $H^1(K, M)$ and the image of $K_{2r-m-1}(\mathfrak{X})\otimes Q$.*

The following is an old conjecture of Beilinson [*Be*] ($H^1_f(K, M)$ was defined in the way of (2.6) (4) by him).

CONJECTURE (2.7). *Assume $wt(M)\neq -1$. Then we have an exact sequence*

$$0 \longrightarrow H^1_f(K,M)\otimes_Q \boldsymbol{R} \overset{a}{\longrightarrow} P \overset{\beta}{\longrightarrow} \mathrm{Hom}_Q(H^0(K,M^*(1)), \boldsymbol{R}) \longrightarrow 0 .$$

where $P=\{(D(M)/D^0(M))\otimes_Q \boldsymbol{R}/\mathrm{Image}\ (H^0(K\otimes_Q \boldsymbol{R}, M_h)\otimes_Q \boldsymbol{R})$

Here a is the regulator map of Beilinson, "Image" is taken with respect to the map (2.2.1), and β is defined as follows: If $M=H^m(X)(r)$ with $m-2r=-2$ and with X purely of dimension n, each element of

$$H^0(K, M^*(1))=(CH^{n-r+1}(X)\otimes Q)/(\mathrm{hom.}\sim 0)$$

defines a cycle class in $fil^{n-r+1}H^{2n-2r+2}_{dR}(X/K)$ which induces

$$D(M)/D^0(M)=H^m_{dR}(X/K)/fil^r H^m_{dR}(X/K) \longrightarrow H^{2n}_{dR}(X/K)/fil^{n+1}H^{2n}_{dR}(X/K)$$

$$\xrightarrow{\text{trace}} K \xrightarrow{\text{trace}} Q .$$

CONJECTURE (2.8). *Assume $wt(M)=-1$. Then:*

(1) $H^1_f(K, M)=H^1(K, M)$.

(2) *The height pairing*

$$(H^1(K, M)\otimes_Q \boldsymbol{R})\times(H^1(K, M^*(1))\otimes_Q \boldsymbol{R}) \longrightarrow R$$

defined by Beilinson [*Be_2*] *and Bloch* [*Bl_2*] *under a certain assumption is defined in general, and it is a perfect pairing of finite dimensional vector spaces over R.*

§ 3. Global p-adic duality.

In this sectin, K denotes a number field and p is a fixed prime number. Let S be a finite set of finite places of K containing all places lying over p. Let O_K be the ring of integers of K, and let $O_{K,S}$ be the ring of S-integers, that is, $O_{K,S}$ is the ring of elements of K which belongs to the local ring of O_K at v for all places v outside 5.

(3.1) Let V be a smooth Q_p-sheaf on $\mathrm{Spec}(O_{K,S})_{et}$, or in other words, a finite dimensional Q_p-vector space with a continuous action of $\mathrm{Gal}(\bar{K}/K)$ which is unramified outside S. Then, by the duality of Poitou-Tate and Artin-Verdier ([*Ser*]II (6.3), [*Po*], [*Ta*$_1$], [*AV*], [*Ma*]), we have an exact sequence of finite dimensional Q_p-vector spaces

$$0 \longrightarrow H^0(O_{K,S}, V) \longrightarrow \bigoplus_{v\in S} H^0(K_v, V) \longrightarrow H^2(O_{K,S}, V^*(1))^*$$

$$\longrightarrow H^1(O_{K,S}, V) \longrightarrow \bigoplus_{v\in S} H^1(K_v, V) \longrightarrow H^1(O_{K,S}, V^*(1))^*$$

$$\longrightarrow H^2(O_{K,S}, V) \longrightarrow \bigoplus_{v\in S} H^2(K_v, V) \longrightarrow H^0(O_{K,S}, V^*(1))^* \longrightarrow 0\,.$$

Here the cohomology groups are etale cohomology groups or Galois cohomology groups, $(\ )^* = \mathrm{Hom}_{Q_p}(\ ,\ Q_p)$, and the maps $\bigoplus_{v\in S} H^i(K_v, V) \to H^{2-i}(O_{K,S}, V^*(1))^*$ are induced by the canonical map

$$H^{2-i}(O_{K,S}, V^*(1)) \longrightarrow \bigoplus_{v\in S} H^{2-i}(K_v, V^*(1))$$

and the local Tate duality

$$H^i(K_v, V) \times H^{2-i}(K_v, V^*(1)) \longrightarrow H^2(K_v, Q_p(1)) \cong Q_p\,.$$

COROLLARY (3.2). *Assume for any place v of K which divides p, V is a de Rham representation of* $\mathrm{Gal}(\bar{K}_v/K_v)$. *We have an exact sequence of finite dimensional Q_p-vector spaces*

$$0 \longrightarrow H^0(K, V) \longrightarrow \bigoplus_{v\in S} H^0(K_v, V) \longrightarrow H^2(O_{K,S}, V^*(1))^*$$

$$\longrightarrow H^1_f(K, V) \longrightarrow \bigoplus_{v\in S} H^1_f(K_v, V) \longrightarrow H^1(O_{K,S}, V^*(1))^*$$

$$\longrightarrow H^1_f(K, V^*(1))^* \longrightarrow 0\,.$$

(3.3) In the rest of this sectin, let M be a pure motive over K of weight $wt(M)$, which is of good reduction outside S. Note that for any place v of K which divides p, $V_p(M)$ is a de Rham representation of $\mathrm{Gal}(\bar{K}_v/K_v)$ by (1.3).

The following conjecture (3.4) is a real theorem of Deligne [*De*$_2$] (resp. of Fontaine, Messing [*FM*] and Faltings [*Fa*$_1$]) if v does not divide p (resp. v divides p) and M is of good reduction at v. For general v which does not divide p (resp. divides p), a geometric analogue of (3.4) was proved in [*De*$_2$] II (resp. [*Fa*$_2$]).

CONJECTURE (3.4). *Let v be a finite place of K. If v does not divide p, let φ_v be the Frobenius automorphism in* $\mathrm{Gal}(K_{v,nr}/K_v)$ *acting on $H^0(K_{nr},\ )$, and let $P_v(t) = \det_{Q_p}(1-\varphi_v^{-1}t \ \ H^0(K_{nr}, V_p(M))) \in Q_p[t]$. If v divides p, let f_v: $D_{crys}(V_p(M)) \to D_{crys}(V_p(M))$ be the Frobenius operator, $\kappa(v)$ the residue field of v, $K_{v,0}$ the field of fractions of the p-Witt ring $W(\kappa(v))$, $d(v) = [\kappa(v): F_p]$, and let*

$$P_v(t)=\det_{K_{v,0}}(1-f_v^{d(v)}t\,;\,D_{crys}(V_p(M)))\in K_{v,0}[t]\,.$$

Then, in any case, $p_v(t)$ *is with* Q*-coefficients,and has the form*

$$\prod_i (1-\alpha_i t) \qquad (\alpha_i\in \boldsymbol{C},\,|\alpha_i|\leqq N(v)^{wt(M)/2})$$

($N(v)$ *is the norm of* v) *in* $\boldsymbol{C}[t]$.

PROPOSITION (3.5). *Assume* $wt(M)\leqq -l$ *and assume Conj.* (3.4) *is true for* M. *Let* v *be a finite place of* K.

(1) *// v divides p, the exponential map of* § 1 *induces an isomorphism*

$$D(M)/D^0(M)\otimes_K K_v \xrightarrow{\cong} H^1_f(K_v,\ V_p(M)).$$

(2) *// v does not divide p, then* $H^1_f(K_v,V_p(M))=0$.

(3) $H^0(K_v, V_p(M))=0$.

Proof. The case $v\backslash p$ follows from (1.6.1). The case v does not divide p follows from the fact that $H^1_f(K_v,\ )$ (resp. $H^0(K_v,\ )$) is isomorphic to the cokernel (resp. kernel) of $1-\varphi_v^{-1}$ on $H^0(K_{v,nr},\ )$.

From (3.2) and (3.5), we can deduce easily:

PROPOSITION (3.6). *Assum* $wt(M)\leqq -l$ *and assume that the conjectures* (2.6) *and* (3.4) *are true for* M. *Then:*

(1) *For any* i, *we have*

$$H^i(O_{K,S},\ V_p(M)^*(1)) \xrightarrow{\cong} H^i_{lim}(K,\ V_p(M)^*(1))\,.$$

(2) *We have an exact sequence of finite dimensional* Q_p*-vector spaces*

$$\begin{aligned}0 &\longrightarrow H^2_{lim}(K,V_p(M)^*(1))^* \longrightarrow H^1_f(K,M)\otimes_Q Q_p\\ &\longrightarrow D(M)/D^0(M)\otimes_Q Q_p \longrightarrow H^1_{lim}(K,V_p(M)^*(1))^*\\ &\longrightarrow \mathrm{Hom}_Q(H^1_f(K,\ M^*(1)),Q_p)\longrightarrow 0.\end{aligned}$$

The above sequence in (2) plays the role of the p-adic version of the sequence in (2.7) of vector spaces over R.

Remark (3.7). In (3.6), assume $wt(M)\leqq -2$. Then $H^1_f(K,M^*(1))=0$. Indeed, since $M^*(1)$ is of weight$\geqq 0$, it is a direct sum of subspaces of $(K_{2r-m-1}(X)\otimes Q)^{(r)}$ with $m-2r\geqq 0$, i.e. with $2r-m-1<0$. Furthermore it is probable that $H^2_{lim}(K,V_p(M)^*(1))$ vanishes (this is conjectured by Jannsen), or equivalently (if we asssume (2.6) and (3.4)), the p-adic regulator map $H^1_f(K,M)\otimes_Q Q_p\rightarrow D(M)/D^0(M)\otimes_Q Q_p$ in (3.6) (2) is injective. If this is the case, the sequence in (3.6) (2) has the simple form

$$0 \longrightarrow H^1_f(K,M)\otimes_Q Q_p \longrightarrow D(M)/D^0(M)\otimes_Q Q_p$$
$$\longrightarrow \mathrm{Hom}_{Q_p}(H^1_{lim}(K,V_p(M)^*(1)),\ Q_p) \longrightarrow 0$$

very similar to the sequence in (2.7).

§ 4. Iwasawa main conjecture.

(4.1) In this section, let K be a number field and let M be a pure motive over K of weight $\leqq -1$. let L be a finite abelian extension of K with Galois group G, and let p be a fixed prime number.

(4.2) We define $Q[G]$-modules $H_h, H_d, H_k, H'_k, H''_k$ and $Q_p[G]$-modules H^i_n $(i\in Z)$ as follows. With the notations in § 2, let

$$H_h=H^0(L\otimes_Q \boldsymbol{R},\ M_h),\quad H_d=(D(M)/D^0(M))\otimes_K L,\quad H_k=H^1_f(L,\ M),$$
$$H'_k=H\ (L,\ M^*(1)),\quad H''_k=H^1_f(L,\ M^*(1)),$$

and let

$$H^i_p=H^i_{lim}(L,\ V_p(M)^*(1))\quad (2.9).$$

Then $H'_k=0$ unless $wt(M)=-2, H''_k=0$ unless $wt(M)=-1$, $H^i_p=0$ for $i\geqq 3$, and $H^0_p=0$ unless $wt(M)=-2$. If $X=H^m(X)(r)$,then

$$H_h=H^m_{cl}((X\otimes_K L)\otimes_Q C,\ Q(2\pi i)^r)^+$$

where+means the fixed part by $\mathrm{Gal}(C/R)$ which acts simultaneously on C and on $Q(2\pi i)^r$,

$$H_d=H^m_{dR}((X\otimes_K L)/L)/fil^r,$$

and H_k is a certain subspace of $K_{2r-m-1}(X\otimes_K L)\otimes Q$.

(4.3) In (4.3) and (4.4) we give purely module theoretic preliminaries.

Here we give a preliminary concerning determinant modules. Let R be a commutative ring. Recall that for a finitely generated projective R-module L, the determinant R-module $\det_R(L)$ is defined to be the exterior power $\wedge^r_R L$ where r is the rank of L which is a locally constant function on Spec(R) (so $\wedge^r_R L$ is defined locally on Spec (R) first and glued globally on Spec (R)). This definition of the determinant module is generalized to perfect complexes as follows.

Let $\mathcal{C}$ be the derived category of the category of R-modules. An object C of $\mathcal{C}$ is called a perfect complex if there is a bounded complex of R-modules consisting of finitely generated projective R-modules which represents C. For a perfect complex C in $\mathcal{C}$, the determinant module $\det_R(C)$ is the invertible R-module defined as follows. Take a representative of C

$$\cdots \longrightarrow L_i \longrightarrow L_{i-1} \longrightarrow \cdots$$

which is bounded and which consists of finitely generated projective R-modules. Then

$$\det_R(C)=\bigotimes_{i\in Z}\{\det_R(L_i)\}^{\otimes(-1)^i}.$$

It is known that $\det_R(C)$ is independent, modulo canonical isomorphisms, of the choice of a representative as above.

(4.4) For a ring R, which will be Q, Q_p, Z_p or $Z_{(p)}=Z_p\cap Q$ below, and for an $R[G]$-module F, let $F^{\#}$ be the $R[G]$-module whose underlying R-module is F but on which $\sigma\in G$ acts by the original action of σ^{-1} on F. Let $F^*=\mathrm{Hom}_R(F, R)$ on which $\sigma\in G$ acts by $h\to h\sigma^{-1}$. Then $F^{*\#}=F^{\#*}$ is identified with the dual module $\mathrm{Hom}_{R[G]}(F, R[G])$ on which $a\in R[G]$ acts by $h\to h\circ a-a\circ h$, via the isomorphism

$$F^{*\#}\longrightarrow \mathrm{Hom}_{R[G]}(F,R[G])\; h\longmapsto (x\longmapsto \sum_{\sigma\in G}h(\sigma^{-1}x)\sigma).$$

(4.5) In the rest of this section except in (4.15)-(4.17), we assume that the conjectures (2.6) (2.7) (2.8) (3.4) are true for the pull back of M over L.

We define a free $Q[G]$-module Φ^{mot} of rank 1 with an isomorphism of $\boldsymbol{R}[G]$-modules

$$\Phi^{mot}\otimes_Q \boldsymbol{R}\xrightarrow{\cong}\boldsymbol{R}[G] \tag{4.5.1}$$

and a free $Q_p[G]$-module Φ_p^{ar} of rank 1 with an isomorphism of $Q_p[G]$-modules

$$\Phi^{mot}\otimes_Q Q_p\xrightarrow{\cong}\Phi_p^{ar}. \tag{4.5.2}$$

Let

$$\Phi^{mot}=\det_{Q[G]}(H_h)\otimes_{Q[G]}\det_{Q[G]}(H_d)^{*\#}\otimes_{Q[G]}\det_{Q[G]}(H_k)$$
$$\otimes_{Q[G]}\det_{Q[G]}(H_k')^{*}\otimes_{Q[G]}\det_{Q[G]}(H_k'')^{\#}.$$

Note that the fourth (resp. the last) det( ) is $\det_{Q[G]}(\{0\})=Q[G]$ and can be cancelled if $wt(M)\neq -2$ ($wt(M)\neq -1$). Let

$$\Phi_p^{ar}=\det_{Q[G]}(H_h)\otimes_{Q[G]}\det_{Q_p[G]}R\Gamma_{lim}(K, V_p(M)^*(1))^*.$$

We define the isomosphisms (4.5.1) (4.5.2) as follows. First we consider the Archimedean side (4.5.1). If $u\,t(M)\leqq -2$, by Conj. (2.7) (with K replaced by L) we have an exact sequence

$$0\longrightarrow H_k\otimes_Q\boldsymbol{R}\longrightarrow (H_d\otimes_Q\boldsymbol{R})/(H_h\otimes_Q\boldsymbol{R})\longrightarrow (H_k')^*\otimes_Q\boldsymbol{R}\longrightarrow 0$$

which gives (4.5.1). If $wt(M)=-l$, the height pairing

$$H_k\otimes_Q\boldsymbol{R}\xrightarrow{\cong}(H_k'')^*\otimes_Q\boldsymbol{R}\quad (2.8)$$

and the isomorphism

$$H_h \otimes_Q \boldsymbol{R} \xrightarrow{\cong} H_d \otimes_Q \boldsymbol{R}$$

give (4.5.1). Next we consider the p-adic side (4.5.2). By (3.6) (2), we have an exact sequence

$$0 \longrightarrow (H_p^2)^* \longrightarrow H_k \otimes_Q Q_p \longrightarrow H_d \otimes_Q Q_p \longrightarrow (H_p^1)^* \longrightarrow (H_k'')^* \otimes_Q Q_p \longrightarrow 0.$$

This sequence and

$$H_k' \otimes_Q Q_p \xrightarrow{\cong} HP \quad (2.6)\ (3)$$

give (4.5.2).

(4.6) We consider the partial L-functions of M relative to the abelian extension L/K. Let S be a finite set of finite places of K containing all finite places at which M has bad reduction and all finite places which ramify in L/K. Let

$$L_S(M, s) = \sum_{\mathfrak{U}} a(\mathfrak{U}) N(\mathfrak{U})^{-s}$$

be the Hasse-Weil L-function of M without Euler factors for places in S. Here $\mathfrak{U}$ ranges over all non-zero ideals of O_K, $a(\mathfrak{U}) \in Q$, and $N(\mathfrak{U})$ denotes the norm of $\mathfrak{U}$.

For $\sigma \in G$, we define the partial L-function

$$L_S(M, \sigma\text{-part}, s)$$

to be $\sum_{\mathfrak{U}} a(\mathfrak{U}) N(\mathfrak{U})^{-s}$ where $\mathfrak{U}$ ranges over all non-zero ideals of O_K which are prime to 5 such that the Artin symbols $((L/K)/\mathfrak{U}) \in G$ coincide with σ. This function $L_S(M, \sigma\text{-part}, s)$ converges absolutely when $Re(s) > wt(M)/2+1$.

We will consider the values of $L_S(M, \sigma\text{-part}, s)$ at $s=0$. If $M = H^m(X)(r)$, these are the values of the partial Hasse-Weil L-functions $L_S(H^m(X), \sigma\text{-part}, s)$ at $s = r \geqq (m+1)/2$ (note $L_S(N(r), \sigma\text{-part}, s) = L_S(N, \sigma\text{-part}, s+r)$ for any motive N), where $(m+1)/2$ is the central point of the conjectural functional equations of $L_S(H^m(X), \sigma\text{-part}, s)$ under the substitution $s \leftrightarrow m+1-s$.

Define "the analytic zeta element"

$$\zeta^{an}_{L/K,S}(M) \in \boldsymbol{R}[G]$$

assuming no conjecture if $wt(M) \leqq -3$, and assuming some conjectures if $wt(M) = -2$ or -1 as follows. If $wt(M) \leqq -3$, define

$$\zeta^{an}_{L/K,S}(M) = \sum_{\sigma \in G} L_S(M, \sigma\text{-part}, 0)\ \sigma.$$

Assume $wt(M) = -2$ (resp. $wt(M) = -1$). We proceed making conjectures. We conjecture that the functions $L_S(M, \sigma\text{-part}, s)$ are extended to the whole complex plane as meromorphic functions. Let $\mathscr{H}'_k$ (resp. H_k'') be the coherent sheaf on Spec$(Q[G])$ associated to the $Q[G]$-module H_k' (resp. H_k''). Take an open set U of Spec$(Q[G])$ on which $\tilde{H}'_k$ (resp. *fig)* has constant rank $r(U)$. Then we conjecture that the image $f_U(s)\in\mathcal{O}(U\otimes_Q \boldsymbol{R})$ of $s^{r(U)\varepsilon}\sum_{\sigma\in G} L_S(M, \sigma\text{-part}, s)\ \sigma \in \boldsymbol{R}[G]$ with $\varepsilon=1$ (resp. -1) is holomorphic at $s=0$ as a vector valued function in s. We define $\zeta^{an}_{L/K,S}(M)$to be the unique element of $\boldsymbol{R}[G]$ such that for any U as above, the image of $\zeta^{an}_{L/K,S}(M)$in $\mathcal{O}(U\otimes_Q \boldsymbol{R})$ coincides with $f_U(0)$.

If S' is a finite set of finite places of K containing S, we have

$$\zeta^{an}_{L/K,S'}(M)=(\prod_{v\in S'-S} P_v(\varphi_v))\zeta^{an}_{L/K,S}(M) \tag{4.6.1}$$

where $P_v(t)\in Q[t]$is the polynomial such that $P_v(N(v)^{-s})^{-1}$ is the Euler factor of $L_S(M, s)$ at v, $\varphi_v\in G$ is the Frobenius of v, and we assumed in the case $wt(M)=-1,-2$ the conjectures needed in the definition of $\zeta^{an}_{L/K,S}(M)$ (which are equvalent to the conjectures needed for $\zeta^{an}_{L/K,S}(M)$) are true.

The following Conj. (4.7) is a famous Beilinson conjecture $[Be_1]$ when $L=K$. The generalization to abelian extensions L/K is proposed by several peoples (Stark, Gross, Beilinson, $\cdots$). In the critical case, (4.7) was conjectured be Deligne $[De_1]$.

CONJECTURE (4.7). *The image of* $\zeta^{an}_{L/K,S}(M)$*under the isomorphism* (4.5.1) *is contained in* $\Phi^{mot}\subset\Phi^{mot}\otimes_Q \boldsymbol{R}$.

Assuming this conjecture we denote by $\zeta^{mot}_{L/K,S}(M)$the element of Φ^{mot} corresponding to $\zeta^{an}_{L/K,S}(M)$via (4.5.1), and call it "the motivic zeta element". We denote by $\zeta^{ar}_{L/K,S}(M)_p$ the image of $\zeta^{mot}_{L/K,S}(M)$ in Φ^{ar}_p and call it "the (p-adic) arithmetic zeta element".

The relation (4.6.1) concerning the change of the zeta element when we enlarge S is extended to motivic zeta elements and arithmetic zeta elements, if the conjectures needed for the definitions of them are true.

(4.8) Fix a Z_p-sheaf T in $V_p(M)$ such that $T\otimes_{Z_p}Q_p=V_p(M)$, in other words, a Gal$(\bar{K}/K)$-stable Z_p-lattice T of $V_p(M)$. Let $H_{h,T}\subset H_h$ be the inverse image of $H^0(L\otimes_Q \boldsymbol{C}, T)\subset H^0(L\otimes_Q \boldsymbol{C}, V_p(M))$under the composite map

$$H_h \longrightarrow H^0(L\otimes_Q \boldsymbol{C}, M_h)\otimes_Q Q_p\cong H^0(L\otimes_Q \boldsymbol{C}, V_p(M)).$$

For example, if $M=H^m(X)(r)$,we can take

$$T=H^m_{et}(\bar{X}, \quad Z_p)(r)/(\text{torsion})\subset V_p(M)=H^m_{et}(XQ_p)(r)$$

(then $H_{h,T}=\{H^m((X\otimes_K L)\otimes_Q \boldsymbol{C}\, Z(2\pi i)^r)/(\text{torsion})\}^+\otimes_Z Z_{(p)}$). We have

$$H_{h,T}\otimes_{Z_{(p)}}Q=H_h, \qquad H_{h,T}\otimes_{Z_{(p)}}Z_p=H^0(L\otimes_Q \boldsymbol{R}, \text{T}).$$

Assume $p\neq 2$. Then $H^0(L\otimes_Q \boldsymbol{R}, T)$ is a finitely generated projective $Z_p[G]$-module as is easily seen. It follows that $H_{h,T}$and hence $H^*_{h,T}=\mathrm{Hom}_{Z_{(p)}}(H^*_{h,T}, Z_{(p)})$ are finitely generated projective $Z_{(p)}[G]$-modules. Let $T^*=\mathrm{Hom}_{Z_p}(T, Z_p)$.

Let $O_{L,S}$ be the ring of elements of L which belong to the local ring of O_L

at v for any finite place v of L not lying over S. Assume S contains all places of K lying over p. We see in (4.17) below that if $p\neq 2$, $R\Gamma(O_{L,S}, T^*(1))$ is a perfect complex in the derived category of the category of $Z_p[G]$-modules. Let

$$\Phi^{ar}_{p,S,T}=(\det_{Z_{(p)}[G]}(H_{h,T}))\otimes_{Z_{(p)}[G]}\{\det_{Z_p[G]}R\Gamma(O_{L,S},\ T^*(1))\}^*$$

where $(\ )^*=\mathrm{Hom}(\ ,\ Z_p)$. Since

$$H^i(O_{L,S},\ T^*(1))\otimes_{Z_p}Q_p\cong H^i_{lim}(IV_p(M)^*(1))\quad (3.6)\ (1),$$

we have $\Phi^{ar}_{p,S,T}\otimes_{Z_p}Q_p=\Phi^{ar}_p$.

Iwasawa main conjecture (4.9). *Assume $p\neq 2$, and assume S contains all places of K lying over p, all finite places of K at which M has bad reduction, and all finite places of K which ramify in L/K. Then, the arithmetic zeta element*

$$\zeta^{ar}_{L/K,S}(M)_p\in\Phi^{ar}_p=\Phi^{ar}_{p,S,T}\otimes_{Z_p}Q_p$$

is a $Z_p[G]$-basis of $\Phi^{ar}_{p,S,T}$

Note that $Z_{(p)}[G]$ and $Z_p[G]$ are semi-local rings and hence any invertible modules over them are free modules.

Remark (4.10). Conj. (4.9) is compatible with isogeny. That is, if T and T' are two Z_p-sheaves in $V_p(M)$ such that $T\otimes_{Z_p}Q_p=V_p(M)=T'\otimes_{Z_p}Q_p$, $\Phi^{ar}_{p,S,T}=\Phi^{ar}_{p,S,T'}$ holds in Φ^{ar}_p by (4.17) (3) below, and hence Conj. (4.9) for the pair (M, T) is equivalent to the Conj. (4.9) for the pair (M, T').

Remark (4.11). Conj. (4.9) is compatible with localization, i.e. with enlarging 5. Let S' be a finite set of finite places of K containing 5. Then we have a distinguished triangle

$$R\Gamma(O_{L,S},\ T^*(1)) \dashrightarrow R\Gamma(O_{L,S'},T^*(1)) \dashrightarrow \bigoplus_{v\in S'-S} R\Gamma(v,\ T^*)[-1] \longrightarrow .$$

From this we have easily

$$\Phi^{ar}_{p,S',T}=(\prod_{v\in S'-S} P_v(\varphi_v))\cdot\Phi^{ar}_{p,S,T}\quad \text{in } \Phi^{ar}_p\,.$$

By comparing this with $\zeta^{ar}_{L/K,S'}(M)_p=(\prod_{v\in S'-S}P_v(\varphi_v))\zeta^{ar}_{L/K,S}(M)_p$, we see that Conj. (4.9) for the pair (M, S) is equivalent to Conj. (4.9) for (M, S').

Remark (4.12). I refer to the case $p=2$ which was excluded in Conj. (4.9). Let $p=2$ and assume that all Archimedean places of K split in L. Then $H_{h,T}$ is a free $Z_{(2)}[G]$-module of finite rank. Under this assumption, I conjecture that the trancation $\tau_{\leq 2}R\Gamma(O_{L,S}T^*(1))$ is a perfect complex in the derived category of the category of $Z_2[G]$-modules, and the statement of Conj. (4.9) holds if we replace $R\Gamma(O_{L,S}, T^*(1))$ by $\tau_{\leq 2}R\Gamma(O_{L,S}, T^*(1))$.

Remark (4.13). To state Conj. (4.9) we needed many conjectures concerning K-theory which are difficult to verify. If $wt(M)\leqq -3$ and M is critical in the sense of Deligne (i.e. $H_h\otimes_Q \boldsymbol{R} \xrightarrow{\cong} H_d\otimes_Q \boldsymbol{R}$), there is a way to get rid of conjectures on K-theory. In this case, define $H_k=0$. Then, once we have expressions of values of partial L-functions of M at $s=0$ in terms of period integrals, Conj. (4.9) becomes purely a problem on *p-adic* Hodge theory and Galois cohomology.

Remark (4.14) (on Euler systems). If L'/K is a subextension of L/K with Galois group G', the norm maps induce isomorphisms $H_h\otimes_{Q[G]}Q[G'] \xrightarrow{\cong} H_h(L')$, $H_d\otimes_{Q[G]}Q[G'] \xrightarrow{\cong} H_d(L')$, etc., where $H_h(L')$, etc. mean the H_h etc. defined for the extension L'/K, and hence isomorphisms

(4.14.1) $$\Phi^{mot}\otimes_{Q[G]}Q[G'] \xrightarrow{\cong} \Phi^{mot}(L')$$

(4.14.2) $$\Phi_p^{ar}\otimes_{Q_p[G]}Q[G'] \xrightarrow{\cong} \Phi_p^{ar}(L').$$

Furthermore, if the conjectures needed for the definitions of $\zeta_{L/K,S}^{mot}(M)$ (resp. $\zeta_{L/K,S}^{ar}(M)_p$) are true, the isomorphism (4.14.1) (resp. (4.14.2)) sends $\zeta_{L/K,S}^{mot}(M)$ to $\zeta_{L'/K,S}^{mot}(M)$ (resp. $\zeta_{L/K,S}^{ar}(M)_p$ to $\zeta_{L'/K,S}^{ar}(M)_p$). This fact and the change of zeta elements with enlarging S described after (4.7) suggest that when L and S vary, the systems $\{\zeta_{L/K,S}^{mot}(M)\}_{L,S}$ and $\{\zeta_{L/K,S}^{ar}(M)_p\}_{L,S}$ should be called "Euler systems of Kolyvagin" ([*Ko*]) for the motive M. Is it possible to apply theories of Kolyvagin on his Euler systems to these general systems?

(4.15) In the next section, we will see that if $K=Q$, $M=Q(1)$ (resp. $M=Q(r)$ with r a positive even integer) and L is the real part of $Q(\alpha)$ with α a root of 1, the algebraic (resp. arithmetic) zeta element is essentially the fundamental cyclotomic unit $(1-\alpha)(1-\alpha^{-1})$ (resp. the p-adic cyclotomic element of Soulé and Deligne in the Galois cohomology of $Z_p(1-r)$).

Zeta functions live in some world. They come to $\boldsymbol{R}[G]$ and become $\zeta_{L/K,S}^{an}(M)$ to be called special values of zeta functions. When we find they come to Φ^{mot} *and* become $\zeta_{L/K,S}^{mot}(M)$, we call them expressions of special values of zeta functions in terms of period integrals, in terms of regulators of K-theory, $\cdots$. We have realized they come to Φ_p^{ar} *and* become $\zeta_{L/K,S}^{ar}(M)_p$ only in very special cases. In such cases, we have called them expressions of special values of zeta functions in terms of explicit reciprocity laws, describing explicitly and map $H_d\otimes_Q Q_p \to (H_p^1)^*$ (cf. [*dS*] for the case $M=H^1(E)(1)$ with E elliptic curves with complex multiplication. This point will be discussed in more detail in [Ka_1].)

There should be many beautiful extensions of the theory of cyclotomic units and of the known theory of explicit reciprocity laws, to motives.

(4.16) To describe how arithmetic zeta elements are important for the study of arithmetic of varieties, I introduce a result of a forthcoming paper [Ka_2]. (We assume no conjecture in this (4.16).) Let E be an elliptic curve over Q dominated by a modular curve, let $M=H^1(E)(1)$and let S be a finite set of primes containg all primes at which E has bad reduction. In [Ka_2], we give a new proof of the theorem of Kolyvagin

$$L(H^1(E), 1)\neq 0 \Longrightarrow E(Q) \quad \text{is a finite group}$$

in the following way without using Heegner points. In [Ka], we construct an element c of $H_h\otimes_Q H^1_p$ such that the Q_p-dual $H^1_p\to H^*_d\otimes_Q Q_p$ of $H_d\otimes_Q Q_p\to(H^1_p)^*$ sends c to

$$(4.16.1) \qquad \gamma\otimes\left\{L_S(H^1(E), 1)\left(\int_\gamma \omega\right)^{-1}\right\}\omega\in H_h\otimes_Q H^*_d\subset H_h\otimes_Q H^*_d\otimes_Q Q_p$$

where $\gamma\in H_h-\{0\}$, $\omega\in\Gamma(E, \Omega^1_{E/Q})-\{0\}$, we identify H_h with $H_1(E(\boldsymbol{R}), Q)$, Hi with $\Gamma(E, \Omega^1_{E/Q})$, and we denote by $\int_\gamma \omega$ the integration of ω against γ. Note H_h and H_d are one dimensional over Q, $L_S(H^1(E), 1)\left(\int_\gamma \omega\right)^{-1}\in Q$, and the element (4.16.1) is independent of the choices of γ and ω.

Assume $L_S(H^1(E), 1)\neq 0$. We obtain $E(Q)$ finite as follows. The property of c implies that the map $H_d\otimes_Q Q_p\to(H^1_p)^*$is injective. By the exact sequence (3.2), we have that $H^1_f(Q, V_p(M))\to H_d\otimes_Q Q_p$is the zero map. On the other hand, the map $E(Q)\to H^1(Q, V_p(M))$ induced by the exact sequences

$$0 \longrightarrow {}_{p^n}E \longrightarrow E \xrightarrow{p^n} E \longrightarrow 0 \qquad ({}_{p^n}E=\mathrm{Ker}(p^n\colon E\to E))$$

lands in $H^1_f(Q, V_p(M))$ ([BK] (3.11)), and the composite map $E(Q)\to H^1_f(Q, V_p(M))\to H_d\otimes_Q Q_p$ coincides with

$$E(Q) \longrightarrow E(Q_p)\otimes Q \xrightarrow[\cong]{\log} \mathrm{Lie}(E)\otimes_Q Q_p=H_d\otimes_Q Q_p\,.$$

From these facts, it follows that $E(Q)\to E(Q_p)\otimes Q$is the zero map. This shows that $E(Q)$ is finite. (This kind of method was used by Coates-Wiles [CW], Bloch [Bl_1] §2, and by K. Rubin (de Shalit [dS] IV §2) for elliptic curves with complex multiplication).

The element c is the zeta element $\zeta^{ar}_{Q/Q,S}(M)$ if $L(H^1(E), 1)\neq 0$ (strictly speaking, we know c is the arithmetic zeta element only after we know $H^1_{\lim}(Q, V_p(M))$ is one dimensional over Q_p and $H^2_{\lim}(Q, V_p(M))=0$ as consequences of Kolyvagin's theorem on the finiteness of the Tate-Shafarevich group; I do not have a new proof of this theorem). The element c above is defined in [Ka_2] by using "modular units in K_2 of modular curves" of Beilinson [Be_1], just as the p-adic cyclotomic elements of Deligne and Soulé are defined (cf. (5.11)) by using classical cyclotomic units in $K_1=G_m$. The fact c is sent to the element (4.16.1) is shown by applying the explicit reciprocity law for two dimensional local fields by Vostokov and Kirillov ([VK]) to the completed modular function field

$$(\varprojlim_n (Z/p^n Z[[q]][q^{-1}]))\left[\frac{1}{p}\right]$$

where q is the q-invariant.

Finally we prove the following result used in (4.8) and (4.10).

PROPOSITION (4.17). *(Here we assume no conjecture.) Assume $p\neq 2$, and let S be a finite set of finite places of K containing all prime divisors of p in K. Let F be a smooth Z_p-sheaf on* $\mathrm{Spec}\,(O_{K,S})_{et}$. *Then:*

(1) *$R\Gamma(O_{L,S}, F)$ is a perfect complex in the derived category of the category of $Z_p[G]$-modules.*

(2) *If L'/K is a subextension of L/K with Galois group G', we have a canonical isomorphism*

$$R\Gamma(O_{L,S}, F)\overset{L}{\otimes}_{Z_p[G]}Z_p[G'] \xrightarrow{\cong} R\Gamma(O_{L',S}, F)\,.$$

(3) *If $p^n F=0$ for some $n\geqq 0$, we have*

$$\det{}_{Z_p[G]}(R\Gamma(O_{L,S},F))\otimes_{Z_p[G]}\det{}_{Z_p[G]}(H^0(L\otimes_Q \boldsymbol{R},\ F^*(1)))^*=Z_p[G] \quad \text{in } Q_p[G]\,,$$

where $(\ )^*=\mathrm{Hom}(\ ,\ Q_p/Z_p)$ and the left hand side is regarded as embedded in its $\otimes_{Z_p}Q_p$ which is identified with

$$\det{}_{Q_p[G]}(\{0\})\otimes_{Q_p[G]}\det{}_{Q_p[G]}(\{0\})=Q_p[G]\,.$$

Proof. (1) and (2) are proved by the methods of Deligne (SGA4, Ch. 17) as follows. It is enough to prove that the morphism

$$h_N:\ R\Gamma(O_{L,S},\ F)\otimes^L_{Z_p[G]}N\longrightarrow R\Gamma(O_{L,S},\ F\otimes^L_{Z_p[G]}N)$$

is an isomorphism for any finitely generated $Z_p[G]$-module N. To prove that the map $H^q(h_N)$ induced on the q-th cohomology groups of these complexes is an isomorphism, take an exact sequence of $Z_p[G]$-modules of the form

$$0\longrightarrow N'\longrightarrow L_r\longrightarrow\cdots\longrightarrow L_0\longrightarrow N\longrightarrow 0$$

with L_i free of finite type and with $r>2-q$. Since h_{L_i} are clearly isomorphisms, the bijectivity of $H^q(h_N)$ is reduced to the bijectivity of $H^{q+r}(h_{N'})$, but the cohomology groups are zero in degree >2.

We prove (3). If $G=\{1\}$, the statement of (3) is equivalent to the formula of Tate ([Ta_3] Thm. (2.2))

$$\prod_{0\leq i\leq 2}\#(H^i(O_{L,S},F))^{(-1)^i}=\#(H^0(L\otimes_Q \boldsymbol{R},\ F^*(1)))^{-1}.$$

($\#(\ )$ denotes the order of the set). Our proof of (3) is essentially the same with the argument of Tate in his proof (suggested by Serre) of this formula.

We are reduced to the case where there is a cyclic extension K' of K of degree prime to p which is unramified outside S, and F corresponds to a finite $\mathrm{Gal}(K'/K)$-module killed by p. By replacing L with the composite field $K'L(\zeta_p)$ where ζ_p is a primitive p-th root of 1, we may assume $F=Z/pZ(1)$. Let $K_0(Z/pZ[G])$ (resp. $K_0'(Z/pZ[G])$) be the Grothendieck group of the category of finitely generated projective (resp. finitely generated) $Z/pZ[G]$-modules. Then for a perfect complex C in the derived category of the category of $Z/pZ[G]$-modules, the $Z_p[G]$-submodule $\det_{Z_p[G]}(C)$ of $Q_p[G]$ is determined by the class of C in $K_0(Z/pZ[G])$. Since $K_0(Z/pZ[G])\to K_0'(Z/pZ[G])$ is injective, it is sufficient to prove that the sum of the class of $R\Gamma(O_{L,S}, Z/pZ(1))$ and the class of $H^0(L\otimes_Q \boldsymbol{R}, Z/pZ)$ in $K_0'(Z/pZ[G])$ is zero. This fact is proved, just as Tate says in [Ta_3], by considering the cohomology sequence of $0\to Z/pZ(1)\to G_m\xrightarrow{p} G_m\to 0$, together with the knowledge of the cohomology of G_m furnished by class field theory.

§ 5. Zeta elements of $Q(r)$ ($r\geqq 1$) for cyclotomic extensions.

In this section, let $K=Q$, $M=Q(r)$ with $r\geqq 1$, let $N\geqq 1$, and let L be the extension of Q generated by a primitive N-th root of 1. We give an explicit description of the motivic zeta element ((5.6), which is a rewriting of known results) and a "half description" of the arithmetic zeta element ((5.14), for which we need a result (5.12) proved in [Ka_1]).

(5.1) For $c\in(Z/NZ)^\times$, let σ_c be the element of $G=\mathrm{Gal}(L/K)$ characterized by the property $\sigma_c(\alpha)=\alpha^c$ for N-th roots of a of 1. Define the rings

$$A=Q[G]/(\sigma_{-1}-(-1)^r),\qquad B=Q[G]/(\sigma_{-1}+(-1)^r).$$

Then

$$Q[G]\xrightarrow{\cong} A\times B.$$

In the case $r=1$, let $B'=Q[G]/(\sigma_{-1}-1,\sum_{\sigma\in G}\sigma)$. Then, $B\xrightarrow{\cong}Q\times B'$ where the part $B\to Q$ sends elements of G to 1.

The following lemma is easiy seen.

LEMMA (5.2). (1) *$H_d=L$, and it is a free $Q[G]$-module of rank* 1.

(2) *H_h is identified with the space of systems $\{a(\iota)\}_\iota$ which associate to each embedding $\iota: L\to \boldsymbol{C}$ an element $a(\iota)$ of $Q(2\pi i)^r$ satisfying $a(\bar\iota)=\overline{a(\iota)}$. (Here — denote the complex conjugation.) The canonical injection*

$$H_h\otimes_Q \boldsymbol{R}\longrightarrow H_d\otimes_Q\boldsymbol{R}=L\otimes_Q\boldsymbol{R}$$

associates to $a=\{a(\iota)\}_\iota\in H_h$, the unique element of $L\otimes_Q\boldsymbol{R}$ whose image in C for any embedding $\iota: L\to\boldsymbol{C}$ coincides with $a(\iota)$. The action of $Q[G]$ on H_h is given by $\sigma(a)(\iota)=a(\iota\circ\sigma)$.

(3) *The action of $Q[G]$ on H_h factors through A, and H_h is a free A-*

module of rank one.

(5.3) Assume r=1. Then $H_k=O_L^\times\otimes Q$, and $H_k'=Q$ with the trivial action of G. The classical theory of regulator shows that Conj. (2.7) is true for $Q(1)$ over any number field, the action of $Q[G]$ on H_k factors through B', and H_k is a free B'-module of rank 1.

For an N-th root a of 1 in L such that $\alpha\neq 1$, we define the cyclotomic element $c_1(\alpha)\in H_k$ as follows. If the order of a is not a power of a prime number, then $1-\alpha$ is a unit, and we define $c_1(\alpha)$ to be the image of $(1-\alpha)^{-1}\in O_L^\times$ in H_k. If the order of a is a power of a prime number l, $1-\alpha$ does not belong to $O_L^\times$, but we have

$$O_L^\times\otimes Q \xrightarrow{\cong} O_L^\times\otimes_{Z[G]}B \xrightarrow{\cong} \left(O_L\left[\frac{1}{l}\right]\right)^\times\otimes_{Z[G]}B\,.$$

We define $c_1(\alpha)\in H_k$ in this case to be the image of $(1-\alpha)^{-1}\in(O_L[1/l])^\times$. It is easily seen that if $N\geqq 2$ and a is a primitive N-th root of 1, $c_1(\alpha)$ is a basis of the B'-module H_k.

(5.4) Assume $r\geqq 2$. Borel proved Conj. (2.7) is true for $Q(r)$ over any number field ([*Bo*] Borel used his regulator map but the coincidence of it with the regulator map of Beilinson is checked in [*Ra*]). It follows that the action of $Q[G]$ on H_k factors through B, and H_k is a free B-module of rank one.

For an N-th root a of 1 in L, Beilinson defined an element $c_r(\alpha)\in H_k$whose image in $(H_d\otimes_Q \boldsymbol{R})/(H_h\otimes_Q \boldsymbol{R})$is the class of

$$\sum_{n\geqq 1}\alpha^n n^{-r}\in L\otimes_Q \boldsymbol{R}=H_d\otimes_Q \boldsymbol{R}\,.$$

If a is a primitive N-th root of 1, $c_r(\alpha)$ is a basis of the B-module H_k.

(5.5) We fix some notations. Let a be an N-th root of 1 in L.

(5.5.1) We define $w_r(\alpha)\in L$ as follows. For $n\geqq 1$, let

$$\nu_n: Q[t]/(t^N-1)\longrightarrow Q[t]/(t^N-1)$$

be the ring homomorphism $t\to t^n$. We define $w_r(\alpha)$ to be the image of $(\prod_{\substack{l\mid N\\ l\text{ prime}}}(1-l^{-r}\nu_l)^{-1})(t)$ under $Q[t]/(t^N-1)\to L$; $t\to\alpha$. Here $1-l^{-r}\nu_l$: $Q[t]/(t^N-1)\to Q[t]/(t^N-1)$ is bijective, since the eigenvalues of ν_l are 0 or a root of 1 (indeed, there are $i>j\geqq 0$ such that $\nu_l^i=\nu_l^j$). If $N\geqq 1$ and a is a primitive N-th root of 1, $w_r(\alpha)$ is the sum of a and a linear combination over Q of powers of a which are not primitive N-th roots of 1, and it is a basis of the $Q[G]$-module L.

(5.5.2) For a primitive N-th root α of 1 and for $x\in Q(2\pi i)^r$, let $x\langle\alpha\rangle$ be the following element of H_h. To an embedding $\iota: L\to \boldsymbol{C}$, $x\langle\alpha\rangle$ associates $x\in Q(2\pi i)^r$ if $\iota(\alpha)=\exp(2\pi i/N)$, $\bar{x}$ if $\iota(\alpha)=\exp(-(2\pi i/N))$ and associates $0\in Q(2\pi i)^r$ otherwise. (Cf. (5.2) (2).)

(5.5.3) We define $d_r(\alpha)\in L=H_d$ as follows. Let $g(t)$ be the rational function in t defined by

$$g(t)=\left(\frac{d}{t^{-1}dt}\right)^{r-1}\left(\frac{t}{1-t}\right)=\sum_{n\geq 1} n^{r-1}t^{n}.$$

If $\alpha\neq 1$, we define $d_r(\alpha)=(-1)^r(r-1)!^{-1}g(\alpha)$ For $\alpha=1$, take any integer c which is different from 0, 1, -1, and let $d_r(1)$ be $(1-c^r)^{-1}$ times the value at $t=1$ of the rational function $(-1)^r(r-1)!^{-1}(g(t)-g(t^c))$. Then rf$_r$(1) is independent of the choice of c.

We have $d_r(\alpha^{-1})=(-1)^r d_r(\alpha)$ if $r\geqq 2$, and $d_1(\alpha^{-1})=-d_1(\alpha)-1$.

Let rf?(α)e(/ί$_d$)* be the Q-linear map

$$H_d=L\longrightarrow Q\,;\; x\longmapsto Tr_{L/Q}(x\,d_r(\alpha)).$$

The motivic zeta element is described as follows.

PROPOSITION (5.6). *Let the notations be as above, and let S be the set of places of Q consisting of ∞ and all prime divisors of N.*

(1) *The image of $\zeta^{an}_{L/K,S}(M)\in \mathbf{R}[G]$ in $\Phi^{mot}\otimes_Q \mathbf{R}$ belongs to Φ^{mot}.*

(2) *The image of $\zeta^{mot}_{L/K,S}(M)$ in*

$$\Phi^{mot}\otimes_{Q[G]}A\cong H_h\otimes_{Q[G]}H_d^{*\#}$$

coincides with

$$\left(\frac{2\pi i}{N}\right)^r\langle\alpha\rangle\otimes d_r^*(\alpha)$$

for any primitive N-th root a of 1 in L.

(3) // $r\geqq 2$ *(resp.* r=1 *and* $N\geqq 2$), *the image of $\zeta^{mot}_{L/K,S}(M)$ in*

$$\Phi^{mot}\otimes_{Q[G]}B\,(\text{resp. }\Phi^{mot}\otimes_{Q[G]}B')\cong H_k\otimes_{Q[G]}H_d^{\otimes(-1)}$$

coincides with

$$c_r(\alpha)\otimes w_r(\alpha)^{-1}$$

for any primitive N-th root a of 1 in L. (Here $H_d^{\otimes(-1)}$ means the inverse of the invertible $Q[G]$-module H_d.) If $r=1$, the image of $\zeta^{mot}_{L/K}(M)$ in $\Phi^{mot}\otimes_{Q[G]}Q=H_d^{\otimes(-1)}\otimes_{Q[G]}Q\cong \mathrm{Hom}_{Q[G]}(H_d, Q)$ coincides with the trace map $L\to Q$.

Proof. All things follow from the results introduced in (5.2)-(5.5) by direct computation, except that; for (2), we have to recall the following fact which is a consequence of the functional equations of partial Riemann zeta functions. The value at $s=r$ of

$$\Big(\sum_{\substack{n\geq 1\\ n\equiv c \bmod N}} n^{-s}\Big)+(-1)^r\Big(\sum_{\substack{n\geq 1\\ n\equiv -c \bmod N}} n^{-s}\Big)$$

($c\in Z$) coincides with

$$2^{-1}\left(\frac{2\pi i}{N}\right)^r (d_r(\alpha)+(-1)^r d_r(\alpha^{-1})) \qquad \text{with} \quad \alpha=\exp\left(\frac{2c\pi i}{N}\right).$$

(5.7) Now we fix a prime number p, and consider the p-adic side. We first recall a result of Soulé.

THEOREM ([So_2] §1). *Let F be a number field, and let $r\geqq 2$. Then the chern class map induces an isomorphism*

$$K_{2r-i}\left(O_F\left[\frac{1}{p}\right]\right)\otimes Q_p \xrightarrow{\cong} H^i\left(O_F\left[\frac{1}{p}\right], Q_p(r)\right) \qquad for \quad i=1, 2$$

Both groups are zero if $i=2$.

By using easy localization arguments and the fact $K_{2r-1}(F)\otimes Q=(K_{2r-1}(F)\otimes Q)^{(r)}$, we deduce from this theorem;

COROLLARY (5.8). *Let F and r be as in (5.7). Then, for any $i\in Z$, the chern class map induces an isomorphism*

(5.8.1) $$H^i(F, Q(r))\otimes_Q Q_p \xrightarrow{\cong} H^i_{lim}(F, Q_p(r)).$$

Both groups are zero if $i\neq 1$.

COROLLARY (5.9). *For any number field F and for any $r\in Z$, the chern class map induces an isomorphism*

(5.9.1) $$H^1_f(F, Q(r))\otimes_Q Q_p \xrightarrow{\cong} H^1_f(F, Q_p(r)).$$

Both groups are zero if $r\leqq 0$.

Proof. As is easily seen, $H^1_f(F, Q_p(r))$ is isomorphic to the kernel of $H^1(O_F[1/p], Q_p(r))\to\bigoplus_{v|p} H^1(F_v, Q_p(r))/H^1_f(F_v, Q_p(r))$. For a place v of F lying over p, $H^1_f(F_v, Q_p(r))$ coincides with $H^1(F_v, Q_p(r))$ if $r\geqq 2$, coincides with the image of $O^\times_{F_v}\otimes Q$ in $H^1(F_v, Q_p(1))$ if $r=1$, coincides with the "unramified part" of $H^1(F_v, Q_p)$ if $r=0$, and is zero if $r<0$ ([BK] §3).

We have from these facts

$$H^1_f(F, Q_p(r))=H^1_{lim}(IQ_p(r)) \qquad \text{if} \quad r\geqq 2,$$

$$H^1_f(F, Q_p)=H^1(O_F, Q_p)=0.$$

These reduce the case $r\geqq 2$ of (5.9) to (5.8), and prove the case $r=0$ of (5.9). (Note $H^1(F, Q(r))\subset K_{2r-1}(F)\otimes Q$ by definition and $K_{2r-1}(F)$ is zero when $r\leqq 0$.) The case $r=1$ of (5.8) is checked easily. Finally assume r<0. Then $H^1_f(K, Q_p(r))$ is isomorphic to the kernel of $H^1(O_F[1/p], Q_p(r))\to\bigoplus_{v|p} H^1(F_v Q_p(r))$ which is isomorphic to $H^2(O_F[1/p], Q_p(1-r))^*$ by (2.1). The last group is zero by (5.7).

(5.10) Now we return to the cyclotomic case. By (3.2) and (5.9), we have an exact sequence (without any conjecture)

$$0 \longrightarrow (H_p^2)^* \longrightarrow H_k \otimes_Q Q_p \longrightarrow \text{ff}_4\text{®}(?_p \longrightarrow (H_p^1)^* \longrightarrow 0\,.$$

One conjectures $H_p^2=0$, but this is not yet proved. However, since $H_k \otimes_{Q[G]} A$ $=0$, we have $H_p^2 \otimes_{Q[G]} A=0$. From this and (5.2) (1), we see that $H_p^1 \otimes_{Q[G]} A$ is a free $A \otimes_Q Q_p$-module of rank 1. We have

$$\Phi_p^{ar} \otimes_{Q[G]} A \cong H_h \otimes_{Q[G]} (H_p^1)^*$$

canonically.

We will give an explicit description of the image of the arithmetic zeta element in $\Phi_p^{ar} \otimes_{Q[G]} A$, by using p-adic cyclotomic elements of Deligne and Soulé, which we recall here.

In the rest of this section, let S be the set of places of Q consisting all prime divisors of N, and let S' be the union of S and $\{p\}$. Of course one has $S=S'$ if $p|N$.

(5.11) Let a be a primitive N-th root of 1 in L and let $m \in Z$. Then, the p-adic cyclotomic elements

$$c_m(\alpha),\ c'_m(\alpha) \in H^1(O_{L,S'},\ Z_p(m))$$

of Deligne and Soule are defined unless $(\alpha, r)=(1,\ 1)$ ([De_3], [So_2]).

In the case $m=1$, $c_1(\alpha)$ will be the image $\{1-\alpha\}$ of $1-\alpha$ under the canonical map $O_{L,S}^\times \to H^1(O_{L,S'}, Z_p(1))$defined by Kummer theory. The elements $c_m(\alpha)$ and $c'_m(\alpha)$ will be related to each other by

$$c'_m(\alpha)=(1-p^{m-1}\varphi_p^{-1})c_m(\alpha) \quad \text{in} \quad H^1(L,\ Q_p(m))$$

where φ_p is the Frobenius of p.

Let $n \geqq 1$. Take a p^n-th root β of α of order $p^n N$. Then, we obtain an element

$$\begin{aligned}\{1-\beta\} \otimes [\beta^N]^{\otimes(m-1)} &\in H^1(Q(\beta),\ Z/p^n Z(1)) \otimes Z/p^n Z(m-1)\\ &\cong H^1(Q(\beta),\ Z/p^n Z(m))\,.\end{aligned}$$

Here $\{1-\beta\}$ denotes the image of $1-\beta$ by Kummer theory, and $[\beta^N]$ is just β^N but one puts [ ] to avoid a confusion for we consider $Z/p^n Z(1)$as an additive group. One sees easily that the elements

$$c'_m(\alpha)_n \overset{\text{def}}{=\!=} N_{Q(\beta)/L}(\{1-\beta\} \otimes [\beta^N]^{\otimes(m-1)}) \in H^1(O_{L,S'},\ Z/p^n Z(m))$$

($N_{Q(\beta)/L}$ denotes the norm map) is independent of the choice of β, and $c'_m(\alpha)_n$ forms a projective system when n varies. Let

$$c'_m(\alpha)=\varprojlim_n c'_m(\alpha)_n \in H^1(O_{L,S'},\ Z_p(m))\,.$$

Now we define $c_m(\alpha)$. In the case $p\backslash N$, we define

$$c_m(\alpha)=c'_m(\alpha).$$

In the case $m=1$, let $c_m(\alpha)$ be the image of $1-\alpha\in O^{\times}_{F,S}$ in $H^1(O_{L,S'}, Z_p(1))$. Then these two definitions agree when $p\backslash N$ and $m=1$. If $(N, p)=1$ and $m\geqq 2$ (resp. if $(N, p)=1$ and $m\leqq 0$), we define

$$c_m(\alpha)=\sum_{i\geqq 0}(p^{m-1})^i c'_m(\alpha^{p^{-i}})$$

where $\alpha^{p^{-i}}$ is the unique p^i-th root of a of order N

$$(\text{resp. } c_m(\alpha)=-\sum_{i\geqq 1}(p^{1-m})^i c'_m(\alpha^{p^i})).$$

The following theorem will be proved in [Ka_1], by using an "explicit reciprocity law" for the motive $Q(r)$.

THEOREM (5.12). *Let a be a primitive N-th root of* 1 *in L. Then, the image of $c_{1-r}(\alpha)$ $(r\geqq 1)$ in $H^*_d\otimes_Q Q_p$ under the dual map $H^1_p\to H^*_d\otimes_Q Q_p$ of $H_d\otimes_Q Q_p\to(H^1_p)^*$ coincides with $-N^{-r}d^*_r(\alpha)$.*

In the case N is prime to p, this result follows easily from [BK] § 2 (2.1).

Remark (5.13). Deligne and Soulé, and also Gros and Kurihara ([Gr]) consider these cyclotomic elements in H^1 of $Q_p(m)$ mainly for positive m, and relate them to special values of p-adic zeta functions, though we consider here these elements with ra^O which are related to special values of complex zeta functions.

By (5.6) (2) and (5.13), we have

THEOREM (5.14). *The image of $\zeta^{\alpha r}_{L/K,S}(M)_p$ (resp. $\zeta^{\alpha r}_{L/K,S'}(M)_p)\in\Phi^{\alpha r}_p$ in $\Phi^{\alpha r}_p\otimes_{Q[G]}A\cong H_h\otimes_{Q[G]}(H^1_p)^{\#}$ coincides with*

$$-(2\pi i)^r\langle\alpha\rangle\otimes c_{1-r}(\alpha)\qquad (\textit{resp. } -(2\pi i)^r\langle\alpha\rangle\otimes c'_{1-r}(\alpha)).$$

for any primitive N-th root a of 1.

§ 6. Relation with classical Iwasawa theory.

In this section, we show that when we consider the situation where $K=Q$, $M=Q(r)$ with r a positive even integer and L is the maximal real subfield of $Q(\alpha)$ with a a root of 1 of order a power of p, our Iwasawa main conjecture coincides with the classical Iwasawa conjecture.

The classical Iwasawa theory uses characteristic polynomials of torsion modules. We first relate this concept to determinant modules.

PROPOSITION (6.1). *Let R be a Noetherian normal ring and let F be the total quotient ring of R (that is, F is obtained from R by inverting all non-zero-divisors in R). Let Y be a finitely generated R-module of finite tor-dimension such that $Y\otimes_R F=0$. Then, the image of*

$$\det_R(Y) \longrightarrow \det_R(Y)\otimes_R F=\det_F(Y\otimes_R F)=\det_F(\{0\})=F$$

coincides with $\mathrm{Char}(Y)^{-1}$, *where* $\mathrm{Char}(Y)$ *is the unique invertible ideal of R such that for any prime ideal $\mathfrak{p}$ of R of height one, the stalk* $\mathrm{Char}(Y)_{\mathfrak{p}}$ *coincides with* $(\mathfrak{p}R_{\mathfrak{p}})^{n(\mathfrak{p})}$ *where*

$$n(\mathfrak{p})=\mathrm{length}_{R_{\mathfrak{p}}}(Y_{\mathfrak{p}}).$$

Proof. Since R is normal, an invertible R-module in F (for example $\det_R(Y)$) is characterized by its stalks in codimension one. Hence we are reduced to the case where R is a discrete valuation ring. In this case, $Y\cong\bigoplus_i R/a_iR$ for a finite family $(a_i)_i$ of non-zero elements of R. By using the resolution $0\to\bigoplus_i R \xrightarrow{\alpha}\bigoplus_i R\to Y\to 0$ with $\alpha=(a_i)_i$, we obtain $\det_R(Y)=(\prod_i a_i)^{-1}R$ in F.

(6.2) We recall the classical Iwasawa main conjecture proved by Mazur and Wiles. (Cf. [*Iw*], [*Wa*] Ch. 13).

Let $K=Q$, $M=Q(r)$, and let r be an even positive integer. Let $S=\{p\}$ with p an odd prime. For $n\geqq 1$, take a primitive p^n-th root α_n of 1 in $\bar{Q}$ satsfying $(\alpha_{n+1})^p=\alpha_n$. Let L_n be the maximal real subfield of $Q(\alpha_n)$, and let $L_\infty=\bigcup_n L_n$, $O_{L_\infty,S}=\bigcup_n O_{L_n,S}$. Unless the contrary is explicitly stated, $(\ )^*$ means $\mathrm{Hom}(\ , Q_p/Z_p)$ in what follows.

Let

$$\mathcal{X}=H^1_{et}(O_{L_\infty,S},\ Q_p/Z_p)^*.$$

Then $\mathcal{X}$ is the Galois group of the maximal abeiian extension of L_∞ which is unramified outside p. Let

$$\begin{aligned}\mathcal{Y}&=H^1(L_\infty\otimes_Q Q_p,\ Q_p/Z_p)^*\\ &=\varprojlim_{m,n}(L_n\otimes_Q Q_p)^\times/\{(L_n\otimes_Q Q_p)^\times\}^{p^m} \quad \text{(by class field theory)}\end{aligned}$$

where $\varprojlim_n$ is taken with respect to norms. Let

$$c=\{(1-\alpha_n)(1-\alpha_n^{-1})\}_{n\geq 1}\in\mathcal{Y}.$$

Let $\Gamma=\mathrm{Gal}(L_\infty/Q)$. Then the completed group ring $Z_p[[\Gamma]]$ is a regular ring and so any finitely generated $Z_p[[\Gamma]]$-module is a perfect complex over $Z_p[[\Gamma]]$ when regarded as an object of the derived category.

In Iwasawa theory, it is well known that $\mathcal{X}$ and $\mathcal{Y}/Z_p[[\Gamma]]c$ are finitely generated torsion $Z_p[[\Gamma]]$-modules (Iwasawa [*Iw*]). The classical Iwasawa main conjecture, in one formulation, is stated as

(6.2.1) $$\mathrm{Char}_{Z_p[[\Gamma]]}(\mathcal{X})=\mathrm{Char}_{Z_p[[\Gamma]]}(\mathcal{Y}/Z_p[[\Gamma]]c).$$

(6.3) Now we relate our conjecture (4.9) to (6.2.1). For $n \geqq 1$, let $\Gamma_n = \mathrm{Gal}(L_\infty/L_n)$, $G_n = \mathrm{Gal}(L_n/Q)$.

LEMMA (6.3).

(1) [illegible] , $_{p}[\;_{n}]) = \begin{cases} \mathscr{X}(-r)_{\Gamma_n} & \text{if } i=0 \\ 0 & \text{otherwise.} \end{cases}$

(2) $\mathrm{Tor}_i^{Z_p[[\Gamma]]}(\mathscr{Y}(-r), Z_p[G_n]) = \begin{cases} \mathscr{Y}(-r)_{\Gamma_n} & \text{if } i=0 \\ 0 & \text{otherwise.} \end{cases}$

(3) *There is a distinguished triangle*

$$R\Gamma(O_{L_n,S}, Z_p(1-r))[1] \longrightarrow \mathscr{Y}(-r)_{\Gamma_n} \longrightarrow \mathscr{X}(-r)_{\Gamma_n} \longrightarrow .$$

Here $(\ )_{\Gamma_n}$ *denote coinvariants by* Γ_n.

Proof. Consider the spectral sequences

$$E_{2,gl}^{i,j} = H^i(\Gamma_n, H^j(O_{L_\infty,S}, (Q_p/Z_p)(r))) \Longrightarrow H^{i+j}(O_{L_n,S}, (Q_p/Z_p)(r))$$

$$E_{2,loc}^{i,j} = H^i(\Gamma_n, H^j(L_\infty \otimes_Q Q_p, (Q_p/Z_p)(r))) \Longrightarrow H^{i+j}(L_n \otimes_Q Q_p, (Q_p/Z_p)(r)).$$

Then we have:

(6.4.1) $E_{2,gl}^{i,j} = 0$ and $E_{2,loc}^{i,j} = 0$ except the cases $(i, j) = (0, 0), (1, 0), (0, 1)$.

(6.4.2) $$E_{2,gl}^{i,j} \xrightarrow{\cong} E_{2,loc}^{i,j} \quad \text{if } j=0.$$

Indeed, (6.4.1) follows from the facts that the cohomological p-dimension of Γ_n is 1, the cohomological dimension of $L_\infty \otimes_Q Q_p$ (resp. $O_{L_\infty,S}$) is $\leqq 2$, and

$$H^2(L_n \otimes_Q Q_p, (Q_p/Z_p)(r)) = H^0(L_n \otimes_Q Q_p, Z_p(1-r))^* = 0,$$

$$H^2(O_{L_n,S}, (Q_p/Z_p)(r)) = 0, \qquad H^2(O_{L_\infty,S}, (Q_p/Z_p)(r)) = 0$$

([So_2]) The proof of (6.4.2) is easy.

Finally (6.3) (3) follows from the above facts and the distinguished triangle

$$R\Gamma(O_{L_n,S}, Z_p(1-r)) \longrightarrow R\Gamma(L_n \otimes_Q Q_p, (Q_p/Z_p)(r))^*[-2]$$
$$\longrightarrow R\Gamma(O_{L_n,S}, (Q_p/Z_p)(r))^*[-2] \longrightarrow ,$$

which comes from Artin-Verdier duality.

(6.5) We relate Conj. (4.9) to (6.2.1). By (5.14), Conj. (4.9) for $K=Q$, $L=L_n$ with $n \geqq 1$ and $M=Q(r)$ is equivalent to the statement that $c_{1-r}(\alpha_n) \in H^1(O_{L_n,S}, Q_p(1-r))$ is a $Z_p[G_n]$-base of $\det_{Z_p[G_n]}(R\Gamma(O_{L_n,S}, Z_p(1-r))[1])$. Let

$$\gamma = \{((1-\alpha_n)(1-\alpha_n^{-1}) \otimes \alpha_n^{\otimes(-r)}\}_{n \geqq 1} \in \mathscr{Y}(-r),$$

and let $\gamma_n \in \mathscr{Y}(-r)_{\Gamma_n}$ be the image of γ. Then the image of $c_{1-r}(\alpha_n)$ in $H^1(L_n \otimes_Q Q_p, Z_p(1-r))$ coincides with the image of $2^{-1}\gamma_n$. By (6.3) (3), (4.9) is equivalent to the statement that γ_n is a $Z_p[G_n]$-basis of $\det_{Z_p[G_n]}(\underset{\deg. 0}{\mathcal{Y}(-r)_{\Gamma_n}} \to \underset{\deg. 1}{\mathcal{X}(-r)_{\Gamma_n}})$. This holds for any $n \geqq 1$ if and only if γ is a $Z_p[[\Gamma]]$-basis of $\det_{Z_p[[\Gamma]]}(\mathcal{Y}(-r) \to \mathcal{X}(-r))$, that is, if and only if (6.2.1) holds.

§ 7. Relation with Tamagawa numbers of motives.

In this section, we see that the conjecture on Tamagawa numbers of motives in [BK] is regarded as the case of trivial abelian extension of our Iwasawa main conjecture. For simplicity, we treat motives of $wt \leqq -3$ and we consider numbers (Tamagawa numbers, values of L-functions, $\cdots$) modulo multiplication by powers of 2.

In this section we assume the conjectures (2.6) (2.7) (3.4) (these conjectures were assumed also in [BK]).

(7.1) We fix notations. In this section, let M be a pure motive over Q of weight $\leqq -3$. Fix an odd prime number p. Take a Z_p-sheaf T in $V_p(M)$ such that $T \otimes_{Z_p} Q_p = V_p(M)$.

Let

$$H_f^1(Q, T) \subset H^1(Q, T) \qquad (\text{resp. } H_f^1(Q_p, T) \subset H^1(Q_p, T))$$

be the inverse image of

$$H_f^1(Q, V_p(M)) \subset H^1(Q, V_p(M)) \qquad (\text{resp. } H_f^1(Q_p, V_p(M)) \subset H^1(Q_p, V_p(M))).$$

Let $H_{k,T} \subset H_f^1(Q, T)$ be the inverse image of $H_k \subset H_f^1(Q, V_p(M))$. Then, $H_{k,T}$ is a finitely generated $Z_{(p)}$-module such that

$$H_{k,T} \otimes_{Z_{(p)}} Q \xrightarrow{\cong} H_k, \qquad H_{k,T} \otimes_{Z_{(p)}} Z_p \xrightarrow{\cong} H_f^1(Q, T).$$

(7.2) We review the definition of the Tamagawa number of the pair (M, T), which is an element of $R^\times / Z_{(p)}^\times$. (In [BK], the Tamagawa number is defined as a number without modulo $Z_{(p)}^\times$, by using l-adic realizations of M for all prime numbers l. We work here only with the p-adic realization, so we have a number modulo $Z_{(p)}^\times$. By varying p, we can recover the Tamagawa number in [BK].)

Take a $Z_{(p)}$-lattice Δ of H_d. By Conj. (2.7) which we assumed to hold, we have an isomorphism

$$\text{(7.2.1)} \qquad H_{k,T} \otimes_{Z_{(p)}} R \xrightarrow{\cong} (\Delta \otimes_{Z_{(p)}} R)/(H_{h,T} \otimes_{Z_{(p)}} R).$$

Let

$$\Phi_{T,\Delta}^{mot} = \det_{Z_{(p)}}(H_{h,T}) \otimes_{Z_{(p)}} \det_{Z_{(p)}}(H_{k,T}) \otimes_{Z_{(p)}} \det_{Z_{(p)}}(\Delta)^*.$$

$((\ )^* = \mathrm{Hom}_{Z_{(p)}}(\ , Z_{(p)}))$. Then (7.2.1) induces

$$\text{(7.2.2)} \qquad \Phi_{T,\Delta}^{mot} \otimes_{Z_{(p)}} R \xrightarrow{\cong} \bar{R}.$$

Let $\alpha\in\mathbf{R}^{\times}/Z_{(p)}^{\times}$ be the image of a $Z_{(p)}$-basis of $\Phi_{T,\Delta}^{mot}$ under (7.2.2). (With the notations in [BK], a is the volume of $A(\mathbf{R})/A(Q)$ modulo $Z_{(p)}^{\times}$. The choice of Δ here corresponds to the choice of $\det_{Q}(H_{a})\xrightarrow{\cong}Q$ in [BK].) On the other hand, let $\beta\in Q^{\times}/Z_{(p)}^{\times}$ be the element such that the image of a $Z_{(p)}$-basis of $\det_{Z_{(p)}}(\Delta)$ under the isomorphism

$$\det_{Z_{(p)}}(\Delta)\otimes_{Z_{(p)}}Q_{p}\xrightarrow{\cong}\det_{Z_{p}}(H_{f}^{1}(Q_{p},\ T))\otimes_{Z_{p}}Q_{p}$$

induced by $\exp:\Delta\otimes_{Z_{(p)}}Q_{p}\xrightarrow{\cong}H_{f}^{1}(Q_{p}T)\otimes_{Z_{p}}Q_{p}$ is (a representative in $Q^{\times}$ of) β times a Z_{p}-basis of $\det_{Z_{p}}(H_{f}^{1}(Q_{p},\mathrm{T}))$. (With the notation of [BK], β is the volume of $A(Q_{p})$ modulo $Z_{(p)}^{\times}$.)

Let 5 be a finite set of places of Q containing ∞, p, and all finite places at which M has bad reduction. Define

$$\mu_{S,f}\in Q^{\times}/Z_{(p)}^{\times},\qquad \mu_{S}\in\mathbf{R}^{\times}/Z_{(p)}^{\times}\qquad\text{by}$$

$$\mu_{S,f}=\beta\prod_{\substack{v\in S\\ v\neq\infty,\,p}}\#(H^{0}(Q_{v},\ T\otimes(Q_{p}/Z_{p}))),\quad \mu_{S}=\alpha\ \mu_{S,f}.$$

Then μ_{S} is independent of the choice of Δ. Define

$$\mathrm{Tam}(M,\ T)=\mu_{S}\cdot L_{S}(M,0)^{-1}\in\mathbf{R}^{\times}/Z_{(p)}^{\times}.$$

This element, called the Tamagawa number of the pair (M, T), is independent of the choice of S.

The following (7.3) is clear.

LEMMA (7.3). *The image of* $L_{S}(M,0)\in\mathbf{R}$ *under the isomorphism* (7.2.2) *is equal to (a representative in* $Q^{\times}$ *of)* $\mu_{S,f}\ \mathrm{Tam}(M,T)^{-1}$ *times a* $Z_{(p)}$*-basis of* $\Phi_{T,\Delta}^{mot}$.

(7.4) By (7.3), we see that Conj. (4.7) is true in this situation (with $K=L=Q$) if and only if $\mathrm{Tam}(M,\ T)\in Q$. Assume $\mathrm{Tam}(M,\ T)\in Q$. Then, the motivic zeta element $\zeta_{Q/Q,S}^{mot}(M)$ is $\mu_{S,f}\ \mathrm{Tam}(M,\ T)^{-1}$ times a $Z_{(p)}$-basis of $\Phi_{T,\Delta}^{mot}$.

(7.5) To state the conjecture in [BK] on Tamagawa numbers, we have to consider the Tate-Shafarevich group of a motive.

Consider the map $\iota:P\rightarrow Q$ where

$$P=H^{1}(Q,\quad T\otimes(Q_{p}/Z_{p}))/(H_{k,T}\otimes(Q_{p}/Z_{p}))$$

$$Q\ =\ H^{1}(Q_{p},\ \ T\otimes(Q_{p}/Z_{p}))/(H_{f}^{1}(Q_{p}T)\otimes(Q_{p}/Z_{p})\oplus(\bigoplus_{v\neq p}H^{1}(Q_{v},\ \ T\otimes(Q_{p}/Z_{p})))$$

The kernel of c is a generalization of the Tate-Shafarevich group of an abelian variety.

PROPOSITION (7.6). $\mathrm{Ker}(\iota)$ *and* $\mathrm{Coker}(\iota)$ *are finite groups.*

The proof of (7.6) is given below. (Recall we assumed the conjectures (2.6) (2.7) (3.4). Otherwise such finiteness becomes very difficult).

The p-primary part of the conjecture on Tamagawa numbers of motives in [BK] is stated as follows.

CONJECTURE (7.7). $\mathrm{Tam}(M, T)=\#(\mathrm{Coker}(\iota))\cdot\#(\mathrm{Ker}(\iota))^{-1}$ *in* $Q^{\times}/Z_{(p)}^{\times}$.

The following (7.8) shows the equivalence between (7.7) and our Iwasawa main conjecture in this situation.

PROPOSITION (7.8). *Under the isomorphism* $\Phi^{mot}\otimes_Q Q_p=\Phi_p^{ar}$, *the image of the zeta element* $\zeta^{mot}_{Q/Q,S}(M)\in\Phi^{mot}$ *is*

$$\#(\mathrm{Coker}(\iota))\ \#(\mathrm{Ker}(\iota))^{-1}\mathrm{Tam}(M, T)^{-1}$$

times a Z_p*-basis of*

$$\Phi^{ar}_{p,T}=\det_{Z_{(p)}}(H_{k,T})\otimes_{Z_{(p)}}\{\det_{Z_p}(R\Gamma(Z_S, T^*(1)))\}^*.$$

Proofs of (7.6) and (7.8).

By Artin-Verdier duality and the loccalization theory for etale cohomology, we have an acyclic complex

$$C: 0\longrightarrow \underset{(\text{deg. }0)}{H^0(Q,}\ T\otimes(Q_p/Z_p))\longrightarrow \underset{v\in S}{0} H^0(Q_v,\ T\otimes(Q_p/Z_p))$$
$$\longrightarrow H^2(Z_S, T^*(1))^*\longrightarrow H^1(Q,\ T\otimes(Q_p/Z_p))$$
$$\longrightarrow \bigoplus_{v\in S} H^1(Q_v,\ T\otimes(Q_p/Z_p))\oplus(\bigoplus_{v\notin S} H^0(v, T\otimes(Q_p/Z_p)(-1)))$$
$$\longrightarrow \mathrm{ff}'(Z_{\beta},\ T^*(1))^*\longrightarrow H^2(Q,\ T\otimes(Q_p/Z_p))$$
$$\overset{\gamma}{\longrightarrow} \bigoplus_{v\in S} H^2(Q_v,\ T\otimes(Q_p/Z_p))\longrightarrow 0\ .$$

Here the * outside the notation of cohomology $H^i(\ )$ are $\mathrm{Hom}(\ ,\ Q_p/Z_p)$,whereas the * inside $H^i(\ )$ are $\mathrm{Hom}(\ ,Z_p)$.

Jannsen proved that the map γ is an isomorphism. ([Ja] §4 Thm. 3d).

We define a subcomplex C'_Z of C. Since the image of

$$\mathrm{Hom}_{Q_p}(H^2(Z_S, V_p(M)^*(1))^*\longrightarrow H^1(Z_S,\ V_p(M))$$

belongs to $H^1_f(Z_S, V_p(M))=H_{k,T}\otimes_{Z_{(p)}}Q_p$we see that there is a Z_p-submodule D of $H^1(Z_S,T^*(1))^*$ of finite index whose image in $H^1(Z_S, T\otimes(Q_p/Z_p))$is contained in $H_{k,T}\otimes(Q_p/Z_p)$. Let C'_D be the complex

$$0\longrightarrow \underset{(\text{deg. }2)}{D}\longrightarrow H_{k,T}\otimes(Q_p/Z_p)\longrightarrow !^1_f(Q_p,\ T)\otimes(Q_p/Z_p)$$
$$\longrightarrow H^1(Z_S,T^*(1))^*\longrightarrow 0\ .$$

We have an exact sequence of complexes

$$0\longrightarrow C'_D\longrightarrow C\longrightarrow C''_D\longrightarrow 0$$

where C''_D is the complex

$$0 \longrightarrow H\,(Q,\, T\otimes(Q_p/Z_p)) \longrightarrow \bigoplus_{v\in S} H^0(Q_v,\, T\otimes(Q_p/Z_p))$$

$$\longrightarrow H^2(Z_S,\, T^*(1))^*/D \longrightarrow P \xrightarrow{\iota} Q \longrightarrow 0\,.$$

Here P and Q are as in (7.5). Since C is acyclic and the cohomology groups of the complex C'_D are finite, it follows that the cohomology groups of the complex C''_D are finite, that is, $\mathrm{Ker}(\iota)$ and $\mathrm{Coker}(\iota)$ are finite.

For a bounded complex E of abelian groups whose cohomology groups are finite, let $\chi(E)=\prod_i \#(H^i(E))^{(-1)^i}$. We have

(7.9.1) $$\chi(C'_D)=\chi(C''_D)^{-1}\,.$$

It is easily seen that the isomorphism $\Phi^{mot}\otimes_Q Q_p \xrightarrow{\cong} \Phi_p^{ar}$ sends a $Z_{(p)}$-basis of $\Phi_{T,\Delta}^{mot}$ to

(7.9.2) $$\beta^{-1}\chi(C'_D)^{-1}\#((H_{k,T})_{tor})^{-1}\#(H^2(Z_S T^*(1))^*/D)^{-1}\#(H^1_f(Q_p,T)_{tor})$$

times a generator of $\Phi_{p,T}^{ar}$. Here tor denote the torsion part. Since

$$(H_{k,T})_{tor}=H^1_f(Q,\, T)_{tor}\cong H^0(Q,\, T\otimes(Q_p/Z_p))$$

and

$$H^1_f(Q_p,\ T)_{tor}\cong H^0(Q_p,\ T\otimes(Q_p/Z_p))\,,$$

we have by (7.9.1) that the element (7.9.2) is equal to

$$\#(\mathrm{Coker}\,(\iota))\cdot\#(\mathrm{Ker}\,(\iota))^{-1}\cdot\mu_{S,f}^{-1}\,.$$

Hence by (7.3), the map $\Phi^{mot}\otimes_Q Q_p\cong\Phi_p^{ar}$ sends $\zeta_{Q/Q,S}^{mot}(M)$ to

$$\#(\mathrm{Coker}\,(\iota))\ \ \#(\mathrm{Ker}(\iota))^{-1}\cdot \mathrm{Tam}\,(M,\, T)^{-1}$$

times a Z_p-basis of $\Phi_{p,T}^{ar}$.

REFERENCES

[AV] ARTIN, M. AND VERDIER, J.-L., Seminar on etale cohomology of number fields, Wood Hole 1964.

[Be_1] BEILINSON, A., Higher regulators and values of L-functions, J. of Sov. Math. 30 (1985) 2036-2070.

[Be_2] BEILINSON, A., Height pairings between algebraic cycles, in Current Trends in Arithmetical Algebraic Geometry, Contemporary Math. 67 (1987) 1-24.

[Bo] BOREL, A., Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers, Ann. Sc. Norm. Sup. 4 (1977) 613-636.

[Bl_1] BLOCH, S., Algebraic cycles and values of L-functions II, Duke Math. J. 52 (1985) 379-397.

[Bl_2] BLOCH, S., Height pairings for algebraic cycles, J. Pure and Appl. Algebra 34 (1984) 119-145.

[*BK*] BLOCH, S. AND KATO, K., Tamagawa numbers of motives and *L*-functions, in The Grothendieck Festschrift vol. I, Birkhauser (1990) 333-400.

[*CW*] COATES, J. AND WILES, A., On the conjecture of Birch and Swinnerton-Dyer, Inventiones math. 39 (1977) 223-251.

[De_1] DELIGNE, P., Valeurs de fonctions *L* et periods d'intégrales, Proc. Symp. Pure Math. 33 (1979) AMS, 313-346.

[De_2] DELIGNE, P., La conjecture de Weil I, Publ. Math. IHES 43 (1972) 273-307, II, ibid. 52 (1981).

[De_3] DELIGNE, P., Le groupe fundamental de la droite projective moins trois points, in Galois group over *Q*, Springer (1989) 79-293.

[Fa_1] FALTINGS, G., Crystalline cohomology and *p*-adic Galois representations, in Algebraic Analysis, Geometry, and Number Theory, The Johns Hopkins Univ. Press (1989) 25-80.

[Fa_2] FALTINGS, G., *F*-isocrystals on open varieties, Results and conjectures, The Grothendieck Festschrift vol. II, Birkhauser (1990) 219-248.

[*Fo*] FONTAINE, J.M., Sur certains types de representations *p*-adiques du groupes de Galois d'un corps local: construction d'un anneau de Barsotti-Tate, Ann. of Math. 115 (1982) 529-577.

[*FM*] FONTAINE, J.M. AND MESSING, W., *p*-adic periods and *p*-adic etale cohomology, in Current Trends in Arithmetical Algebraic Geometry, Cont. Math. 67 (1987) AMS, 179-207.

[*Gr*] GROS, M., Régulateurs syntomiques et valeurs de fonctions *L* *p*-adiques I (with Appendix by Kurihara, M.), Inventiones math. 99 (1990) 293-320.

[*Iw*] IWASAWA, K., On Z_l extensions of algebraic number fields, Ann. of Math. 98 (1973) 246-326.

[*Ja*] JANNSEN, U., On the *l*-adic cohomology of varieties over number fields and its Galois cohomology, in Galois group over *Q*, Springer (1989).

[Ka_1] KATO, K., Lectures on the approach to Iwasawa theory of Hasse-Weil *L*-functions via B_{dR}. In preparation.

[Ka_2] KATO, K., *p*-adic Hodge theory and special values of zeta functions of elliptic cusp forms, in preparation.

[*Ko*] KOLYVAGIN, V.A., Euler systems, The Grothendieck Festschrift vol. II, Birkhauser (1990) 435-484.

[*Ma*] MAZUR, B., Notes on etale cohomology of number fields, Ann. Sci. EC. Norm. Sup. 6 (1973) 521-556.

[*MW*] MAZUR, B. AND WILES, A., Class fields of abelian extensions of *Q*, Inventiones math. 76 (1984) 179-330.

[*Po*] POITOU, Séminaire Lille, 1962-1963.

[*Ra*] RAPOPORT, M., Comparison of the regulators of Beilinson and of Borel, in Beilinson's conjectures on special values of L-functions, Academic Press (1988) 169-192.

[*Sei*] SEILER, W. K., λ-rings and Adams operations in algebraic *K*-theory, in Beilinson's conjectures on special values of L-functions, Academic Press (1988) 93-102.

[*Ser*] SERRE, J.-P., Cohomologie Galoisienne, Lecture Notes in Math. 5, Springer (1973).

[*dS*] de SHALIT, E., Iwasawa theory of elliptic curves with complex multiplication, Academic Press (1987).

[So_1] SOULÉ, C., *K*-theorie des anneaux d'entiers de corps de nombres et cohomologie étale, Inventiones math. 55 (1979) 251-295.

[So_2] SOULÉ, C., On higher p-adic regulators, in Algebraic K-theory, Lecture Notes Math. 854 (1981) 371-401, Springer.

[Ta_1] TATE, J., Duality theorems in Galois cohomology over number fields, Proc. ICM. Stockholm, 1962 Institute Mittag-Leffler (1963) 288-295.

[Ta_2] TATE, J., p-divisible groups, in Proc. of a Conf. on local fields, 1966 Springer (1967) 153-183.

[Ta_3] TATE, J., On the conjectures of Birch and Swinnerton-Dyer and a geometric analog, Sem. Bourbaki exposé 306, 1965-66, in Dix exposés sur la Cohomologie des Schémas, North Holland (1968).

[VK] VOSTOKOV, S. V. AND KIRILLOV, A. N., Normed pairing in a two dimensional local field, J. Sov. Math. 30 (1985).

[Wa] WASHINGTON, L. C., Introduction to cyclotomic fields, Springer (1982).

[$ASPM$, 17] Advanced Studies in Pure Math. 17, Algebraic Number Theory in honor of K. IWASAWA, Academic Press (1989).

[FP] FONTAINE, J.-M. AND PERRIN-RIOU, B., Autour des conjectures de Bloch et Kato, I, II, III, C. R. Acad. Sci. Paris, 313, Serie I, 189-196, 349-356, 421-428 (1991).