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HOLOMORPHIC EQUIVALENCE PROBLEM FOR A CERTAIN CLASS OF UNBOUNDED REINHARDT DOMAINS IN C^2, II

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Introduction.

For a pair (a, b) of real constants with $(a, b)\neq(0, 0)$ and a positive constant r, we define an unbounded Reinhardt domain $D_{a,b}^*(r)$ in $(C^*)^2$ by

$$D_{a,b}^*(r)=\{(z, w)\in(C^*)^2 \mid |z|^a|w|^b<r\}.$$

Also, for a pair (a, b) of non-negative constants with $(a, b)\neq(0, 0)$ and a positive constant r, we define an unbounded Reinhardt domain $D_{a,b}(r)$ in C^2 by

$$D_{a,b}(r)=\{(z, w)\in C^2 \mid |z|^a|w|^b<r\}.$$

Here, when $ab=0$, for example, when $b=0$, the domain $D_{a,0}(r)$ is understood as

$$D_{a,0}(r)=\{(z, w)\in C^2 \mid |z|^a<r\}.$$

In our previous paper [3], we investigated the holomorphic automorphisms and the equivalence of the domains $D_{a,b}(r)$ with $(a, b)\in Z^2$ as well as those of the domains $D_{a,b}^*(r)$ with $(a, b)\in Z^2$. The purpose of the present paper is to continue our study in the case where a and b are arbitrary real constants.

Our main results of this paper are as follows (see Section 1 for terminologies).

MAIN THEOREM 1. *If $D_{a,b}^*(r)$ and $D_{u,v}^*(s)$ are holomorphically equivalent, then they are algebraically equivalent.*

MAIN THEOREM 2. *If $D_{a,b}(r)$ and $D_{u,v}(s)$ are holomorphically equivalent, then they are algebraically equivalent under a transformation given by*

$$C^2\ni(z, w)\longmapsto(\alpha z, \beta w)\in C^2$$

or

$$C^2\ni(z, w)\longmapsto(\gamma w, \delta z)\in C^2,$$

where α, β, γ, δ are non-zero complex constants.

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This paper is organized as follows. In Section 1, we recall basic concepts and results on Reinhardt domains. In Section 2, we discuss a correspondence between Reinhardt domains and tube domains, which is needed later. Section 3 is devoted to the study of the holomorphic automorphisms of domains $D^*_{a,b}(r)$. In Section 4, we first introduce the notion of a *plurisubharmonic Liouville foliation,* and then apply this to the study of the holomorphic automorphisms of domains $D_{a,b}(r)$. The results of Sections 3 and 4 are used in Section 5 to prove Main Theorems 1 and 2 stated above. In Section 6, we give a concluding remark on our results.

1. Basic concepts on Reinhardt domains.

We first recall notation and terminologies. The set of non-zero complex numbers is denoted by $\boldsymbol{C}^*$. The multiplicative group of complex numbers of absolute value 1 is denoted by $U(1)$. An automorphism of a complex manifold M means a biholomorphic mapping of M onto itself. The group of all automorphisms of M is denoted by $\mathrm{Aut}(M)$. Two complex manifolds are said to be holomorphically equivalent if there is a biholomorphic mapping between them.

We now recall some basic concepts and results on Reinhardt domains (cf. [2, Section 2]). Write $T=(U(1))^n$. The group T acts as a group of automorphisms on $\boldsymbol{C}^n$ by

$$(\alpha_1, \cdots, \alpha_n)\cdot(z_1, \cdots, z_n)=(\alpha_1 z_1, \cdots, \alpha_n z_n)$$
$$\text{for } (\alpha_1, \cdots, \alpha_n)\in T \text{ and } (z_1, \cdots, z_n)\in \boldsymbol{C}^n .$$

By definition, a Reinhardt domain D in $\boldsymbol{C}^n$ is a domain in $\boldsymbol{C}^n$ which is stable under the action of T; that is, $\alpha\cdot D\subset D$ for all $\alpha\in T$. The subgroup of $\mathrm{Aut}(D)$ induced by T is denoted by $T(D)$.

An automorphism φ of $(\boldsymbol{C}^*)^n$ is called an algebraic automorphism of $(\boldsymbol{C}^*)^n$ if the components of φ are given by Laurent monomials; that is, φ is of the form

$$\varphi: (\boldsymbol{C}^*)^n\ni(z_1, \cdots, z_n)\longmapsto(w_1, \cdots, w_n)\in(\boldsymbol{C}^*)^n ,$$
$$w_i=\alpha_i {z_1}^{a_{1i}} \cdots {z_n}^{a_{ni}} , \qquad i=1, \cdots, n ,$$

where $(a_{ij})\in GL(n, \boldsymbol{Z})$ and $(\alpha_i)\in(\boldsymbol{C}^*)^n$. The set $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$ of all algebraic automorphisms of $(\boldsymbol{C}^*)^n$ forms a subgroup of $\mathrm{Aut}((\boldsymbol{C}^*)^n)$. The group $\mathrm{Aut}_{alg}((\boldsymbol{C}^*)^n)$ is a Lie group with respect to the compact-open topology.

Let φ be an algebraic automorphism of $(\boldsymbol{C}^*)^n$ and write $\varphi(z)=(\varphi_1(z), \cdots, \varphi_n(z))$. In general, the components $\varphi_1, \cdots, \varphi_n$ have zero or poles along each coordinate hyperplane. Let D and D' be domains in $\boldsymbol{C}^n$, not necessarily contained in $(\boldsymbol{C}^*)^n$. If $\varphi_1, \cdots, \varphi_n$ have no poles on D and $\varphi: D\to\boldsymbol{C}^n$ maps D biholomorphically onto D', then we say that φ induces a biholomorphic mapping of D onto D'.

Two Reinhardt domains in $\boldsymbol{C}^n$ are said to be algebraically equivalent if there is a biholomorphic mapping between them induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$.

PROPOSITION 1.1 ([2, Section 2, Proposition 1]). *Let $\varphi: D\to D'$ be a biholomorphic mapping between two Reinhardt domains D and D' in $\boldsymbol{C}^n$. If $\varphi T(D)\varphi^{-1}=T(D')$, then φ is induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$.*

LEMMA 1.1 (cf. [1, Section 4]). *Let φ be a biholomorphic mapping between two domains in $\boldsymbol{C}^n$ both containing the origin. If the components of φ are given by Laurent monomials, then φ is induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$ of the form*

$$(\boldsymbol{C}^*)^n \ni (z_1, \cdots, z_n) \longmapsto (w_1, \cdots, w_n) \in (\boldsymbol{C}^*)^n,$$

$$w_i=\alpha_i z_{\sigma(i)}, \qquad i=1, \cdots, n,$$

where σ is a permutation of $\{1, \cdots, n\}$ and $(\alpha_1, \cdots, \alpha_n)\in(\boldsymbol{C}^*)^n$.

The concept of an algebraic automorphism of a Reinhardt domain will be needed later. An automorphism of a Reinhardt domain D in $\boldsymbol{C}^n$ is called an algebraic automorphism of D if it is induced by an algebraic automorphism of $(\boldsymbol{C}^*)^n$. The set $\mathrm{Aut}_{\mathrm{alg}}(D)$ of all algebraic automorphisms of D forms a subgroup of $\mathrm{Aut}(D)$. The group $\mathrm{Aut}_{\mathrm{alg}}(D)$ may be viewed as a subgroup of $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$. It then follows that $\mathrm{Aut}_{\mathrm{alg}}(D)$ is closed in $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$, and therefore that $\mathrm{Aut}_{\mathrm{alg}}(D)$ is a Lie group with respect to the compact-open topology. We observe that the identity component of $\mathrm{Aut}_{\mathrm{alg}}(D)$ is given by that of the subgroup of $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^n)$ consisting of those transformations f which has the form

$$f: \boldsymbol{C}^n \ni (z_1, \cdots, z_n) \longmapsto (\alpha_1 z_1, \cdots, \alpha_n z_n) \in \boldsymbol{C}^n$$

and satisfy $f(D)=D$, where $(\alpha_1, \cdots, \alpha_n)\in(\boldsymbol{C}^*)^n$.

2. Reinhardt domains and tube domains.

There is a useful correspondence between Reinhardt domains and tube domains (cf. [1, Section 2]). First we recall the definition of a tube domain and fix notation. If Ω is a domain in $\boldsymbol{R}^n$, the tube domain $T_\Omega=\Omega+\sqrt{-1}\boldsymbol{R}^n$ over Ω is the domain in $\boldsymbol{C}^n$ consisting of all points $\zeta=\xi+\sqrt{-1}\eta\in\boldsymbol{C}^n=\boldsymbol{R}^n+\sqrt{-1}\boldsymbol{R}^n$ ($\xi, \eta\in\boldsymbol{R}^n$) such that $\xi\in\Omega$. For each element η of $\boldsymbol{R}^n$, we set the translation $\sigma_\eta\in\mathrm{Aut}(T_\Omega)$ as

$$\sigma_\eta(\zeta)=\zeta+\sqrt{-1}\eta .$$

Now, we define the mapping ord : $(\boldsymbol{C}^*)^n \to \boldsymbol{R}^n$ by

$$\mathrm{ord}(z_1, \cdots, z_n) = (-(2\pi)^{-1} \log|z_1|, \cdots, -(2\pi)^{-1} \log|z_n|).$$

Clearly ord is an open mapping. If E is a subset of $\boldsymbol{C}^n$, the image of $E^* := E \cap (\boldsymbol{C}^*)^n$ under ord is called the logarithmic image of E. To each Reinhardt domain D in $(\boldsymbol{C}^*)^n$, there is associated a tube domain T_Ω for which Ω is the logarithmic image ord(D) of D. The tube domain T_Ω naturally becomes a covering manifold of D. Indeed, introduce the covering $\varpi : \boldsymbol{C}^n \to (\boldsymbol{C}^*)^n$ defined by

$$\tilde{\omega}(\zeta_1, \cdots, \zeta_n) = (e^{-2\pi\zeta_1}, \cdots, e^{-2\pi\zeta_n}).$$

Then we have $T_\Omega = \varpi^{-1}(D)$, and the restriction $\varpi : T_\Omega \to D$ is a covering projection. The covering transformation group for $\tilde{\omega}$ is given by $\sigma_{Z^n} := \{\sigma_\eta \mid \eta \in \boldsymbol{Z}^n\}$. The tube domain T_Ω is called the covering tube domain of D and the covering projection $\varpi : T_\Omega \to D$ is called the canonical covering projection.

Let D be a pseudoconvex Reinhardt domain in $(\boldsymbol{C}^*)^n$ and T_Ω the covering tube domain of D. It follows that T_Ω is pseudoconvex, and therefore that T_Ω is convex. As a consequence, T_Ω is simply connected. This implies that the covering $\varpi : T_\Omega \to D$ is the universal covering of D. Let f be an automorphism of D. Then a lifting $\tilde{f}$ of f is an automorphism of T_Ω. Note that, since the covering transformation group for ϖ is given by σ_{Z^n}, there exists an element $P \in GL(n, \boldsymbol{Z})$ such that

$$\tilde{f} \circ \sigma_\eta = \sigma_{\eta P} \circ \tilde{f} \qquad \text{for every } \eta \in \boldsymbol{Z}^n. \tag{2.1}$$

The following lemma gives a criterion for f to be an algebraic automorphism of D.

LEMMA 2.1. *If $\tilde{f}$ is a complex affine transformation, that is, $\tilde{f}$ can be written in the form*

$$\tilde{f}(\zeta) = \zeta A + \beta \qquad \text{for } \zeta \in T_\Omega,$$

where $A = (a_{ij}) \in GL(n, \boldsymbol{C})$ and $\beta = (\beta_i) \in \boldsymbol{C}^n$, then f is an algebraic automorphism of D.

Proof. It follows from the relation (2.1) that $A = P$, so that $A \in GL(n, \boldsymbol{Z})$. In view of the definition of the covering projection ϖ, we see that f is given by

$$f : D \ni (z_1, \cdots, z_n) \longmapsto (w_1, \cdots, w_n) \in D,$$

$$w_i = e^{-2\pi\beta_i} z_1^{a_{1i}} \cdots z_n^{a_{ni}}, \qquad i = 1, \cdots, n,$$

which implies our assertion. q.e.d.

We conclude this section with a description of the automorphisms of a two-dimensional tube domain T_{Ω} for which Ω is a half-plane.

For convenience we denote by $\boldsymbol{G}$ the right half-plane $T_{(0,\infty)}=\{\xi+\sqrt{-1}\eta\in \boldsymbol{C}\mid\xi>0,\ \eta\in\boldsymbol{R}\}$ in the complex plane $\boldsymbol{C}$.

LEMMA 2.2. *If* $\Omega_0=\{(\xi,\rho)\in\boldsymbol{R}^2\mid\xi>0\}$, *then* $\mathrm{Aut}(T_{\Omega_0})$ *consists of all transformations of the form*

$$T_{\Omega_0}\ni(\zeta,\omega)\longmapsto(\tau(\zeta),\ \lambda(\zeta)\omega+\mu(\zeta))\in T_{\Omega_0},$$

where $\tau\in\mathrm{Aut}(\boldsymbol{G})$, λ *is a nowhere-vanishing holomorphic function on* $\boldsymbol{G}$ *and* μ *is a holomorphic function on* $\boldsymbol{G}$.

Proof. Since $T_{\Omega_0}=\boldsymbol{G}\times\boldsymbol{C}$ and since $\boldsymbol{G}$ is holomorphically equivalent to the unit disk $\{z\in\boldsymbol{C}\mid|z|<1\}$, our assertion is an immediate consequence of [3, Theorem 4.1(i)]. q.e.d.

PROPOSITION 2.1. *If c is a real constant, and if* $\Omega_c=\{(\xi,\rho)\in\boldsymbol{R}^2\mid\xi+c\rho>0\}$, *then* $\mathrm{Aut}(T_{\Omega_c})$ *consists of all transformations of the form*

$$T_{\Omega_c}\ni(\zeta,\omega)\longmapsto(\zeta',\omega')\in T_{\Omega_c},$$

$$(2.2)\qquad\begin{cases}\zeta'=\zeta'(\zeta,\omega)=\tau(\zeta+c\omega)-c\{\lambda(\zeta+c\omega)\omega+\mu(\zeta+c\omega)\},\\ \omega'=\omega'(\zeta,\omega)=\lambda(\zeta+c\omega)\omega+\mu(\zeta+c\omega),\end{cases}$$

where $\tau\in\mathrm{Aut}(\boldsymbol{G})$, λ *is a nowhere-vanishing holomorphic function on* $\boldsymbol{G}$, *and* μ *is a holomorphic function on* $\boldsymbol{G}$.

Proof. We define a complex linear transformation φ of $\boldsymbol{C}^2$ by

$$\varphi:\boldsymbol{C}^2\ni(\zeta,\omega)\longmapsto(\zeta+c\omega,\ \omega)\in\boldsymbol{C}^2.$$

Noting that $T_{\Omega_c}=\{(\zeta,\omega)\in\boldsymbol{C}^2\mid\zeta+c\omega\in\boldsymbol{G}\}$, we see that $\varphi(T_{\Omega_c})=T_{\Omega_0}$, and hence that $\mathrm{Aut}(T_{\Omega_c})=\varphi^{-1}\mathrm{Aut}(T_{\Omega_0})\varphi$. Our assertion follows from Lemma 2.2 and a straightforward computation. q.e.d.

3. Automorphisms of domains $D^*_{a,b}$.

We begin with preliminary observations. Firstly, for every positive constant r, the domain $D^*_{a,b}(r)$ is algebraically equivalent to the domain $D^*_{a,b}(1)$ under a suitable transformation of the form

$$\boldsymbol{C}^2\ni(z,w)\longmapsto(\alpha z,\ \beta w)\in\boldsymbol{C}^2,$$

where $(\alpha,\beta)\in(\boldsymbol{C}^*)^2$. Hence, in order to discuss the automorphisms and the equivalence of domains $D^*_{a,b}(r)$, it is sufficient to deal with domains $D^*_{a,b}(1)$.

For brevity, we set $D^*_{a,b}=D^*_{a,b}(1)$. Secondly, if necessary, we may replace $D^*_{a,b}$ by $D^*_{\delta a,\delta b}$, where δ is a positive constant. In fact, we have $D^*_{\delta a,\delta b}=D^*_{a,b}$.

Now, we classify the domains $D^*_{a,b}$ into the following three classes:

(I) $ab=0$;
(II) $ab\neq 0$ and $b/a\in\boldsymbol{Q}$;
(III) $ab\neq 0$ and $b/a\notin\boldsymbol{Q}$.

If $D^*_{a,b}$ is of class (I) or of class (II), then it is algebraically equivalent to a domain $D^*_{p,q}$ for which $(p, q)\in\boldsymbol{Z}^2$. Therefore, in this case, a description of the automorphisms of $D^*_{a,b}$ follows from [3, Proposition 3.2]. For domains $D^*_{a,b}$ of class (III), we have the following.

THEOREM 3.1. *If $D^*_{a,b}$ is of class* (III), *then* $\mathrm{Aut}(D^*_{a,b})=\mathrm{Aut}_{\mathrm{alg}}(D^*_{a,b})$. *Furthermore, the identity component $G(D^*_{a,b})$ of $\mathrm{Aut}_{\mathrm{alg}}(D^*_{a,b})$ consists of all transformations of the form*

$$D^*_{a,b}\ni(z, w)\longmapsto(\delta^{-b}\alpha z, \delta^a\beta w)\in D^*_{a,b}, \tag{3.1}$$

where α and β are complex constants of absolute value 1 *and δ is a positive constant.*

Proof. We begin by proving the first assertion. Put $c=b/a$. Then $D^*_{a,b}$ is algebraically equivalent to $D^*_{1,c}$. Hence it is sufficient to prove that $\mathrm{Aut}(D^*_{1,c})=\mathrm{Aut}_{\mathrm{alg}}(D^*_{1,c})$. Note that $c\notin\boldsymbol{Q}$ by assumption.

It is readily verified that the covering tube domain of $D^*_{1,c}$ is given by T_{Ω_c}. Let $\varpi: T_{\Omega_c}\to D^*_{1,c}$ be the canonical covering projection. Then the covering $\varpi: T_{\Omega_c}\to D^*_{1,c}$ is the universal covering of $D^*_{1,c}$.

Let f be any element of $\mathrm{Aut}(D^*_{1,c})$ and let $\tilde{f}$ be an element of $\mathrm{Aut}(T_{\Omega_c})$ given as a lifting of f. By Lemma 2.1, to see that $f\in\mathrm{Aut}_{\mathrm{alg}}(D^*_{1,c})$, it suffices to show that $\tilde{f}$ is a complex affine transformation. According to Proposition 2.1, we write $\tilde{f}$ in the form (2.2), and put

$$P=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\in GL(2, \boldsymbol{Z})$$

in (2.1). Then (2.1) implies that

$$\zeta'(\zeta+\sqrt{-1}k, \omega+\sqrt{-1}l)=\zeta'(\zeta, \omega)+\sqrt{-1}(pk+rl), \tag{3.2}$$

$$\omega'(\zeta+\sqrt{-1}k, \omega+\sqrt{-1}l)=\omega'(\zeta, \omega)+\sqrt{-1}(qk+sl), \tag{3.3}$$

for all $(\zeta, \omega)\in T_{\Omega_c}$ and all $(k, l)\in\boldsymbol{Z}^2$. Set $Z=\zeta+c\omega$. Then (3.2) and (3.3) are written as

$$\begin{aligned}&\tau(Z+\sqrt{-1}(k+cl))\\ &\quad -c\{\lambda(Z+\sqrt{-1}(k+cl))(\omega+\sqrt{-1}l)+\mu(Z+\sqrt{-1}(k+cl))\}\\ &=\tau(Z)-c\{\lambda(Z)\omega+\mu(Z)\}+\sqrt{-1}(pk+rl),\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\lambda(Z+\sqrt{-1}(k+cl))(\omega+\sqrt{-1}l)+\mu(Z+\sqrt{-1}(k+cl))\\ &=\lambda(Z)\omega+\mu(Z)+\sqrt{-1}(qk+sl),\end{aligned} \tag{3.5}$$

for all $(\zeta, \omega)\in T_{\Omega_c}$ and all $(k, l)\in \boldsymbol{Z}^2$.

We show that λ is a constant function. Fix a point Z_0 of $\boldsymbol{G}$ and consider the complex affine line

$$L=\{(-cW+Z_0, W)\in \boldsymbol{C}^2 \mid W\in \boldsymbol{C}\}$$

contained in T_{Ω_c}. The restriction to L of the left hand side of (3.5) is a complex affine function of W whose linear part is given by $\lambda(Z_0+\sqrt{-1}(k+cl))$, while the restriction to L of the right hand side of (3.5) is a complex affine function of W whose linear part is given by $\lambda(Z_0)$. Since these two complex affine functions of W must coincide, it follows that

$$\lambda(Z_0+\sqrt{-1}(k+cl))=\lambda(Z_0). \tag{3.6}$$

We recall here that $c\notin \boldsymbol{Q}$. This relation implies that the set $\{Z_0+\sqrt{-1}(k+cl)\in \boldsymbol{G} \mid (k, l)\in \boldsymbol{Z}^2\}$ has an accumulation point in $\boldsymbol{G}$. Since (3.6) holds for all $(k, l)\in \boldsymbol{Z}^2$, we see by a uniqueness theorem for holomorphic functions that $\lambda(Z)=\lambda_0$ for a constant λ_0.

We show that μ is a complex affine function. By the result of the preceding paragraph, (3.5) becomes

$$\mu(Z+\sqrt{-1}(k+cl))+\sqrt{-1}\lambda_0 l=\mu(Z)+\sqrt{-1}(qk+sl)$$

for all $Z\in \boldsymbol{G}$ and all $(k, l)\in \boldsymbol{Z}^2$. Differentiating the both sides of this equation with respect to the variable $\boldsymbol{Z}$, we obtain

$$\mu'(Z+\sqrt{-1}(k+cl))=\mu'(Z) \tag{3.7}$$

for all $Z\in \boldsymbol{G}$. If we fix a point $Z\in \boldsymbol{G}$, then the right hand side of (3.7) is a constant. Since (3.7) holds for all $(k, l)\in \boldsymbol{Z}^2$, it follows from the same argument as in the preceding paragraph that μ' is a constant, and therefore that $\mu(Z)=\mu_1 Z+\mu_0$ for constants μ_0 and μ_1.

We show that τ is a complex affine function. Substituting $\lambda(Z)=\lambda_0$ and $\mu(Z)=\mu_1 Z+\mu_0$ into (3.4) yields that

$$\tau(Z+\sqrt{-1}(k+cl))-\sqrt{-1}c\lambda_0 l-\sqrt{-1}c\mu_1(k+cl)$$
$$=\tau(Z)+\sqrt{-1}(pk+rl)$$

for all $Z\in\boldsymbol{G}$ and all $(k, l)\in\boldsymbol{Z}^2$. Differentiating the both sides of this equation with respect to the variable Z, we obtain

$$\tau'(Z+\sqrt{-1}(k+cl))=\tau'(Z) \tag{3.8}$$

for all $Z\in\boldsymbol{G}$. If we fix a point $Z\in\boldsymbol{G}$, then the right hand side of (3.8) is a constant. Since (3.8) holds for all $(k, l)\in\boldsymbol{Z}^2$, it follows from the same argument as in the preceding paragraphs that τ' is a constant, and therefore that $\tau(Z)=\tau_1 Z+\tau_0$ for constants τ_0 and τ_1.

Since $\lambda(Z)=\lambda_0$, $\mu(Z)=\mu_1 Z+\mu_0$ and $\tau(Z)=\tau_1 Z+\tau_0$, it follows from (2.2) that both $\zeta'(\zeta, \omega)$ and $\omega'(\zeta, \omega)$ are complex affine functions of ζ, ω, so that $\tilde{f}$ is a complex affine transformation. This proves the first assertion.

By the observation made in Section 1, $G(D^*_{a,b})$ is given by the identity component of the subgroup H of $\mathrm{Aut}_{\mathrm{alg}}((\boldsymbol{C}^*)^2)$ consisting of those transformations f which has the form

$$f:\boldsymbol{C}^2\ni(z, w)\longmapsto(\gamma z, \theta w)\in\boldsymbol{C}^2$$

and satisfy $f(D^*_{a,b})=D^*_{a,b}$, where $(\gamma, \theta)\in(\boldsymbol{C}^*)^2$. It is readily verified that $f(D^*_{a,b})=D^*_{a,b}$ precisely when $|\gamma|^a|\theta|^b=1$. This implies that H consists of all transformations of the form (3.1). Since, in particular, H is connected, we have $G(D^*_{a,b})=H$, and the second assertion is proved. q.e.d.

4. Plurisubharmonic Liouville foliation and the automorphisms of domains $D_{a,b}$

We first introduce the notion of a plurisubharmonic Liouville foliation.

Let M be a complex manifold. A collection $\{\Sigma_\alpha\}_{\alpha\in A}$ of subsets Σ_α, $\alpha\in A$, of M is called a *plurisubharmonic Liouville foliation* on M if the following four conditions are satisfied:

(S1) If $\alpha_1, \alpha_2\in A$ and $\alpha_1\neq\alpha_2$, then $\Sigma_{\alpha_1}\cap\Sigma_{\alpha_2}=\emptyset$;

(S2) $\bigcup_{\alpha\in A}\Sigma_\alpha=M$;

(S3) For each subset Σ_α, any bounded plurisubharmonic function on M takes a constant value on Σ_α;

(S4) For every $\alpha_1, \alpha_2\in A$ with $\alpha_1\neq\alpha_2$, there exists a bounded plurisubharmonic function ϕ on M such that the constant values of ϕ on Σ_{α_1} and Σ_{α_2} are different.

If there exists a plurisubharmonic Liouville foliation on M, then we say that M has a plurisubharmonic Liouville foliation. The following lemma shows that M has at most one plurisubharmonic Liouville foliation.

LEMMA 4.1. *If $\{\Sigma_\alpha\}_{\alpha\in A}$ and $\{\Sigma'_{\alpha'}\}_{\alpha'\in A'}$ are two plurisubharmonic Liouville* *foliations on a complex manifold M, then they coincide, that is, there exists a bijective correspondence* $\tau: A \to A'$ *between the index sets* A *and* A' *such that* $\Sigma_\alpha = \Sigma'_{\tau(\alpha)}$ *for every* $\alpha \in A$.

Proof. We first show that if $\Sigma_\alpha \cap \Sigma'_{\alpha'} \neq \emptyset$, say $p \in \Sigma_\alpha \cap \Sigma'_{\alpha'}$, then $\Sigma_\alpha = \Sigma'_{\alpha'}$. Suppose contrarily that $\Sigma_\alpha \neq \Sigma'_{\alpha'}$. Then there exists a point $q \in M$ such that $q \in \Sigma_\alpha \setminus \Sigma'_{\alpha'}$ or $q \in \Sigma'_{\alpha'} \setminus \Sigma_\alpha$, where the notation $\Sigma_\alpha \setminus \Sigma'_{\alpha'}$ stands for the intersection of Σ_α and the complement of $\Sigma'_{\alpha'}$ in M. We may assume without loss of generality that $q \in \Sigma'_{\alpha'} \setminus \Sigma_\alpha$. Since $p \in \Sigma_\alpha$ and $q \notin \Sigma_\alpha$, it follows from (S4) that there exists a bounded plurisubharmonic function ψ on M such that $\psi(p) \neq \psi(q)$. But, since $p \in \Sigma'_{\alpha'}$ and $q \in \Sigma'_{\alpha'}$, this contradicts (S3).

Now, it follows from (S1), (S2) and what we have shown above that, for each element $\alpha \in A$, there is a unique element $\tau(\alpha) \in A'$ with $\Sigma_\alpha = \Sigma'_{\tau(\alpha)}$. The required correspondence is given by $A \ni \alpha \mapsto \tau(\alpha) \in A'$. q.e.d.

The next proposition is useful in the investigation of the automorphisms of domains $D_{a,b}(r)$.

PROPOSITION 4.1. *If* $\varphi: M \to M'$ *is a biholomorphic mapping between two complex manifolds M and M', and if M and M' have plurisubharmonic Liouville foliations* $\{\Sigma_\alpha\}_{\alpha \in A}$ *and* $\{\Sigma'_{\alpha'}\}_{\alpha' \in A'}$, *respectively, then there exists a bijective correspondence* $\tau: A \to A'$ *between the index sets* A *and* A' *such that* $\varphi(\Sigma_\alpha) = \Sigma'_{\tau(\alpha)}$ *for every* $\alpha \in A$.

Proof. It is readily verified that $\{\varphi(\Sigma_\alpha)\}_{\alpha \in A}$ is a plurisubharmonic Liouville foliation on M'. We have only to apply Lemma 4.1 to the plurisubharmonic Liouville foliations $\{\varphi(\Sigma_\alpha)\}_{\alpha \in A}$ and $\{\Sigma'_{\alpha'}\}_{\alpha' \in A'}$ on M'. q.e.d.

Now, before discussing the automorphisms of domains $D_{a,b}(r)$, we make some preparations.

We set $D_{a,b} = D_{a,b}(1)$. As in the preceding section, in order to discuss the automorphisms and the equivalence of domains $D_{a,b}(r)$, it is sufficient to deal with the domains $D_{a,b}$. Also, if necessary, we may replace $D_{a,b}$ by $D_{\delta a, \delta b}$, where δ is a positive constant.

In a manner similar to the case of domains $D^*_{a,b}$, we classify the domains $D_{a,b}$ into the following three classes:

(I) $ab = 0$;
(II) $ab \neq 0$ and $b/a \in \boldsymbol{Q}$;
(III) $ab \neq 0$ and $b/a \notin \boldsymbol{Q}$.

A description of the automorphisms of domains $D_{a,b}$ follows from the above classification. In fact, if $D_{a,b}$ is of class (I), then it is algebraically equivalent to the domain $D_{1,0}$; if $D_{a,b}$ is of class (II), then it is algebraically equivalent to a domain $D_{p,q}$ with $(p, q) \in \boldsymbol{Z}^2$ and $(p, q) \neq (0, 0)$. Therefore, in these cases, the description of automorphisms of $D_{a,b}$ is a consequence of [3, Theorem 4.1]. To describe automorphisms of domains $D_{a,b}$ of class (III), we first prove the following lemma, which is basic in an application of the notion of a plurisubharmonic Liouville foliation to our investigation.

LEMMA 4.2. *Let c be a real constant with $c \notin \boldsymbol{Q}$ and Z be a point of $\boldsymbol{C}$. Then the image of the complex affine line*

$$L_Z = \{(\zeta, \omega) \in \boldsymbol{C}^2 \mid \zeta + c\omega = Z\} \subset \boldsymbol{C}^2 \tag{4.1}$$

under the covering projection $\varpi : \boldsymbol{C}^2 \to (\boldsymbol{C}^*)^2$ given in Section 2 is a dense subset of the set

$$\Sigma = \{(z, w) \in \boldsymbol{C}^2 \mid |z||w|^c = e^{-2\pi X}\} \subset (\boldsymbol{C}^*)^2 ,$$

where $Z = X + \sqrt{-1}Y$ $(X, Y \in \boldsymbol{R})$.

Proof. The set $\varpi(L_Z)$ is given by

$$\begin{aligned} \varpi(L_Z) &= \{(e^{-2\pi(-c\omega+Z)}, e^{-2\pi\omega}) \in \boldsymbol{C}^2 \mid \omega \in \boldsymbol{C}\} \\ &= \{(re^{2\pi c\rho}\gamma e^{\sqrt{-1}2\pi c\eta}, e^{-2\pi\rho}e^{\sqrt{-1}(-2\pi\eta)}) \in \boldsymbol{C}^2 \mid \rho, \eta \in \boldsymbol{R}\} , \end{aligned}$$

where $r = e^{-2\pi X}$ and $\gamma = e^{\sqrt{-1}(-2\pi Y)}$, while if, for each $\delta > 0$, we set

$$\Pi_\delta = \{(r\delta^{-c}\alpha, \delta\beta) \in \boldsymbol{C}^2 \mid (\alpha, \beta) \in T = (U(1))^2\} ,$$

then $\Sigma = \bigcup_{\delta>0} \Pi_\delta$. As a consequence, we have $\varpi(L_Z) \subset \Sigma$. To prove that $\varpi(L_Z)$ is dense in Σ, it is sufficient to show that, for every $\delta > 0$, the set $\varpi(L_Z) \cap \Pi_\delta$ is dense in Π_δ. For this, fix $\delta > 0$ and consider the mappings $\iota : \boldsymbol{R} \to T$, $h : T \to T$ and $g : T \to \Pi_\delta$ given by

$$\begin{aligned} &\iota(\eta) = (e^{\sqrt{-1}2\pi c\eta}, e^{\sqrt{-1}(-2\pi\eta)}) && \text{for } \eta \in \boldsymbol{R} , \\ &h(\alpha, \beta) = (\gamma\alpha, \beta) && \text{for } (\alpha, \beta) \in T , \\ &g(\alpha, \beta) = (\alpha r\delta^{-c}, \beta\delta) && \text{for } (\alpha, \beta) \in T . \end{aligned}$$

Clearly, h is a homeomorphism of T onto itself, and g is a homeomorphism of T onto Π_δ. On the other hand, it is well-known that, since $(2\pi c)/(-2\pi) = -c \notin \boldsymbol{Q}$, the set $\iota(\boldsymbol{R})$ is dense in T. Therefore we see that $(g \circ h \circ \iota)(\boldsymbol{R})$ is dense in Π_δ. Since $\varpi(L_Z) \cap \Pi_\delta = (g \circ h \circ \iota)(\boldsymbol{R})$, this proves our assertion, and the proof of the lemma is complete. q.e.d.

As a consequence of this lemma, we obtain the following result.

LEMMA 4.3. *Every domain $D_{a,b}$ of class (III) has a plurisubharmonic Liouville foliation.*

Proof. We may assume without loss of generality that $D_{a,b}=D_{1,c}$. Then $c\notin\boldsymbol{Q}$. For each $r\in I:=\{t\in\boldsymbol{R}\mid 0\leqq t<1\}$, set $\Sigma_r=\{(z, w)\in D_{1,c}\mid |z||w|^c=r\}$. Clearly, the collection $\{\Sigma_r\}_{r\in I}$ of the subsets Σ_r, $r\in I$, of $D_{1,c}$ satisfies (S1) and (S2).

To see (S3), we need Liouville's theorem of the following type:

Liouville's Theorem. If a subharmonic function defined on the whole complex plane is bounded above, then it is constant.

Let u be any bounded plurisubharmonic function on $D_{1,c}$. Since $\Sigma_0=\{(z, w)\in\boldsymbol{C}^2\mid zw=0\}$, the fact that u takes a constant value on Σ_0 is an immediate consequence of Liouville's theorem. Consider the case of the subset Σ_r for which $r\neq 0$. Then we have $\Sigma_r\subset D_{1,c}^*\subset D_{1,c}$. As we saw in the proof of Theorem 3.1, the covering tube domain of $D_{1,c}^*$ is given by T_{Ω_c}. Let $\varpi: T_{\Omega_c}\to D_{1,c}^*$ be the canonical covering projection. Now, take two points p and q of Σ_r. Then we can find a complex affine line $L_Z\subset T_{\Omega_c}$ of the form (4.1) such that $p\in\varpi(L_Z)\subset\Sigma_r$. Since the restriction to L_Z of the function $u\circ\varpi$ gives a bounded subharmonic function on the whole complex plane, it follows from Liouville's theorem that $u\circ\tilde{\omega}$ takes a constant value on L_Z, so that u takes a constant value on $\varpi(L_Z)$. Since, by Lemma 4.2, $\varpi(L_Z)$ is a dense subset of Σ_r containing p, and since u is upper semicontinuous, we see that $u(p)\leqq u(q)$. A similar argument shows that $u(q)\leqq u(p)$. Therefore we obtain $u(p)=u(q)$. This implies that u takes a constant value on Σ_r, and (S3) is verified.

It remains to see (S4). Consider the function ϕ on $D_{1,c}$ given by $\phi(z, w)=|z||w|^c$. It is readily verified that ϕ is a bounded plurisubharmonic function on $D_{1,c}$. For every $r\in I$, we have $\Sigma_r=\{(z, w)\in D_{1,c}\mid\phi(z, w)=r\}$. This implies that if $r, r'\in I$ and $r\neq r'$, then the constant values of ϕ on Σ_r and $\Sigma_{r'}$ are different, and (S4) is verified. q.e.d.

For automorphisms of domains $D_{a,b}$ of class (III), we have the following.

THEOREM 4.1. *If* $D_{a,b}$ *is of class* (III), *then* $\mathrm{Aut}(D_{a,b})=\mathrm{Aut}_{\mathrm{alg}}(D_{a,b})$. *Furthermore,* $\mathrm{Aut}_{\mathrm{alg}}(D_{a,b})$ *consists of all transformations of the form*

$$D_{a,b}\ni(z, w)\longmapsto(\delta^{-b}\alpha z, \delta^{a}\beta w)\in D_{a,b},$$

where α *and* β *are complex constants of absolute value* 1 *and* δ *is a positive constant.*

Proof. To prove the first assertion, we may assume without loss of generality that $D_{a,b}=D_{1,c}$. Let $\{\Sigma_r\}_{r\in I}$ be the plurisubharmonic Liouville foliation on $D_{1,c}$ given in Lemma 4.3. If f is an element of $\mathrm{Aut}(D_{1,c})$, then, by Proposition 4.1, there exists a bijective mapping $\tau: I\to I$ such that $f(\Sigma_r)=\Sigma_{\tau(r)}$ for $r\in I$. As a consequence, Σ_0 and $\Sigma_{\tau(0)}$ are homeomorphic. Clearly, if $r\neq 0$, then Σ_r is not homeomorphic to Σ_0. Therefore we must have $\Sigma_{\tau(0)}=\Sigma_0$, so that $f(\Sigma_0)=\Sigma_0$. Since $D_{1,c}$ is the disjoint union of $D_{1,c}^*$ and Σ_0, this implies that $f(D^*_{1,c})=D^*_{1,c}$, and hence that the restriction f^* of f to $D^*_{1,c}$ gives an automorphism of $D^*_{1,c}$. By Theorem 3.1, we have $\mathrm{Aut}(D^*_{1,c})=\mathrm{Aut}_{alg}(D^*_{1,c})$. Using this fact, we see that f^* is induced by an algebraic automorphism of $(\boldsymbol{C}^*)^2$, which shows that $f\in\mathrm{Aut}_{alg}(D_{1,c})$. Thus we obtain $\mathrm{Aut}(D_{1,c})=\mathrm{Aut}_{alg}(D_{1,c})$.

To prove the second assertion, we take an element f of $\mathrm{Aut}_{alg}(D_{a,b})$. Note that $D_{a,b}$ contains the origin and that $a\neq b$ by the assumption that $D_{a,b}$ is of class (III). Hence, using Lemma 1.1, we see that f can be written in the form

$$f: D_{a,b}\ni(z, w) \longmapsto (\gamma z, \theta w)\in D_{a,b},$$

where $(\gamma, \theta)\in(\boldsymbol{C}^*)^2$. It is readily verified that (γ, θ) satisfies $|\gamma|^a|\theta|^b=1$, and this implies the second assertion. q.e.d.

5. Proof of Main Theorems 1 and 2.

We begin with a lemma concerning a domain of class (III).

LEMMA 5.1. *If $D^*_{a,b}$ is of class (III), then any bounded holomorphic function on $D^*_{a,b}$ is constant. Consequently, if $D_{a,b}$ is of class (III), then any bounded holomorphic function on $D_{a,b}$ is constant.*

Proof. Since $D^*_{a,b}$ is an open subset of $D_{a,b}$, the second assertion is an immediate consequence of the first assertion. To prove the first assertion, we may assume without loss of generality that $D^*_{a,b}=D^*_{1,c}$. Let h be a bounded holomorphic function on $D^*_{1,c}$. Fix a constant r with $0<r<1$ and set $\Sigma=\{(z, w)\in D^*_{1,c}\,|\,|z||w|^c=r\}$. Then h takes a constant value α on Σ. Indeed, as in the proof of Lemma 4.3, consider the covering tube domain T_{Ω_c} of $D^*_{1,c}$ and let $\varpi: T_{\Omega_c}\to D^*_{1,c}$ be the canonical covering projection. If L_Z is a complex affine line in $\boldsymbol{C}^2$ given by (4.1) and if $Z=-(2\pi)^{-1}\log r\in\boldsymbol{G}$, then $L_Z\subset T_{\Omega_c}$. Since the restriction to L_Z of the function $h\circ\varpi$ gives a bounded holomorphic function on the whole complex plane, it follows from usual Liouville's theorem that $u\circ\varpi$ takes a constant value on L_Z, so that h takes a constant value on $\varpi(L_Z)$. Since, by Lemma 4.2, $\varpi(L_Z)$ is a dense subset of Σ, we see that h takes a constant value α on Σ, as desired. Now suppose that h is not constant and write $V=\{(z, w)\in D^*_{a,b}\,|\,h(z, w)-\alpha=0\}$. Then V is a proper analytic subset of $D^*_{a,b}$, and hence $D^*_{a,b}-V=\{(z, w)\in D^*_{a,b}\,|\,(z, w)\notin V\}$ is connected. But, since $D^*_{a,b}-\Sigma$ is disconnected, the relation $V\supset\Sigma$ implies that $D^*_{a,b}-V$ is disconnected. This is a contradiction, and we conclude that h is constant. q.e.d.

COROLLARY. *If $D^*_{a,b}$ is of class (I) or of class (II) and if $D^*_{u,v}$ is of class (III), then $D^*_{a,b}$ and $D^*_{u,v}$ are not holomorphically equivalent. Similarly, if $D_{a,b}$ is of class (I) or of class (II) and if $D_{u,v}$ is of class (III), then $D_{a,b}$ and $D_{u,v}$ are not holomorphically equivalent.*

Proof. By assumption, the domain $D_{a,b}^{*}$ is algebraically equivalent to a domain $D_{p,q}^{*}$ for which $(p, q)\in \boldsymbol{Z}^{2}$. Since $h(z, w)=z^{p}w^{q}$ gives a non-constant bounded holomorphic function on $D_{p,q}^{*}$, there exists a non-constant bounded holomorphic function on $D_{a,b}^{*}$. On the other hand, Lemma 5.1 asserts that any bounded holomorphic function on the domain $D_{u,v}^{*}$ of class (III) is constant. Thus $D_{a,b}^{*}$ and $D_{u,v}^{*}$ are not holomorphically equivalent. A similar argument shows that the domains $D_{a,b}$ and $D_{u,v}$ are not holomorphically equivalent.

q. e. d.

We now prove Main Theorem 1. It is sufficient to deal with the case where $D_{a,b}^{*}(r)=D_{a,b}^{*}$ and $D_{u,v}^{*}(s)=D_{u,v}^{*}$.

If $D_{a,b}^{*}$ is of class (I) or of class (II), then, by the above corollary, so is $D_{u,v}^{*}$. In this case, since $D_{a,b}^{*}$ and $D_{u,v}^{*}$ are algebraically equivalent (see [3, Proposition 3.1]), there is nothing to prove.

Suppose that $D_{a,b}^{*}$ is of class (III). Then, again by the above corollary, $D_{u,v}^{*}$ is of class (III). Let $\varphi: D_{a,b}^{*}\rightarrow D_{u,v}^{*}$ be a biholomorphic mapping between $D_{a,b}^{*}$ and $D_{u,v}^{*}$. We show that φ is induced by an algebraic automorphism of $(\boldsymbol{C}^{*})^{2}$. By Proposition 1.1, it is sufficient to show that $\varphi T(D_{a,b}^{*})\varphi^{-1}=T(D_{u,v}^{*})$.

As in Theorem 3.1, let $G(D_{a,b}^{*})$ and $G(D_{u,v}^{*})$ denote the identity components of the Lie groups $\mathrm{Aut}_{\mathrm{alg}}(D_{a,b}^{*})$ and $\mathrm{Aut}_{\mathrm{alg}}(D_{u,v}^{*})$, respectively. Then $T(D_{a,b}^{*})$ (resp. $T(D_{u,v}^{*})$) is a two-dimensional compact subgroup of $G(D_{a,b}^{*})$ (resp. $G(D_{u,v}^{*})$). On the other hand, we have $\varphi G(D_{a,b}^{*})\varphi^{-1}=G(D_{u,v}^{*})$. Indeed, it is clear that $\varphi\,\mathrm{Aut}(D_{a,b}^{*})\varphi^{-1}=\mathrm{Aut}(D_{u,v}^{*})$. Since $\mathrm{Aut}(D_{a,b}^{*})=\mathrm{Aut}_{\mathrm{alg}}(D_{a,b}^{*})$ and $\mathrm{Aut}(D_{u,v}^{*})=\mathrm{Aut}_{\mathrm{alg}}(D_{u,v}^{*})$ by the first assertion of Theorem 3.1, it follows that $\varphi\,\mathrm{Aut}_{\mathrm{alg}}(D_{a,b}^{*})\varphi^{-1}=\mathrm{Aut}_{\mathrm{alg}}(D_{u,v}^{*})$. By the definition of $G(D_{a,b}^{*})$ and $G(D_{u,v}^{*})$, we have $\varphi G(D_{a,b}^{*})\varphi^{-1}=G(D_{u,v}^{*})$.

We show that $\varphi T(D_{a,b}^{*})\varphi^{-1}=T(D_{u,v}^{*})$. It follows from the second assertion of Theorem 3.1 that $G(D_{u,v}^{*})$ is isomorphic to $T(D_{u,v}^{*})\times \boldsymbol{R}$ as a Lie group, where $\boldsymbol{R}$ is regarded as the additive group of real numbers. This implies that if there is a two-dimensional compact subgroup of $G(D_{u,v}^{*})$, then it coincides with $T(D_{u,v}^{*})$. Since $\varphi T(D_{a,b}^{*})\varphi^{-1}$ is a two-dimensional compact subgroup of $G(D_{u,v}^{*})$ by the relation $\varphi G(D_{a,b}^{*})\varphi^{-1}=G(D_{u,v}^{*})$, we see that $\varphi T(D_{a,b}^{*})\varphi^{-1}=T(D_{u,v}^{*})$, and the proof of Main Theorem 1 is completed.

To prove Main Theorem 2, it is sufficient to show that if $D_{a,b}(r)=D_{a,b}$ and $D_{u,v}(s)=D_{u,v}$, then $D_{a,b}(r)$ and $D_{u,v}(s)$ are algebraically equivalent under the identity transformation or the transformation of the form

$$\boldsymbol{C}^{2}\ni(z, w)\longmapsto(w, z)\in \boldsymbol{C}^{2}\,.$$

If $D_{a,b}$ is of class (I) or of class (II), then, by the corollary to Lemma 5.1, so is $D_{u,v}$. In this case, our assertion follows from [3, Theorem 4.2].

Suppose that $D_{a,b}$ is of class (III). Then, again by the corollary to Lemma 5.1, $D_{u,v}$ is of class (III). Let $\varphi: D_{a,b}\rightarrow D_{u,v}$ be a biholomorphic mapping between $D_{a,b}$ and $D_{u,v}$. By using Theorem 4.1 in place of Theorem 3.1, an application to the mapping φ of the same argument as in the proof of Main

Theorem 1 yields that φ is induced by an algebraic automorphism of $(\boldsymbol{C}^*)^2$. Since both $D_{a,b}$ and $D_{u,v}$ contain the origin, it follows from Lemma 1.1 that φ is given by

$$\varphi: D_{a,b} \ni (z, w) \longmapsto (\alpha z, \beta w) \in D_{u,v}$$

or

$$\varphi: D_{a,b} \ni (z, w) \longmapsto (\gamma w, \theta z) \in D_{u,v},$$

where $(\alpha, \beta) \in (\boldsymbol{C}^*)^2$ and $(\gamma, \theta) \in (\boldsymbol{C}^*)^2$. When φ is given by the former transformation, (α, β) satisfies $|\alpha|^u |\beta|^v = 1$, and

$$f: D_{u,v} \ni (z, w) \longmapsto (\alpha^{-1} z, \beta^{-1} w) \in D_{u,v}$$

is an automorphism of $D_{u,v}$. Therefore, in this case, there exists a biholomorphic mapping between $D_{a,b}$ and $D_{u,v}$ given by the identity transformation $f \circ \varphi$. On the other hand, when φ is given by the latter transformation, a similar argument shows that there exists a biholomorphic mapping between $D_{a,b}$ and $D_{u,v}$ given by the transformation of the form

$$D_{a,b} \ni (z, w) \longmapsto (w, z) \in D_{u,v}.$$

We thus conclude our assertion, and the proof of Main Theorem 2 is completed.

6. A concluding remark.

For each $t \in \boldsymbol{R}_{>0}$, we write $M_t = D_{1,t}$, where $\boldsymbol{R}_{>0}$ denotes the set of positive real numbers. The results of this paper together with that of our previous paper [3] assert that the differentiable family $\{M_t\}_{t \in \boldsymbol{R}_{>0}}$ of the complex manifolds M_t, $t \in \boldsymbol{R}_{>0}$, has the following properties:

(i) M_t and $M_{t'}$ are holomorphically equivalent precisely when $tt'=1$.

(ii) When $t \in \boldsymbol{Q}$, the group $\mathrm{Aut}(M_t)$ is infinite-dimensional, while, when $t \notin \boldsymbol{Q}$, the group $\mathrm{Aut}(M_t)$ is finite-dimensional.

Indeed, recall from the proof of Theorem 2 that if M_t and $M_{t'}$ are holomorphically equivalent, then they are algebraically equivalent under the identity transformation or the transformation φ of the form

$$\varphi: \boldsymbol{C}^2 \ni (z, w) \longmapsto (w, z) \in \boldsymbol{C}^2.$$

Since M_t and $M_{t'}$ do not coincide as sets whenever $t \neq t'$, we see that M_t and $M_{t'}$ are holomorphically equivalent precisely when $\varphi(M_t) = M_{t'}$. Since $\varphi(M_t) = D_{t,1} = D_{1,1/t} = M_{1/t}$, the property (1) follows. The property (2) is an immediate consequence of the theorem of Section 4 and [3, Theorem 4.1]. Finally, we observe that $D_{1,0}$ may be viewed as a degeneration of the family $\{M_t\}_{t \in \boldsymbol{R}_{>0}}$ as t tends to zero.

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