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KODAI MATH. J.
15 (1992), 403—429

SINGULAR VARIATION OF DOMAIN AND SPECTRA OF THE LAPLACIAN WITH SMALL ROBIN CONDITIONAL BOUNDARY II

Dedicated to Professor Takeshi Watanabe on his 60th birthday

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1. Introduction.

This paper is a continuation of previous paper [6].

Let Ω be a bounded domain in $\boldsymbol{R}^2$ with smooth boundary $\partial\Omega$. Let $\tilde{w}$ be a fixed point point in Ω. Let $B(\varepsilon, \tilde{w})$ be the disk of radius ε with the center $\tilde{w}$. We put $\Omega_\varepsilon=\Omega\setminus\overline{B(\varepsilon, \tilde{w})}$. Consider the following eigenvalue problem

$$(1.1)\qquad \begin{gathered} -\Delta u(x)=\lambda u(x) \qquad x\in\Omega_\varepsilon \\ u(x)=0 \qquad x\in\partial\Omega \\ u(x)+k\varepsilon^\sigma\frac{\partial u}{\partial\nu_x}(x)=0 \qquad x\in\partial B(\varepsilon, \tilde{w}). \end{gathered}$$

Here k denotes the positive constant. And σ is a real number. Here $\partial/\partial\nu_x$ denotes the derivative along the exterior normal direction with respect to Ω_ε.

Let $\mu_j(\varepsilon)>0$ be the j-th eigenvalue of (1.1). Let μ_j be the j-th eigenvalue of the problem

$$(1.2)\qquad \begin{gathered} -\Delta u(x)=\lambda u(x) \qquad x\in\Omega \\ u(x)=0 \qquad x\in\partial\Omega . \end{gathered}$$

Let $G(x, y)$ be the Green function of the Laplacian in Ω associated with the boundary condition (1.2).

Main aim of this paper is to show the following Theorems. Let $\varphi_j(x)$ be the L^2-normalized eigenfunction associated with μ_j. We have the following.

THEOREM 1. *Assume that μ_j is a simple eigenvalue. Then,*

$$\mu_j(\varepsilon)=\mu_j-2\pi\varphi_j(\tilde{w})^2/(\log\varepsilon)+O(|\log\varepsilon|^{-2}),$$

for $\sigma\geqq 1$.

Received March 24, 1992.

THEOREM 2. *Assume that μ_j is a simple eigenvalue. Then,*

$$\mu_j(\varepsilon)=\mu_j+Q_j\varepsilon^{1-\sigma}+R_j\varepsilon^2+O(\varepsilon^{2-\sigma}) \qquad (-1<\sigma<0)$$

$$\mu_j(\varepsilon)=\mu_j+R_j\varepsilon^2+Q_j\varepsilon^{1-\sigma}+O(\varepsilon^3|\log\varepsilon|) \qquad (-2<\sigma\leqq -1)$$

$$\mu_j(\varepsilon)=\mu_j+R_j\varepsilon^2+O(\varepsilon^3|\log\varepsilon|) \qquad (\sigma\leqq -2),$$

where

$$Q_j=(2\pi/k)\varphi_j(\tilde{w})^2$$

$$R_j=-\pi(2|\operatorname{grad}\varphi_j(\tilde{w})|^2-\mu_j\varphi_j(\tilde{w})^2).$$

Remark. The case $\sigma\in[0, 1)$ is treated in [6]. It is curious to the authors that the asymptotic behaviour of $\mu_j(\varepsilon)-\mu_j$ is the same when $\sigma\leqq -2$. For the related papers we have Ozawa [7], [8], [9], Rauch-Taylor [10], Besson [3], Chavel [4] and the references in the above papers.

For other related problems on singular variation of domains the readers may be referred to Anné [1], Arrieta, Hale and Han [2], Jimbo [5].

2. Outline of proof of Theorem 1 and Theorem 2.

We introduce the following kernel $p_\varepsilon(x, y)$.

$$p_\varepsilon(x, y)=G(x, y)+g(\varepsilon)G(x, \tilde{w})G(\tilde{w}, y) \tag{2.1}$$
$$+h(\varepsilon)\langle\nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle$$
$$+i(\varepsilon)\langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle,$$

where

$$\langle\nabla_w u(\tilde{w}), \nabla_w v(\tilde{w})\rangle=\sum_{n=1}^{2}\frac{\partial u}{\partial w_n}\frac{\partial v}{\partial w_n}\,|\,w=\tilde{w}$$

$$\langle H_w u(\tilde{w}), H_w v(\tilde{w})\rangle=\sum_{m,n=1}^{2}\frac{\partial^2 u}{\partial w_m\partial w_n}\frac{\partial^2 v}{\partial w_m\partial w_n}\,|\,w=\tilde{w}$$

when $w=(w_1, w_2)$ is an orthonormal frame of $\boldsymbol{R}^2$. Here $g(\varepsilon)$, $h(\varepsilon)$, $i(\varepsilon)$ are determined so that

$$p_\varepsilon(x, y)+k\varepsilon^\sigma\frac{\partial}{\partial\nu_x}p_\varepsilon(x, y) \qquad x\in\partial B(\varepsilon, \tilde{w}) \tag{2.2}$$

is small in some sense.

If we put

$$g(\varepsilon)=-(\gamma-(2\pi)^{-1}\log\varepsilon+k(2\pi)^{-1}\varepsilon^{\sigma-1})^{-1} \tag{2.3}$$

(2.4) $$h(\varepsilon)=(k\varepsilon^{\sigma}-\varepsilon)/((2\pi\varepsilon)^{-1}+k(2\pi)^{-1}\varepsilon^{\sigma-2}) \qquad (\sigma<0)$$
$$=0 \qquad (\sigma\geqq 1)$$

and

(2.5) $$i(\varepsilon)=k\varepsilon^{\sigma+1}/(\pi^{-1}\varepsilon^{-2}+2k\pi^{-1}\varepsilon^{\sigma-3}) \qquad (\sigma<0)$$
$$=0 \qquad (\sigma\geqq 1),$$

the above aim for (2.2) to be small is attained. Here

$$\gamma=\lim_{x\to\tilde{w}}(G(x,\ \tilde{w})+(2\pi)^{-1}\log|x-\tilde{w}|).$$

Let $G_{\varepsilon}(x, y)$ be the Green function of the Laplacian in Ω_{ε} associated with the boundary condition (1.1).

We put

$$(\boldsymbol{G}f)(x)=\int_{\Omega}G(x,\ y)f(y)dy$$

$$(\boldsymbol{G}_{\varepsilon}f)(x)=\int_{\Omega_{\varepsilon}}G_{\varepsilon}(x,\ y)f(y)dy$$

and

$$(\boldsymbol{P}_{\varepsilon}f)(x)=\int_{\Omega_{\varepsilon}}p_{\varepsilon}(x,\ y)f(y)dy \qquad (\sigma<0)$$
$$=\int_{\Omega}p_{\varepsilon}(x,\ y)f(y)dy \qquad (\sigma\geqq 1).$$

In case of $\sigma<0$, $\boldsymbol{P}_{\varepsilon}$ cannot operate on $L^{p}(\Omega)$ because of the existence of $h(\varepsilon)$-term and $i(\varepsilon)$-term in (2.1).

Let T and T_{ε} be operators on Ω and Ω_{ε}, respectively. Then, $\|T\|_{p}$, $\|T_{\varepsilon}\|_{p,\varepsilon}$ denote the operator norm on $L^{p}(\Omega)$, $L^{p}(\Omega_{\varepsilon})$, respectively. Let f and f_{ε} be functions on Ω and Ω_{ε}, respectively. Then, $\|f\|_{p}$, $\|f_{\varepsilon}\|_{p,\varepsilon}$ denotes the norm on $L^{p}(\Omega)$, $L^{p}(\Omega_{\varepsilon})$, respectively.

At first we outline the proof of Theorem 1. A crucial part of our proof of Theorem 1 is the following.

THEOREM 3. *Fix $\sigma\geqq 1$. Then, there exists a constant C such that*

(2.6) $$\|\chi_{\varepsilon}\boldsymbol{P}_{\varepsilon}\chi_{\varepsilon}-\boldsymbol{G}_{\varepsilon}\|_{2,\varepsilon}\leqq C\varepsilon|\log\varepsilon|^{-1}$$

holds. Here χ_{ε} is the characteristic function of $\bar{\Omega}_{\varepsilon}$.

Since $\boldsymbol{G}_{\varepsilon}$ is approximated by $\chi_{\varepsilon}\boldsymbol{P}_{\varepsilon}\chi_{\varepsilon}$ and the difference between $\boldsymbol{P}_{\varepsilon}$ and $\chi_{\varepsilon}\boldsymbol{P}_{\varepsilon}\chi_{\varepsilon}$ is small in some sense, we know that everything reduces to our investigation of the perturbative analysis of $\boldsymbol{G}\to\boldsymbol{P}_{\varepsilon}$. This is the outline of our proof of Theorem 1.

Next we outline the proof of Theorem 2. One important part of our proof of Theorem 2 is the following.

THEOREM 4. *Fix* $\sigma<0$. *Then, there exists a constant* C *such that*

$$\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\varphi_j)\|_{2,\varepsilon}\leqq C\varepsilon^{2-\sigma} \qquad (-1<\sigma<0)$$
$$\leqq C\varepsilon^3|\log\varepsilon| \qquad (\sigma\leqq-1) \tag{2.7}$$

holds.

We fix j *and put*

$$\begin{aligned}\bar{p}_\varepsilon(x, y)=G(x, y)&-\pi\mu_j\varepsilon^2\cdot G(x, \tilde{w})G(\tilde{w}, y)\\&+g(\varepsilon)G(x, \tilde{w})G(\tilde{w}, y)\\&+h(\varepsilon)\langle\nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\xi_\varepsilon(x)\xi_\varepsilon(y)\\&+i(\varepsilon)\langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle\xi_\varepsilon(x)\xi_\varepsilon(y)\end{aligned} \tag{2.8}$$

where $\xi_\varepsilon(x)\in C^\infty(\boldsymbol{R}^2)$ *satisfies* $|\xi_\varepsilon(x)|\leqq1$, $\xi_\varepsilon(x)=1$ *for* $x\in\boldsymbol{R}^2\setminus\overline{B(\varepsilon, \tilde{w})}$, $\xi_\varepsilon(x)=0$ *for* $x\in B(\varepsilon/2, \tilde{w})$ *and* $\xi_\varepsilon(x-\tilde{w})$ *is rotationary invariant. Furthermore we put*

$$(\bar{\boldsymbol{P}}_\varepsilon f)(x)=\int_\Omega\bar{p}_\varepsilon(x, y)f(y)dy.$$

The other important part of our proof of Theorem 2 is the following.

THEOREM 5. *Fix* $\sigma<0$. *Then, there exist a constant* C *such that*

$$\|(\chi_\varepsilon\bar{\boldsymbol{P}}_\varepsilon-\boldsymbol{P}_\varepsilon\chi_\varepsilon)\varphi_j\|_{2,\varepsilon}\leqq C\varepsilon^{2-\sigma} \qquad (-1<\sigma<0)$$
$$\leqq C\varepsilon^3|\log\varepsilon| \qquad (\sigma\leqq-1) \tag{2.9}$$

holds.

Since (2.7) and (2.9) are both $o(\varepsilon^2)$, we know that everything reduces to our investigation of the perturbative analysis of $\boldsymbol{G}\to\boldsymbol{P}_\varepsilon$. This is the outline of our proof of Theorem 2.

3. Preliminary Lemmas.

We write $B(\varepsilon, \tilde{w})=B_\varepsilon$. Next Lemma is proved in Ozawa [6].

LEMMA 3.1. *Fix* $\sigma<1$. *Assume that* $u_\varepsilon(x)\in C^\infty(\bar{\Omega}_\varepsilon)$ *satisfies*

$$\Delta u_\varepsilon(x)=0 \qquad x\in\boldsymbol{\Omega}_\varepsilon$$

$$u_\varepsilon(x)=0 \qquad x\in\partial\Omega$$

$$\mathrm{Max}\left\{\left|u_\varepsilon(x)+k\varepsilon^\sigma\frac{\partial u_\varepsilon}{\partial\nu_x}(x)\right| ;\ x\in\partial B_\varepsilon\right\}=M_\varepsilon,$$

then

$$\|u_\varepsilon\|_{p,\varepsilon}\leqq C\varepsilon^{1-\sigma}M_\varepsilon \qquad (1\leqq p<+\infty) \tag{3.1}$$

holds for a constant C independent of ε.

Remark. In Ozawa [6], $\sigma\geqq 0$ is assumed. But this assumption is not required to get the above Lemma.

Now we want to estimate $\|u_\varepsilon\|_{p,\varepsilon}$ for $\sigma\geqq 1$ under the same assumption of u_ε as above. We have the following.

LEMMA 3.2. *Fix* $M\in C^\infty(\partial B_\varepsilon)$, $\sigma\geqq 1$ *and* $q>\sigma$. *Then there exists at least one solution of*

$$\Delta v_\varepsilon(x)=0 \qquad x\in \boldsymbol{R}^2\setminus\bar{B}_\varepsilon \tag{3.2}$$

$$v_\varepsilon(x)+k\varepsilon^\sigma\frac{\partial v_\varepsilon}{\partial\nu_x}(x)=M(\theta) \qquad x=\tilde{w}+\varepsilon(\cos\theta,\ \sin\theta) \tag{3.3}$$

satisfying

$$|v_\varepsilon(x)|\leqq C\varepsilon^{1-\sigma}\,\mathrm{Max}_\theta|M(\theta)|(1+|\log r|) \qquad \text{for } r\geqq\varepsilon \tag{3.4}$$

$$|v_\varepsilon(x)|\leqq C\,\mathrm{Max}_\theta|M(\theta)|(|\log r|/|\log\varepsilon|+\varepsilon^{(1/2)(1-\sigma/q)}(r-\varepsilon)^{-1/2q'}) \tag{3.5}$$

for $r>\varepsilon$, *where* $r=|x-\tilde{w}|$ *and* q' *satisfies* $(1/q)+(1/q')=1$.

Proof. We put $x=\tilde{w}+r(\cos\theta,\ \sin\theta)$ and

$$v_\varepsilon(x)=a_0\log r+\sum_{j=1}^{\infty}(b_j\sin j\theta+c_j\cos j\theta)(-j)^{-1}r^{-j}.$$

Then it satisfies $\Delta v_\varepsilon(x)=0$ for $x\in\boldsymbol{R}^2\setminus\bar{B}_\varepsilon$. We see tnat

$$v_\varepsilon(x)+k\varepsilon^\sigma\frac{\partial v_\varepsilon}{\partial\nu_x}(x)\Big|_{x\in\partial B_\varepsilon}=s_0+\sum_{j=1}^{\infty}(s_j\sin j\theta+t_j\cos j\theta)=M(\theta)$$

implies

$$a_0(\log\varepsilon-k\varepsilon^{\sigma-1})=s_0$$

$$b_j\varepsilon^{-j}(-(1/j)-k\varepsilon^{\sigma-1})=s_j$$

$$c_j\varepsilon^{-j}(-(1/j)-k\varepsilon^{\sigma-1})=t_j$$

for $j\geqq 1$.

Thus we have

$$|v_\varepsilon(x)| \leqq |s_0 \log r|/(k\varepsilon^{\sigma-1}+|\log \varepsilon|) + \Big(\sum_{j=1}^{\infty}(s_j^2+t_j^2)\Big)^{1/2}\Big(\sum_{j=1}^{\infty}(\varepsilon/r)^{2j}(1+jk\varepsilon^{\sigma-1})^{-2}\Big)^{1/2}. \tag{3.6}$$

Using the Hölder's inequality, we have

$$\begin{aligned}\sum_{j=1}^{\infty}&(\varepsilon/r)^{2j}(1+jk\varepsilon^{\sigma-1})^{-2}\\ &\leqq\Big(\sum_{j=1}^{\infty}(\varepsilon/r)^{2jq'}\Big)^{1/q'}\Big(\sum_{j=1}^{\infty}(1+jk\varepsilon^{\sigma-1})^{-2q}\Big)^{1/q}\\ &\leqq(\varepsilon^{2q'}/(r^{2q'}-\varepsilon^{2q'}))^{1/q'}\Big(\int_0^{\infty}(1+k\varepsilon^{\sigma-1}s)^{-2q}ds\Big)^{1/q}\\ &\leqq C(\varepsilon/(r-\varepsilon))^{1/q'}\varepsilon^{-(\sigma-1)/q}\\ &=C\varepsilon^{1-\sigma/q}(r-\varepsilon)^{-1/q'} \qquad \text{for } r>\varepsilon .\end{aligned} \tag{3.7}$$

By (3.6), (3.7) and the inequaiity

$$s_0^2+\sum_{j=1}^{\infty}(s_j^2+t_j^2)\leqq C\int_0^{2\pi}|M(\theta)|^2 d\theta \leqq C'(\mathrm{Max}|M(\theta)|)^2 ,$$

we get

$$\begin{aligned}|v_\varepsilon(x)|&\leqq|s_0|\cdot|\log r|/(k\varepsilon^{\sigma-1})\\ &\quad+\Big(\sum_{j=1}^{\infty}(s_j^2+t_j^2)\Big)^{1/2}\Big(\sum_{j=1}^{\infty}j^{-2}\Big)^{1/2}k^{-1}\varepsilon^{1-\sigma}\\ &\leqq C\,\mathrm{Max}|M(\theta)|\varepsilon^{1-\sigma}(1+|\log r|) \qquad \text{for } r\geqq\varepsilon ,\end{aligned}$$

and

$$|v_\varepsilon(x)|\leqq C\,\mathrm{Max}_{\theta}|M(\theta)|((|\log r|/|\log \varepsilon|)+\varepsilon^{(1/2)(1-\sigma/q)}(r-\varepsilon)^{-1/2q'})$$

for $r>\varepsilon$. Thus the proof is now complete. q.e.d.

We have the following.

LEMMA 3.3. *Fix $\sigma\geqq 1$ and $q>\sigma$. Under the same assumptions of u_ε in Lemma* 3.1,

$$\|u_\varepsilon\|_{p,\varepsilon}\leqq CM_\varepsilon(|\log \varepsilon|^{-1}+\varepsilon^{(1/2)(1-\sigma/q)}) \qquad (1<p<2q') \tag{3.10}$$

holds for a constant C independent of ε.

Proof. By Lemma 3.2 and using the same repeating construction of the functions $v_\varepsilon^{(n)}$ in Proposition 1 of Ozawa [7], we have

$$|u_\varepsilon(x)|\leqq CM_\varepsilon(|\log r|/|\log \varepsilon|+\varepsilon^{(1/2)(1-\sigma/q)}(r-\varepsilon)^{-1/2q'}) \tag{3.11}$$

for $r>\varepsilon$.

We fix $R>0$ such that $\Omega\subset B(R, \tilde{w})$. Then, we have

$$\int_{\Omega_\varepsilon}(r-\varepsilon)^{-p/2q'}dx\leqq 2\pi\int_\varepsilon^R r(r-\varepsilon)^{-p/2q'}dr \tag{3.12}$$

$$\leqq 2\pi R\int_\varepsilon^{R+\varepsilon}(r-\varepsilon)^{-p/2q'}dr\leqq C \qquad \text{for } 1<p<2q'.$$

By (3.11) and (3.12), we get (3.10). q.e.d.

4. Proof of Theorem 3.

From this section to section 7, we assume $\sigma\geqq 1$. By (2.3) we know that

$$g(\varepsilon)=2\pi(\log\varepsilon)^{-1}+O(|\log\varepsilon|^{-2}). \tag{4.1}$$

We take an arbitrary fixed point $x\in\partial B_\varepsilon$. Without loss of generality we may assume that $\tilde{w}=(0, 0)$ and $x=(\varepsilon, 0)$.

We put

$$S(x, y)=G(x, y)+(1/2\pi)\log|x-y|.$$

Then, $S(x, y)\in C^\infty(\Omega\times\Omega)$.

We put $p_\varepsilon(x, y)$ as before. Then, we have

$$p_\varepsilon(x, y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}p_\varepsilon(x, y)\Big|_{x=(\varepsilon,0)}$$

$$=G(x, y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}G(x, y)-g(\varepsilon)k\varepsilon^\sigma\frac{\partial}{\partial x_1}S(x, \tilde{w})G(\tilde{w}, y)$$

$$+g(\varepsilon)(-(2\pi)^{-1}\log\varepsilon+S(x, \tilde{w})+k(2\pi)^{-1}\varepsilon^{\sigma-1})G(\tilde{w}, y).$$

Let $\gamma=S(\tilde{w}, \tilde{w})$. Then, $S(x, \tilde{w})=\gamma+O(\varepsilon)$ as $\varepsilon\to 0$. Since

$$g(\varepsilon)(-(2\pi)^{-1}\log\varepsilon+\gamma+k(2\pi)^{-1}\varepsilon^{\sigma-1})=-1,$$

we get the following.

$$p_\varepsilon(x, y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}p_\varepsilon(x, y)\Big|_{x=(\varepsilon,0)} \tag{4.2}$$

$$=G(x, y)-G(\tilde{w}, y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}G(x, y)$$

$$+g(\varepsilon)\Big(O(\varepsilon)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}S(x, \tilde{w})\Big)G(\tilde{w}, y).$$

We take an arbitrary $f\in L^p(\Omega_\varepsilon)$ and put $\tilde{f}=\chi_\varepsilon f$. From (4.2), we get

$$(4.3)\qquad (\boldsymbol{P}_\varepsilon\tilde{f})(x)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon\tilde{f})(x)\Big|_{x=(\varepsilon,0)}$$

$$=(\boldsymbol{G}\tilde{f})(x)-(\boldsymbol{G}\tilde{f})(\tilde{w})-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})(x)$$

$$+g(\varepsilon)\Big(O(\varepsilon)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}S(x,\,w)\Big)(\boldsymbol{G}\tilde{f})(\tilde{w}).$$

By the Sobolev embedding theorem

$$\|\boldsymbol{G}\tilde{f}\|_{C^{1+\tau}(\Omega)}\leqq C\|\tilde{f}\|_p=C\|f\|_{p,\varepsilon}$$

if $\tau=1-2/p$, $2<p<\infty$. Therefore we get

$$(4.4)\qquad |(\boldsymbol{G}\tilde{f})(x)-(\boldsymbol{G}\tilde{f})(\tilde{w})|\leqq C\varepsilon\|f\|_{p,\varepsilon}$$

$$|(\boldsymbol{G}\tilde{f})(\tilde{w})|\leqq C\|f\|_{p,\varepsilon}$$

$$\Big|\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})(x)\Big|\leqq C\|f\|_{p,\varepsilon}$$

for $p>2$, $x=(\varepsilon,\,0)$ and $\tilde{w}=(0,\,0)$.

From (4.1), (4.3) and (4.4) we have the following.

$$\Big|(\boldsymbol{P}_\varepsilon\tilde{f})(x)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon\tilde{f})(x)\Big|_{x=(\varepsilon,0)}\Big|$$

$$\leqq C(\varepsilon+\varepsilon^\sigma+|g(\varepsilon)|(\varepsilon+\varepsilon^\sigma))\|f\|_{p,\varepsilon}$$

$$\leqq C\varepsilon\|f\|_{p,\varepsilon}\,.$$

We put $(\chi_\varepsilon\boldsymbol{P}_\varepsilon\chi_\varepsilon-\boldsymbol{G}_\varepsilon)f=v$. Then, $v=\chi_\varepsilon\boldsymbol{P}_\varepsilon\tilde{f}-\boldsymbol{G}_\varepsilon f$ and v satisfies the assumptions in Lemma 3.3 with $M_\varepsilon=C\varepsilon\|f\|_{p,\varepsilon}$, because $\boldsymbol{G}_\varepsilon f$ satisfies the given Robin condition on ∂B_ε. By Lemma 3.3 we have

$$\|v\|_{p,\varepsilon}\leqq C(|\log\varepsilon|^{-1}+\varepsilon^{(1/2)(1-\sigma/q)})\varepsilon\|f\|_{p,\varepsilon}$$

$$\leqq C\varepsilon|\log\varepsilon|^{-1}\|f\|_{p,\varepsilon}$$

for $p>2$ and $q>\sigma$. Therefore,

$$\|\chi_\varepsilon\boldsymbol{P}_\varepsilon\chi_\varepsilon-\boldsymbol{G}_\varepsilon\|_{p,\varepsilon}\leqq C\varepsilon|\log\varepsilon|^{-1}$$

for $p>2$.

By the duality argument

$$\|\chi_\varepsilon\boldsymbol{P}_\varepsilon\chi_\varepsilon-\boldsymbol{G}_\varepsilon\|_{p',\varepsilon}\leqq C\varepsilon|\log\varepsilon|^{-1}$$

for p' satisfying $(1/p)+(1/p')=1$. Now by the Riesz-Thorin interpolation theorem we get Theorem 3.

5. Convergence of eigenvalues for $\sigma \geqq 1$.

At first we want to estimate $\|\boldsymbol{P}_\varepsilon-\boldsymbol{G}\|_2$. We take an arbitrary $v \in L^2(\Omega)$. Then, by the definition and the Sobolev embedding theorem we have

$$(5.1) \qquad (\boldsymbol{P}_\varepsilon v)(x)=(\boldsymbol{G}v)(x)+g(\varepsilon)G(x, \tilde{w})(\boldsymbol{G}v)(\tilde{w})$$

$$(5.2) \qquad \|\boldsymbol{G}v\|_\infty \leqq C\|v\|_2 .$$

Thus,

$$\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G})v\|_2 \leqq C|g(\varepsilon)|\,\|G(\cdot, \tilde{w})\|_2\|v\|_2$$
$$\leqq C|g(\varepsilon)|\,\|v\|_2 \leqq C|\log \varepsilon|^{-1}\|v\|_2 .$$

Therefore we get the following.

LEMMA 5.1. *There exists a constants C independent of* ε *such that*

$$(5.3) \qquad \|\boldsymbol{P}_\varepsilon-\boldsymbol{G}\|_2 \leqq C|\log \varepsilon|^{-1}$$

holds.

Next we want to estimate $\|\boldsymbol{P}_\varepsilon-\chi_\varepsilon \boldsymbol{P}_\varepsilon \chi_\varepsilon\|_2$. *Since*

$$\boldsymbol{P}_\varepsilon-\chi_\varepsilon \boldsymbol{P}_\varepsilon \chi_\varepsilon=(1-\chi_\varepsilon)\boldsymbol{P}_\varepsilon \chi_\varepsilon+\boldsymbol{P}_\varepsilon(1-\chi_\varepsilon),$$

we have

$$(5.4) \qquad \|\boldsymbol{P}_\varepsilon-\chi_\varepsilon \boldsymbol{P}_\varepsilon \chi_\varepsilon\|_2 \leqq \|(1-\chi_\varepsilon)\boldsymbol{P}_\varepsilon \chi_\varepsilon\|_2+\|\boldsymbol{P}_\varepsilon(1-\chi_\varepsilon)\|_2 .$$

By (5.1) *and* (5.2) *we have*

$$\|(1-\chi_\varepsilon)(\boldsymbol{P}_\varepsilon v)\|_2 \leqq \|(1-\chi_\varepsilon)(\boldsymbol{G}v)\|_2+|g(\varepsilon)|\,\|(1-\chi_\varepsilon)G(\cdot, \tilde{w})(\boldsymbol{G}v)(\tilde{w})\|_2$$
$$\leqq C|B_\varepsilon|^{1/2}\|v\|_2+C|g(\varepsilon)|\Big(\int_{B_\varepsilon}|G(x, \tilde{w})|^2 dx\Big)^{1/2}\|v\|_2$$
$$\leqq C(\varepsilon+|g(\varepsilon)|\varepsilon|\log \varepsilon|)\|v\|_2$$
$$\leqq C\varepsilon\|v\|_2 .$$

Therefore we get

$$(5.5) \qquad \|(1-\chi_\varepsilon)\boldsymbol{P}_\varepsilon\|_2 \leqq C\varepsilon$$
$$\|(1-\chi_\varepsilon)\boldsymbol{P}_\varepsilon \chi_\varepsilon\|_2 \leqq C\varepsilon$$

Since we have the duality

$$((1-\chi_\varepsilon)\boldsymbol{P}_\varepsilon)^*=\boldsymbol{P}_\varepsilon(1-\chi_\varepsilon),$$

we get

$$(5.6) \qquad \|\boldsymbol{P}_\varepsilon(1-\chi_\varepsilon)\|_2 \leqq C\varepsilon .$$

By (5.4), (5.5), (5.6) *we get the following.*

LEMMA 5.2. *There exists a constant C independent of ε such that*

$$\|\boldsymbol{P}_\varepsilon-\chi_\varepsilon \boldsymbol{P}_\varepsilon \chi_\varepsilon\|_2 \leqq C\varepsilon$$

holds.

By virtue of Theorem 3, *Lemma* 5.1, *Lemma* 5.2, *we see that there exists a constant C independent of j such that*

$$(5.7) \qquad |\mu_j(\varepsilon)^{-1}-\mu_j^{-1}| \leqq C(\varepsilon|\log\varepsilon|^{-1}+|\log\varepsilon|^{-1}+\varepsilon) \leqq C|\log\varepsilon|^{-1}$$

holds.

We need more precise estimate for the left hand side of (5.7) to get Theorem 1. By (5.7) we know that the multiplicity of $\mu_j(\varepsilon)$ is one for small ε when the multiplicity of μ_j is one.

6. Perturbational Calculus for $\boldsymbol{P}_\varepsilon$.

In this section we consider the behaviour of eigenvalues of $\boldsymbol{P}_\varepsilon$ as ε tends to 0.

We put $A_0=G$ and

$$(A_1 f)(x)=G(x, \tilde{w})(\boldsymbol{G}f)(\tilde{w}).$$

Then,

$$\boldsymbol{P}_\varepsilon=A_0+g(\varepsilon)A_1.$$

It is easy to see

$$\|A_1\|_p \leqq C \qquad (1<p<\infty).$$

Furthermore we put

$$\lambda(\varepsilon)=\lambda_0+g(\varepsilon)\lambda_1$$

$$\phi(\varepsilon)=\phi_0+g(\varepsilon)\phi_1$$

so that $\lambda(\varepsilon)$ and $\phi(\varepsilon)$ is an approximate eigenvalue of $\boldsymbol{P}_\varepsilon$ and an approximate eigenfunction of $\boldsymbol{P}_\varepsilon$, respectively.

As the standard techniques of perturbation theory, we solve the following equations.

Let λ_0 be a simple eigenvalue of A_0. At first

$$(6.1) \qquad (A_0-\lambda_0)\phi_0=0, \qquad \|\phi_0\|_2=1.$$

Next we solve the following equations;

$$(6.2) \qquad (A_0-\lambda_0)\phi_1=(\lambda_1-A_1)\phi_0$$

(6.3) $$(\psi_0, \psi_1)_2=0\,,$$

where $(\ ,\ )_2$ denotes the inner product on $L^2(\Omega)$.

By the Fredholm alternative theory, we see that

(6.4) $$\lambda_1=(A_1\psi_0, \psi_0)_2$$

is the condition such that the unique solution ψ_1 of (6.2), (6.3) exists.

Hereafter we put $\lambda_0=\mu_j^{-1}$. Then $\psi_0=\varphi_j$. We see that

(6.5) $$\lambda_1=|(\boldsymbol{G}\psi_0)(\tilde{w})|^2=\mu_j^{-2}\varphi_j(\tilde{w})^2$$

(6.6) $$(\boldsymbol{P}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)=g(\varepsilon)^2(A_1-\lambda_1)\psi_1\,.$$

By the Fredholm theory, we see that

(6.7) $$\|\psi_1\|_2\leqq C\|\lambda_1-A_1\|_2\|\psi_0\|_2\leqq C\,.$$

By (6.6), (6.7), we have

$$\begin{aligned}\|(\boldsymbol{P}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)\|_2&\leqq|g(\varepsilon)|^2\|A_1-\lambda_1\|_2\|\psi_1\|_2\\&\leqq C|g(\varepsilon)|^2\leqq C|\log\varepsilon|^{-2}\,.\end{aligned}$$

Therefore, we get the following.

LEMMA 6.1. *There exists a constant C independent of ε such that*

(6.9) $$\|(\boldsymbol{P}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)\|_2\leqq C|\log\varepsilon|^{-2}$$

holds.

Next we want to estimate $\|(\boldsymbol{P}_\varepsilon-\lambda(\varepsilon))(1-\chi_\varepsilon)\psi(\varepsilon)\|_{2,\varepsilon}$. We put $\hat{\chi}_\varepsilon=1-\chi_\varepsilon$. Then, we have

(6.10) $$(\boldsymbol{P}_\varepsilon-\lambda(\varepsilon))\hat{\chi}_\varepsilon\psi(\varepsilon)=\sum_{h=1}^{4}T_h\,,$$

where

$$\begin{aligned}T_1&=\boldsymbol{G}\hat{\chi}_\varepsilon\psi_0\\T_2&=g(\varepsilon)\boldsymbol{G}\hat{\chi}_\varepsilon\psi_1\\T_3&=g(\varepsilon)A_1\hat{\chi}_\varepsilon\psi_0\\T_4&=g(\varepsilon)^2A_1\hat{\chi}_\varepsilon\psi_1\end{aligned}$$

on Ω_ε, since $\lambda(\varepsilon)\hat{\chi}_\varepsilon\psi(\varepsilon)=0$ on Ω_ε.

We get

(6.11) $$\|T_1\|_{2,\varepsilon}\leqq\|T_1\|_\infty\leqq C\cdot\|\hat{\chi}_\varepsilon\varphi_j\|_2\leqq C\varepsilon\,.$$

Also,

$$\|T_2\|_{2,\varepsilon} \leqq C|g(\varepsilon)| \cdot \|\hat{\chi}_\varepsilon \phi_1\|_2 .$$

Notice that

$$\phi_1 = (-\lambda_0)^{-1}((\lambda_1 - A_1)\phi_0 - A_0\phi_1) .$$

Then,

$$\begin{aligned} \|\hat{\chi}_\varepsilon \phi_1\|_2 &\leqq C(\|\hat{\chi}_\varepsilon \phi_0\|_2 + \|\hat{\chi}_\varepsilon A_1 \phi_0\|_2 + \|\hat{\chi}_\varepsilon A_0 \phi_1\|_2) \\ &\leqq C\Big(\|\hat{\chi}_\varepsilon\|_2 + \Big(\int_{B_\varepsilon} |G(x, \tilde{w})|^2 dx\Big)^{1/2} + \|\hat{\chi}_\varepsilon\|_2\Big) \\ &\leqq G(\varepsilon + \varepsilon|\log \varepsilon| + \varepsilon) \leqq C\varepsilon|\log \varepsilon| . \end{aligned}$$

Therefore, we get

$$\|T_2\|_{2,\varepsilon} \leqq C|g(\varepsilon)|\varepsilon|\log \varepsilon| \leqq C\varepsilon . \tag{6.12}$$

Furthermore, we have

$$\begin{aligned} \|T_3 + T_4\|_{2,\varepsilon} &\leqq |g(\varepsilon)| \|A_1 \hat{\chi}_\varepsilon \phi_0\|_2 + |g(\varepsilon)|^2 \|A_1 \hat{\chi}_\varepsilon \phi_1\|_2 \\ &\leqq C(|g(\varepsilon)| \|\hat{\chi}_\varepsilon\|_2 + |g(\varepsilon)|^2) \\ &\leqq C(\varepsilon|\log \varepsilon|^{-1} + |\log \varepsilon|^{-2}) \\ &\leqq C|\log \varepsilon|^{-2} . \end{aligned} \tag{6.13}$$

Summing up (6.10), (6.11), (6.12) *and* (6.13), *we have the following inequality.*

$$\|(6.10)\|_{2,\varepsilon} \leqq C(\varepsilon + \varepsilon + |\log \varepsilon|^{-2}) \leqq C|\log \varepsilon|^{-2} .$$

Therefore, we get the following.

LEMMA 6.2. *There exists a constant C independent of ε such that*

$$\|(\boldsymbol{P}_\varepsilon - \lambda(\varepsilon))(1 - \chi_\varepsilon)\phi(\varepsilon)\|_{2,\varepsilon} \leqq C|\log \varepsilon|^{-2}$$

holds.

7. Proof of Theorem 1.

Now we are in a position to prove Theorem 1. By Theorem 3, Lemma 6.1 and 6.2, we have

$$\begin{aligned} \|(\boldsymbol{G}_\varepsilon - \lambda(\varepsilon))(\chi_\varepsilon \phi(\varepsilon))\|_{2,\varepsilon} &\leqq \|\boldsymbol{G}_\varepsilon - \chi_\varepsilon \boldsymbol{P}_\varepsilon \chi_\varepsilon\|_{2,\varepsilon} \|\phi(\varepsilon)\|_{2,\varepsilon} + \|(\boldsymbol{P}_\varepsilon - \lambda(\varepsilon))\phi(\varepsilon)\|_{2,\varepsilon} \\ &\quad + \|(\boldsymbol{P}_\varepsilon - \lambda(\varepsilon))(1 - \chi_\varepsilon)\phi(\varepsilon)\|_{2,\varepsilon} \\ &\leqq C(\varepsilon|\log \varepsilon|^{-1} \|\phi(\varepsilon)\|_{2,\varepsilon} + |\log \varepsilon|^{-2} + |\log \varepsilon|^{-2}) \\ &\leqq C|\log \varepsilon|^{-2} . \end{aligned}$$

Here we used the fact that $\|\psi(\varepsilon)\|_{2,\varepsilon}\in(1/2, 2)$ for small ε. Therefore, there exists at least one eigenvalue $\lambda^*(\varepsilon)$ of $\boldsymbol{G}_\varepsilon$ satisfying

$$|\lambda^*(\varepsilon)-\lambda(\varepsilon)|\leqq C|\log\varepsilon|^{-2}. \tag{7.1}$$

We here represent $\lambda(\varepsilon)$ explicitly as follows:

$$\begin{aligned}\lambda(\varepsilon)&=\mu_j^{-1}+g(\varepsilon)\mu_j^{-2}\varphi_j(\tilde{w})^2\\&=\mu_j^{-1}+2\pi\mu_j^{-2}\varphi_j(\tilde{w})^2(\log\varepsilon)^{-1}+O(|\log\varepsilon|^{-2}).\end{aligned}\tag{7.2}$$

By (7.1), (7.2) and the fact (5.7), we see that $\lambda^*(\varepsilon)$ must be $\mu_j(\varepsilon)^{-1}$. Then, we get

$$|\mu_j(\varepsilon)^{-1}-(\mu_j^{-1}+2\pi\mu_j^{-2}\varphi_j(\tilde{w})^2(\log\varepsilon)^{-1})|\leqq C|\log\varepsilon|^{-2}.$$

Therefore, we get the desired Theorem 1.

8. Proof of Theorem 4.

From this section we assume $\sigma<0$. By (2.3), (2.4) and (2.5), we see that

$$\begin{aligned}g(\varepsilon)&=-(2\pi/k)\varepsilon^{1-\sigma}+O(\varepsilon^{2-2\sigma}|\log\varepsilon|)\\h(\varepsilon)&=2\pi\varepsilon^2+O(\varepsilon^{3-\sigma})\\i(\varepsilon)&=(\pi/2)\varepsilon^4+O(\varepsilon^{5-\sigma}).\end{aligned}\tag{8.1}$$

At first we want to estimate $\|\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon\|_{2,\varepsilon}$. We take an arbitrary fixed point $x\in\partial B_\varepsilon$. Without loss of generality we may assume that $\tilde{w}=(0, 0)$ and $x=(\varepsilon, 0)$.

We put $S(x, y)$ as before. Then, we have the following formulas (8.2), (8.3) in p. 263 and (8.4) in p. 264 of Ozawa [7], respectively.

$$\begin{aligned}&\langle\nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\\&=(2\pi\varepsilon)^{-1}\frac{\partial}{\partial w_1}G(\tilde{w}, y)+\langle\nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\end{aligned}\tag{8.2}$$

for $x=(\varepsilon, 0)$, $\tilde{w}=(0, 0)$.

$$\begin{aligned}&\frac{\partial}{\partial x_1}\langle\nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\\&=-(2\pi)^{-1}\varepsilon^{-2}\frac{\partial}{\partial w_1}G(\tilde{w}, y)+\frac{\partial}{\partial x_1}\langle\nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\end{aligned}\tag{8.3}$$

for $x=(\varepsilon, 0)$, $\tilde{w}=(0, 0)$.

$$(8.4)\qquad \frac{\partial}{\partial x_1}\langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle$$

$$=-2\pi^{-1}\varepsilon^{-3}\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y)+\frac{\partial}{\partial x_1}\langle H_w S(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle$$

for $x=(\varepsilon, 0)$, $\tilde{w}=(0, 0)$.

The same calculation yields

$$(8.5)\qquad \langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle$$

$$=\pi^{-1}\varepsilon^{-2}\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y)+\langle H_w S(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle$$

for $x=(\varepsilon, 0)$, $\tilde{w}=(0, 0)$.

We put $p_\varepsilon(x, y)$ as before. By (8.2), (8.3), (8.4) and (8.5), we have

$$(8.6)\qquad p_\varepsilon(x, y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}p_\varepsilon(x, y)\,|_{x=(\varepsilon,0)}=\sum_{j=1}^{7}L_j\,,$$

where

$$L_1=G(x, y)$$

$$L_2=g(\varepsilon)(-(2\pi)^{-1}\log\varepsilon+\gamma+(2\pi)^{-1}k\varepsilon^{\sigma-1})G(\tilde{w}, y)$$

$$L_3=g(\varepsilon)\Big(O(\varepsilon)-k\varepsilon^{\sigma-1}\frac{\partial}{\partial x_1}S(x, \tilde{w})\Big)G(\tilde{w}, y)$$

$$L_4=(2\pi)^{-1}(\varepsilon^{-1}+k\varepsilon^{\sigma-1})h(\varepsilon)\frac{\partial}{\partial w_1}G(\tilde{w}, y)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}G(x, y)$$

$$L_5=\pi^{-1}(\varepsilon^{-2}+2k\varepsilon^{\sigma-3})i(\varepsilon)\frac{\partial^2}{\partial w_1^2}G(\tilde{w}, y)$$

$$L_6=h(\varepsilon)\langle\nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle$$

$$-k\varepsilon^\sigma h(\varepsilon)\frac{\partial}{\partial x_1}\langle\nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle$$

$$L_7=i(\varepsilon)\langle H_w S(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle$$

$$-k\varepsilon^\sigma i(\varepsilon)\frac{\partial}{\partial x_1}\langle H_w S(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle$$

for $x=(\varepsilon, 0)$, $\tilde{w}=(0, 0)$.

Here we used the fact that

$$S(x, \tilde{w})=\gamma+O(\varepsilon)\qquad\text{as } \varepsilon\to 0\,.$$

By (2.3), (2.4), (2.5) and (8.6), we get the following.

(8.7)
$$p_\varepsilon(x, y)-k\varepsilon^\sigma \frac{\partial}{\partial x_1} p_\varepsilon(x, y)|_{x=(\varepsilon, 0)}$$
$$=G(x, y)-G(\tilde{w}, y)-\varepsilon \frac{\partial}{\partial w_1} G(\tilde{w}, y)$$
$$+g(\varepsilon)\Big(O(\varepsilon)-k\varepsilon^\sigma \frac{\partial}{\partial x_1} S(x, \tilde{w})\Big)G(\tilde{w}, y)$$
$$-k\varepsilon^\sigma\Big(\frac{\partial}{\partial x_1} G(x, y)-\frac{\partial}{\partial w_1} G(\tilde{w}, y)-\varepsilon\frac{\partial^2}{\partial w_1{}^2} G(\tilde{w}, y)\Big)$$
$$+L_6+L_7\,.$$

We take an arbitrary $\tilde{f}\in L^p(\Omega)$ which is zero on B_ε. By (8.7), we have

(8.8)
$$\boldsymbol{P}_\varepsilon\tilde{f}(x)-k\varepsilon^\sigma \frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon\tilde{f})(x)|_{x=(\varepsilon, 0)}$$
$$=(\boldsymbol{G}\tilde{f})(x)-(\boldsymbol{G}\tilde{f})(\tilde{w})-\varepsilon \frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})$$
$$+g(\varepsilon)\Big(O(\varepsilon)-k\varepsilon^\sigma \frac{\partial}{\partial x_1} S(x, \tilde{w})\Big)(\boldsymbol{G}\tilde{f})(\tilde{w})$$
$$-k\varepsilon^\sigma\Big(\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})(x)-\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})-\varepsilon\frac{\partial^2}{\partial w_1{}^2}(\boldsymbol{G}\tilde{f})(\tilde{w})\Big)$$
$$+h(\varepsilon)\langle\nabla_w S(x, \tilde{w}), \nabla_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle$$
$$-k\varepsilon^\sigma h(\varepsilon)\frac{\partial}{\partial x_1}\langle\nabla_w S(x, \tilde{w}), \nabla_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle$$
$$+i(\varepsilon)\langle H_w S(x, \tilde{w}), H_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle$$
$$-k\varepsilon^\sigma i(\varepsilon)\frac{\partial}{\partial x_1}\langle H_w S(x, \tilde{w}), H_w(\boldsymbol{G}\tilde{f})(\tilde{w})\rangle\,.$$

We want to estimate (8.8). By the Sobolev embedding theorem,
$$\|G\tilde{f}\|_{C^{1+\tau}(\Omega)}\leqq C\|\tilde{f}\|_{p,\varepsilon}$$
for $p>2$, $\tau=1-2/p$.

Therefore, we have

(8.9)
$$|(\boldsymbol{G}\tilde{f})(\tilde{w})|\leqq C\|\tilde{f}\|_{p,\varepsilon}$$
$$\Big|(\boldsymbol{G}\tilde{f})(x)-(\boldsymbol{G}\tilde{f})(\tilde{w})-\varepsilon\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})\Big|\leqq C\varepsilon^{1+\tau}\|\tilde{f}\|_{p,\varepsilon}$$
$$\Big|\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})(x)-\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})(\tilde{w})\Big|\leqq C\varepsilon^{\tau}\|\tilde{f}\|_{p,\varepsilon}$$

for $p>2$, $x=(\varepsilon, 0)$.

Furthermore,

$$\left|\frac{\partial}{\partial w_n}(G\tilde{f})(\tilde{w})\right| \leqq C\Big(\int_{\Omega_\varepsilon} |y-\tilde{w}|^{-p'} dy\Big)^{1/p'} \|\tilde{f}\|_{p,\varepsilon} \tag{8.10}$$

$$\leqq C|\log \varepsilon|^{1/2}\|\tilde{f}\|_{2,\varepsilon} \qquad (p=2)$$

$$\leqq C\|\tilde{f}\|_{p,\varepsilon} \qquad (p>2)$$

for $n=1, 2$, where p' satisfies $(1/p)+(1/p')=1$. Also,

$$\left|\frac{\partial^2}{\partial w_m \partial w_n}(G\tilde{f})(\tilde{w})\right| \leqq C\Big(\int_{\Omega_\varepsilon} |y-\tilde{w}|^{-2p'} dy\Big)^{1/p'} \|\tilde{f}\|_{p,\varepsilon} \tag{8.11}$$

$$\leqq C\varepsilon^{-2/p}\|\tilde{f}\|_{p,\varepsilon} \qquad (p>1)$$

for $1\leqq m$, $n\leqq 2$.

Summing up (8.8), (8.9), (8.10) and (8.11), we get

$$\left|(\boldsymbol{P}_\varepsilon\tilde{f})(x)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon\tilde{f})(x)|_{x=(\varepsilon,0)}\right|$$

$$\leqq C(\varepsilon^{1+\tau}+\varepsilon^{1-\sigma+\sigma}+\varepsilon^\sigma(\varepsilon^\tau+\varepsilon^{1-2/p})+\varepsilon^{\sigma+2}+\varepsilon^{4+\sigma-2/p})\|\tilde{f}\|_{p,\varepsilon}$$

$$\leqq C\varepsilon^{\sigma+1-2/p}\|\tilde{f}\|_{p,\varepsilon}$$

for $p>2$.

We put $(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)\tilde{f}=v$. Then, v satisfies the assumption in Lemma 3.1 with $M_\varepsilon=C\varepsilon^{\sigma+1-2/p}\|\tilde{f}\|_{p,\varepsilon}$, because $\boldsymbol{G}_\varepsilon\tilde{f}$ satisfies the given Robin condition on ∂B_ε. By Lemma 3.1, we have

$$\|v\|_{p,\varepsilon}\leqq C\varepsilon^{1-\sigma}\varepsilon^{1+\sigma-2/p}\|\tilde{f}\|_{p,\varepsilon}\leqq C\varepsilon^{2-2/p}\|\tilde{f}\|_{p,\varepsilon}$$

for $p>2$. Therefore,

$$\|\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon\|_{p,\varepsilon}\leqq C\varepsilon^{2-2/p} \qquad (p>2).$$

By the duality argument and the Riesz-Thorin interpolation theorem, we get

$$\|\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon\|_{2,\varepsilon}\leqq C\varepsilon^{2-2/p} \qquad (p>2).$$

We take an arbitrary $\beta\in(0, 1)$ and put $p=2/(1-\beta)$. Then, we have the following.

PROPOSITION 8.1. *Fix $\beta\in(0, 1)$. Then, there exists a constant C independent of ε such that*

$$\|\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon\|_{2,\varepsilon}\leqq C\varepsilon^{1+\beta}$$

holds.

Next we estimate $\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\varphi_j)\|_{2,\varepsilon}$. We put $(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\varphi_j)=v_\varepsilon$. As we get (8.8), we have

$$v_\varepsilon(x)-k\varepsilon^\sigma \frac{\partial v_\varepsilon}{\partial x_1}(x)|_{x=(\varepsilon,0)}=I_0(\varepsilon)-k\varepsilon^\sigma(I_1(\varepsilon)-I_2(\varepsilon))+I_3(\varepsilon), \tag{8.12}$$

where

$$I_0(\varepsilon)=(G\chi_\varepsilon\varphi_j)(x)-(G\chi_\varepsilon\varphi_j)(\tilde{w})-\varepsilon\frac{\partial}{\partial w_1}(G\chi_\varepsilon\varphi_j)(\tilde{w})$$

$$I_1(\varepsilon)=\frac{\partial}{\partial x_1}(G\varphi_j)(x)-\Big(\frac{\partial}{\partial w_1}+\varepsilon\frac{\partial^2}{\partial w_1^2}\Big)(G\varphi_j)(\tilde{w})$$

$$I_2(\varepsilon)=\frac{\partial}{\partial x_1}(G\hat{\chi}_\varepsilon\varphi_j)(x)-\Big(\frac{\partial}{\partial w_1}+\varepsilon\frac{\partial}{\partial w_1^2}\Big)(G\hat{\chi}_\varepsilon\varphi_j)(\tilde{w})$$

$$\begin{aligned}I_3(\varepsilon)=&g(\varepsilon)\Big(O(\varepsilon)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}S(x,\tilde{w})\Big)(G\chi_\varepsilon\varphi_j)(\tilde{w})\\&+h(\varepsilon)\langle\nabla_w S(x,\tilde{w}),\nabla_w(G\chi_\varepsilon\varphi_j)(\tilde{w})\rangle\\&-k\varepsilon^\sigma h(\varepsilon)\frac{\partial}{\partial x_1}\langle\nabla_w S(x,\tilde{w}),\nabla_w(G\chi_\varepsilon\varphi_j)(\tilde{w})\rangle\\&+i(\varepsilon)\langle H_w S(x,\tilde{w}),H_w(G\chi_\varepsilon\varphi_j)(\tilde{w})\rangle\\&-k\varepsilon^\sigma i(\varepsilon)\frac{\partial}{\partial x_1}\langle H_w S(x,\tilde{w}),H_w(G\chi_\varepsilon\varphi_j)(\tilde{w})\rangle\\&\qquad\text{for } x=(\varepsilon,0),\ \tilde{w}=(0,0).\end{aligned}$$

Here we put $\hat{\chi}_\varepsilon=1-\chi_\varepsilon$. Using (8.9), (8.10), (8.11) with $\tilde{f}=\chi_\varepsilon\varphi_j$, we have

$$|I_0(\varepsilon)|\leqq C\varepsilon\|\varphi_j\|_{p,\varepsilon}\leqq C\varepsilon \tag{8.13}$$

$$\begin{aligned}|I_3(\varepsilon)|&\leqq C(|g(\varepsilon)|\varepsilon^\sigma+|h(\varepsilon)|\varepsilon^\sigma+|i(\varepsilon)|\varepsilon^\sigma\varepsilon^{-2/p})\|\varphi_j\|_{p,\varepsilon}\\&\leqq C(\varepsilon+\varepsilon^{2+\sigma}+\varepsilon^{4+\sigma-2/p})\\&\leqq C(\varepsilon+\varepsilon^{2+\sigma})\qquad\text{for } p>2.\end{aligned} \tag{8.14}$$

Since $G\varphi_j(x)=\mu_j^{-1}\varphi_j(x)$, we have

$$|I_1(\varepsilon)|\leqq C\varepsilon^2. \tag{8.15}$$

Furthermore, we have the following estimation (8.16) in p. 267 of Ozawa [7].

$$|I_2(\varepsilon)|\leqq C\varepsilon^2|\log\varepsilon|. \tag{8.16}$$

Summing up (8.12), (8.13), (8.14), (8.15) and (8.16), we have

$$\begin{aligned}\Big|v_\varepsilon(x)-k\varepsilon^\sigma\frac{\partial v_\varepsilon}{\partial x_1}(x)|_{x=(\varepsilon,0)}\Big|&\leqq C(\varepsilon+\varepsilon^\sigma(\varepsilon^2+\varepsilon^2|\log\varepsilon|)+\varepsilon+\varepsilon^{2+\sigma})\\&\leqq C(\varepsilon+\varepsilon^{2+\sigma}|\log\varepsilon|).\end{aligned}$$

By Lemma 3.1, we have

$$\|v_\varepsilon\|_{2,\varepsilon} \leqq C\varepsilon^{1-\sigma}(\varepsilon+\varepsilon^{2+\sigma}|\log\varepsilon|)=C\cdot H(\varepsilon).$$

Here,

$$H(\varepsilon)=\varepsilon^{2-\sigma}+\varepsilon^3|\log\varepsilon| \tag{8.17}$$

$$\leqq C\varepsilon^{2-\sigma} \qquad (-1<\sigma<0)$$

$$\leqq C\varepsilon^3|\log\varepsilon| \qquad (\sigma\leqq -1).$$

Therefore, we get Theorem 4.

9. Convergence of eigenvalues for $\sigma<0$.

We introduce the following kernel $\tilde{p}_\varepsilon(x, y)$.

$$\tilde{p}_\varepsilon(x, y)=G(x, y)+g(\varepsilon)G(x, \tilde{w})G(\tilde{w}, y) \tag{9.1}$$

$$+h(\varepsilon)\langle\nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\chi_\varepsilon(x)\chi_\varepsilon(y)$$

$$+i(\varepsilon)\langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle\chi_\varepsilon(x)\chi_\varepsilon(y)$$

And we put

$$(\tilde{\boldsymbol{P}}_\varepsilon f)(x)=\int_\Omega \tilde{p}_\varepsilon(x, y)f(y)dy.$$

Notice that $(1-\chi_\varepsilon)\chi_\varepsilon=0$ in $h(\varepsilon)$-term and $i(\varepsilon)$-term of (9.1). Therefore, as we get Lemma 5.1, we get the following.

LEMMA 9.1. *There exists a constant C independent of ε such that*

$$\|\boldsymbol{P}_\varepsilon-\chi_\varepsilon\tilde{\boldsymbol{P}}_\varepsilon\chi_\varepsilon\|_2 \leqq C(\varepsilon+|g(\varepsilon)|\varepsilon|\log\varepsilon|) \tag{9.2}$$

$$\leqq C\varepsilon.$$

holds.

Next we want to estimate $\|\tilde{\boldsymbol{P}}_\varepsilon-\boldsymbol{G}\|_2$. We take an arbitrary $v\in L^p(\Omega)$. Then, we see that

$$((\tilde{\boldsymbol{P}}_\varepsilon-\boldsymbol{G})v)(x)=g(\varepsilon)G(x, \tilde{w})(\boldsymbol{G}v)(\tilde{w})$$

$$+h(\varepsilon)\langle\nabla_w G(x, \tilde{w}), \nabla_w(\boldsymbol{G}\chi_\varepsilon v)(\tilde{w})\rangle\chi_\varepsilon(x)$$

$$+i(\varepsilon)\langle H_w G(x, \tilde{w}), H_w(\boldsymbol{G}\chi_\varepsilon v)(\tilde{w})\rangle\chi_\varepsilon(y).$$

Therefore,

$$(9.3)\qquad \|(\tilde{\boldsymbol{P}}_\varepsilon-\boldsymbol{G})v\|_p$$

$$\leqq |g(\varepsilon)|\,\|G(\cdot,\, w)\|_p\|\boldsymbol{G}v\|_\infty$$

$$+|h(\varepsilon)|\sum_{n=1}^{2}\Big(\int_{\Omega_\varepsilon}\Big|\frac{\partial}{\partial w_n}G(x,\,\tilde{w})\Big|^p dx\Big)^{1/p}\Big|\frac{\partial}{\partial w_n}(\boldsymbol{G}\chi_\varepsilon v)(\tilde{w})\Big|$$

$$+|i(\varepsilon)|\sum_{m,n=1}^{2}\Big(\int_{\Omega_\varepsilon}\Big|\frac{\partial^2}{\partial w_m\partial w_n}G(x,\,\tilde{w})\Big|^p dx\Big)^{1/p}\Big|\frac{\partial^2}{\partial w_m\partial w_n}(\boldsymbol{G}\chi_\varepsilon v)(\tilde{w})\Big|$$

holds for $p<1$.

We have

$$(9.4)\qquad \|\boldsymbol{G}v\|_\infty\leqq C\|v\|_p \qquad (p>1),$$

$$(9.5)\qquad \Big(\int_{\Omega_\varepsilon}\Big|\frac{\partial}{\partial w_n}G(x,\,\tilde{w})\Big|^p dx\Big)^{1/p}\leqq C\Big(\int_{\Omega_i}|x-\tilde{w}|^{-p}dx\Big)^{1/p}$$

$$\leqq C|\log\varepsilon|^{1/2}\qquad (p=2)$$

$$\leqq C\varepsilon^{2/p-1}\qquad (p>2),$$

for $n=1,\, 2$, and

$$(9.6)\qquad \Big(\int_{\Omega_\varepsilon}\Big|\frac{\partial^2}{\partial w_m\partial w_n}G(x,\,\tilde{w})\Big|^p dx\Big)^{1/p}\leqq C\Big(\int_{\Omega_\varepsilon}|x-\tilde{w}|^{-2p}dx\Big)^{1/p}$$

$$\leqq C\varepsilon^{2/p-2}\qquad (p>1)$$

for $1\leqq m,\, n\leqq 2$.

By (9.3), (9.4), (9.5), (9.6) and using the estimation (8.10), (8.11) with $\tilde{f}=\chi_\varepsilon v$, we see that

$$\|(\tilde{\boldsymbol{P}}_\varepsilon-\boldsymbol{G})v\|_2\leqq C(|g(\varepsilon)|\,\|v\|_2+|h(\varepsilon)|\,|\log\varepsilon|^{1/2}|\log\varepsilon|^{1/2}\|v\|_{2,\varepsilon}$$

$$+|i(\varepsilon)|\varepsilon^{-1}\varepsilon^{-1}\|v\|_{2,\varepsilon}).$$

$$\leqq C(\varepsilon^{1-\sigma}+\varepsilon^2|\log\varepsilon|+\varepsilon^2)\|v\|_2$$

$$\leqq C(\varepsilon^{1-\sigma}+\varepsilon^2|\log\varepsilon|)\|v\|_2$$

holds for an arbitrary $v\in L^2(\Omega)$. Therefore, we get the following.

LEMMA 9.2. *There exists a constant C independent of ε such that*

$$\|\boldsymbol{P}_\varepsilon-\boldsymbol{G}\|_2\leqq C(\varepsilon^{1-\sigma}+\varepsilon^2|\log\varepsilon|)$$

holds.

Notice that the j-th eigenvalue of $\boldsymbol{P}_\varepsilon$ is equal to the j-th eigenvalue of $\chi_\varepsilon\tilde{\boldsymbol{P}}_\varepsilon\chi_\varepsilon$. We fix $\beta\in(0,\,1)$. Then, by virtue of Proposition 8.1, *Lemma* 9.1 *and* 9.2, *we see that there exists a constant C independent of j such that*

$$(9.7)\qquad |\mu_j(\varepsilon)^{-1}-\mu_j^{-1}|\leqq C(\varepsilon^{1+\beta}+\varepsilon+\varepsilon^{1-\sigma}+\varepsilon^2|\log\varepsilon|)\leqq C\varepsilon$$

holds.

We need more precise estimate estimate for the left hand side of (9.7) to get Theorem 2. By (9.7), we know that the multiplicity of $\mu_j(\varepsilon)$ is one for small ε when the multiplicity of μ_j is one.

10. Perturbational Calculus for P_ε.

In this section we consider the behaviour of eigenvalues of $\boldsymbol{P}_\varepsilon$ as ε tends to 0. We put A_0, A_1 as before. And we put

$$(A_2 f)(x)=\langle\nabla_w G(x, \tilde{w}), \nabla_w(G\xi_\varepsilon f)(\tilde{w})\rangle\xi_\varepsilon(x)$$

$$(A_3 f)(x)=\langle H_w G(x, \tilde{w}), H_w(G\xi_\varepsilon f)(\tilde{w})\rangle\xi_\varepsilon(x).$$

Then,

$$\overline{\boldsymbol{P}}_\varepsilon=A_0+\bar{g}(\varepsilon)A_1+h(\varepsilon)A_2+i(\varepsilon)A_3.$$

where

$$\bar{g}(\varepsilon)=g(\varepsilon)-\pi\mu_j\varepsilon^2. \tag{10.1}$$

Furthermore, we put

$$\lambda(\varepsilon)=\lambda_0+\bar{g}(\varepsilon)\lambda_1+h(\varepsilon)\lambda_2+i(\varepsilon)\lambda_3$$

$$\psi(\varepsilon)=\psi_0+\bar{g}(\varepsilon)\psi_1+h(\varepsilon)\psi_2+i(\varepsilon)\psi_3$$

so that $\lambda(\varepsilon)$ and $\psi(\varepsilon)$ is an approximate eigenvalue of $\overline{\boldsymbol{P}}_\varepsilon$ and an approximate eigenfunction of $\overline{\boldsymbol{P}}_\varepsilon$, respectively.

Let λ_0 be a simple eigenvalue of A_0. At first we set

$$(A_0-\lambda_0)\psi_0=0, \qquad \|\psi_0\|_2=1 \tag{10.2}$$

Next we solve the following equations:

$$(A_0-\lambda_0)\psi_1=(\lambda_1-A_1)\psi_0, \qquad (\psi_0, \psi_1)_2=0 \tag{10.3}$$

$$(A_0-\lambda_0)\psi_2=(\lambda_2-A_2)\psi_0, \qquad (\psi_0, \psi_2)_2=0 \tag{10.4}$$

$$(A_0-\lambda_0)\psi_3=(\lambda_3-A_3)\psi_0, \qquad (\psi_0, \psi_3)_2=0 \tag{10.5}$$

where $(\ ,\ )_2$ denotes the inner product on $L^2(\Omega)$. By the Fredholm alternative theory we see that

$$\lambda_n=(A_n\psi_0, \psi_0)_2 \qquad (n=1, 2, 3) \tag{10.6}$$

is the condition such that the unique solution ψ_1, ψ_2, ψ_3 of (10.3), (10.4), (10.5) exists, respectively.

Hereafter we put $\lambda_0=\mu_j^{-1}$. Then $\psi_0=\varphi_j$.

We have the following:

LEMMA 10.1. *For a constant C independent of* ε,

$$\|A_1\|_p \leqq C \qquad (p>1)$$

$$\|A_2\|_p \leqq C|\log \varepsilon| \qquad (p=2)$$

$$\leqq C\xi^{2/p-1} \qquad (p>2)$$

$$\|A_3\|_p \leqq C\varepsilon^{-2} \qquad (p>1)$$

hold.

Proof. The same estimate as (9.4), (9.5) and (9.6) yields

$$\|A_1 f\|_p \leqq C\|f\|_p \qquad (p>1)$$

$$\|A_2 f\|_p \leqq \sum_{n=1}^{2}\left(\int_{\Omega\setminus B_{\varepsilon/2}}\left|\frac{\partial}{\partial w_n}G(x, \tilde{w})\right|^p dx\right)^{1/p}\left|\frac{\partial}{\partial w_n}(G\xi_\varepsilon f)(\tilde{w})\right|$$

$$\leqq C|\log \varepsilon|\,\|f\|_2 \qquad (p=2)$$

$$\leqq C\xi^{2/p-1}\|f\|_p \qquad (p>2)$$

$$\|A_3 f\|_p \leqq \sum_{m,n=1}^{2}\left(\int_{\Omega\setminus B_{\varepsilon/2}}\left|\frac{\partial^2}{\partial w_m \partial w_n}G(x, \tilde{w})\right|^p dx\right)^{1/p}\left|\frac{\partial^2}{\partial w_m \partial w_n}(G\xi f)(\tilde{w})\right|$$

$$\leqq C\varepsilon^{-2}\|f\|_p \qquad (p>1),$$

because $\xi_\varepsilon(x)=0$ for $x \in B_{\varepsilon/2}$. Therefore, we get the desired result. q.e.d..

By (10.6) we see that

$$|\lambda_n| \leqq |(A_n\phi_0, \phi_0)_2| \leqq C\|A_n\|p \qquad (n=1, 2, 3) \tag{10.6}$$

for $p>1$.
Then, by the Fredholm theory and the estimate of the $L^p(\Omega)$ norm of the right hand side of (10.3), (10.4) and (10.5), we get the following·

LEMMA 10.2. *For a constant C independent of* ε,

$$\|\phi_1\|_p \leqq C \qquad (p>1)$$

$$\|\phi_2\|_p \leqq C|\log \varepsilon| \qquad (p=2)$$

$$\leqq C\varepsilon^{2/p-1} \qquad (p>2)$$

$$\|\phi_3\|_p \leqq C\varepsilon^{-2} \qquad (p>1)$$

hold.

In view of (10.2), (10.3), (10.4) and (10.5), we have

$$(\overline{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)=\bar{g}(\varepsilon)^2(A_1-\lambda_1)\psi_1+h(\varepsilon)^2(A_2-\lambda_2)\psi_2+i(\varepsilon)^2(A_3-\lambda_3)\psi_3$$
$$+\bar{g}(\varepsilon)h(\varepsilon)((A_1-\lambda_1)\psi_2+(A_2-\lambda_2)\psi_1)$$
$$+h(\varepsilon)i(\varepsilon)((A_2-\lambda_2)\psi_3+(A_3-\lambda_3)\psi_2)$$
$$+i(\varepsilon)\bar{g}(\varepsilon)((A_3-\lambda_3)\psi_1+(A_1-\lambda_1)\psi_3). \tag{10.8}$$

By (10.7), (10.8), Lemmas 10.1 and 10.2, we see that

$$\|(\overline{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)\|_2\leqq C(\bar{g}(\varepsilon)^2+\varepsilon^4|\log\varepsilon|^2+|\bar{g}(\varepsilon)|\varepsilon^2|\log\varepsilon|)$$
$$\leqq C(|\bar{g}(\varepsilon)|+\varepsilon^2|\log\varepsilon|)^2. \tag{10.9}$$

By (10.1) we have

$$(|\bar{g}(\varepsilon)|+\varepsilon^2|\log\varepsilon|)^2$$
$$\leqq C(\varepsilon^{1-\sigma}+\varepsilon^2|\log\varepsilon|)^2\leqq C(\varepsilon^{2-\sigma}+\varepsilon^3|\log\varepsilon|)=C\cdot H(\varepsilon).$$

Therefore, we get the following.

PROPOSITION 10.3. *There exists a constant C independent of ε such that*

$$\|(\overline{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\psi(\varepsilon)\|_2\leqq C\cdot H(\varepsilon) \tag{10.10}$$

holds.

Furthermore we want to estimate $\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}$. We fix $\beta\in(0, 1)$. Then, by Proposition 8.1, Lemma 10.2, Theorem 4 and (10.1), we have

$$\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_2$$
$$\leqq\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\varphi_j)\|_2,$$
$$+\|\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon\|_{2,\varepsilon}(|\bar{g}(\varepsilon)|\,\|\psi_1\|_2+|h(\varepsilon)|\,\|\psi_2\|_2+|i(\varepsilon)|\,\|\psi_3\|_2)$$
$$\leqq C(H(\varepsilon)+\varepsilon^{1+\beta}(\varepsilon^{1-\sigma}+\varepsilon^2|\log\varepsilon|))$$
$$=C(1+\varepsilon^\beta)H(\varepsilon)\leqq C\cdot H(\varepsilon).$$

Therefore, we get the following.

PROPOSITION 10.4. *There exists a constant C independent of ε such that*

$$\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}\leqq C\cdot H(\varepsilon)$$

holds.

11. Proof of Theorem 5.

We put

(11.1) $$J_{\varepsilon}(x\,;v)=(\chi_{\varepsilon}\bar{P}_{\varepsilon}v-P_{\varepsilon}\chi_{\varepsilon}v)(x) \qquad \text{for } v\in L^{p}(\Omega).$$

Then, we see that

(11.2) $$\begin{aligned}&\Delta J_{\varepsilon}(x\,;v)=0 \qquad x\in\Omega_{\varepsilon}\\ &J_{\varepsilon}(x\,;v)=0 \qquad x\in\partial\Omega.\end{aligned}$$

As we get (8.8), we have

(11.3) $$\begin{aligned}&J_{\varepsilon}(x\,;v)-k\varepsilon^{\sigma}\frac{\partial}{\partial x_{1}}J_{\varepsilon}(x\,;v)|_{x=(\varepsilon,0)}\\ &=\sum_{n=4}^{6}I_{n}(\varepsilon\,;v)+\sum_{n=8}^{9}I_{n}(\varepsilon\,;v)-k\varepsilon^{\sigma}(I_{7}(\varepsilon\,;v)+I_{10}(\varepsilon\,;v))\end{aligned}$$

where

$$\begin{aligned}
&I_{4}(\varepsilon\,;v)=(G\hat{\chi}_{\varepsilon}v)(x)-(G\hat{\chi}_{\varepsilon}v)(\tilde{w})-\varepsilon\frac{\partial}{\partial w_{1}}(G\xi_{\varepsilon}\hat{\chi}_{\varepsilon}v)(\tilde{w})\\
&I_{5}(\varepsilon\,;v)=g(\varepsilon)\Big(O(\varepsilon)-k\varepsilon^{\sigma}\frac{\partial}{\partial x_{1}}S(x,\tilde{w})\Big)(G\hat{\chi}_{\varepsilon}v)(\tilde{w})\\
&I_{6}(\varepsilon\,;v)=-\pi\mu_{j}\varepsilon^{2}G(x,\tilde{w})(Gv)(\tilde{w})\\
&I_{7}(\varepsilon\,;v)=\frac{\partial}{\partial x_{1}}(G\hat{\chi}_{\varepsilon}v)(x)-\Big(\frac{\partial}{\partial w_{1}}+\varepsilon\frac{\partial^{2}}{\partial w_{1}^{2}}\Big)(G\xi_{\varepsilon}\hat{\chi}_{\varepsilon}v)(\tilde{w})\\
&I_{8}(\varepsilon\,;v)=h(\varepsilon)\langle\nabla_{w}S(x,\tilde{w}),\nabla_{w}(G\xi_{\varepsilon}\hat{\chi}_{\varepsilon}v)(\tilde{w})\rangle\\
&\qquad\qquad -k\varepsilon^{\sigma}h(\varepsilon)\frac{\partial}{\partial x_{1}}\langle\nabla_{w}S(x,w),\nabla_{w}(G\xi_{\varepsilon}\hat{\chi}_{\varepsilon}v)(\tilde{w})\rangle\\
&I_{9}(\varepsilon\,;v)=i(\varepsilon)\langle H_{w}S(x,\tilde{w}),H_{w}(G\xi_{\varepsilon}\hat{\chi}_{\varepsilon}v)(\tilde{w})\rangle\\
&\qquad\qquad -k\varepsilon^{\sigma}i(\varepsilon)\frac{\partial}{\partial x_{1}}\langle H_{w}S(x,\tilde{w}),H_{w}(G\xi_{\varepsilon}\hat{\chi}_{\varepsilon}v)(\tilde{w})\rangle\\
&I_{10}(\varepsilon\,;v)=-\pi\mu_{j}\varepsilon^{2}\frac{\partial}{\partial x_{1}}G(x,\tilde{w})(Gv)(\tilde{w})
\end{aligned}$$

for $x=(\varepsilon\,;0)$, $\tilde{w}=(0,0)$.

By the Sobolev embedding theorem, we have

(11.4) $$\begin{aligned}|I_{4}(\varepsilon\,;v)|&\leqq C\varepsilon\|\hat{\chi}_{\varepsilon}v\|_{p}+C\varepsilon\|\xi_{\varepsilon}\hat{\chi}_{\varepsilon}v\|_{p}\\ &\leqq C\varepsilon\|v\|_{p} \qquad (p>2).\end{aligned}$$

Also,

(11.5) $$\begin{aligned}|I_{5}(\varepsilon\,;v)|&\leqq C|g(\varepsilon)|\varepsilon^{\sigma}\Big(\int_{B_{\varepsilon}}|\log|y-w||^{p'}dy\Big)^{1/p'}\|v\|_{p}\\ &\leqq C\varepsilon^{3-2/p}|\log\varepsilon|\,\|v\|_{p} \qquad (p>1)\end{aligned}$$

$$|I_6(\varepsilon\,;v)|\leqq C\varepsilon^2|\log\varepsilon|\,\|v\|_p\qquad(p>1)\tag{11.6}$$

$$|I_{10}(\varepsilon\,;v)|\leqq C\varepsilon\|v\|_p\qquad(p>1)\tag{11.7}$$

$$\begin{aligned}|I_8(\varepsilon\,;v)|&\leqq C|h(\varepsilon)|\varepsilon^\sigma\Big(\int_{B_\varepsilon\setminus B_{\varepsilon/2}}|y-w|^{-p'}dy\Big)^{1/p'}\|v\|_p\\&\leqq C\varepsilon^{3+\sigma-2/p}\|v\|_p\qquad(p>2)\end{aligned}\tag{11.8}$$

$$\begin{aligned}|I_9(\varepsilon\,;v)|&\leqq C|i(\varepsilon)|\varepsilon^\sigma\Big(\int_{B_\varepsilon\setminus B_{\varepsilon/2}}|y-w|^{-2p'}dy\Big)^{1/p'}\|v\|_p\\&\leqq C\varepsilon^{4+\sigma-2/p}\|v\|_p\qquad(p>1),\end{aligned}\tag{11.9}$$

where p' satisfies $(1/p)+(1/p')=1$.

Since $B(\varepsilon,w)\subset B(2\varepsilon,x)$ for $x=(\varepsilon,0)$ and $\tilde{w}=(0,0)$,

$$\begin{aligned}|I_7(\varepsilon\,;v)|\leqq{}&C\Big(\int_{B(2\varepsilon,x)}|x-y|^{-p'}dy\Big)^{1/p'}\|v\|_p\\&+C\Big(\int_{B_\varepsilon\setminus B_{\varepsilon/2}}|y-w|^{-p'}dy\Big)^{1/p'}\|v\|_p\\&+C\varepsilon\Big(\int_{B_\varepsilon\setminus B_{\varepsilon/2}}|y-w|^{-2p'}dy\Big)^{1/p'}\|v\|_p\\\leqq{}&C\varepsilon^{1-2/p}\|v\|_p\qquad(p>2).\end{aligned}\tag{11.10}$$

Summing up these facts, we have

$$\begin{aligned}&\Big|J_\varepsilon(x\,;v)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}J_\varepsilon(x\,;v)|_{x=(\varepsilon,0)}\Big|\\&\qquad\leqq C\varepsilon^{1+\sigma-2/p}\|v\|_p\qquad(p>2).\end{aligned}\tag{11.11}$$

By (11.2), (11.11) and Lemma 3.1, we have

$$\|J_\varepsilon(\cdot\,;v)\|_{2,\varepsilon}\leqq C\varepsilon^{2-2/p}\|v\|_p\qquad(p>2).$$

Therefore we get the following.

LEMMA 11.1. *There exists a constant C independent of ε such that*

$$\|J_\varepsilon(\cdot\,;v)\|_{2,\varepsilon}\leqq C\varepsilon^{2-2/p}\|v\|_p\tag{11.12}$$

holds for any $v\in L^p(\Omega)$ $(p>2)$.

By the way, we have the following formula (11.13) in p. 271 of Ozawa [7].

$$I_7(\varepsilon\,;\varphi_j)=-(\varepsilon/2)\varphi_j(w)+O(\varepsilon^2|\log\varepsilon|)\tag{11.13}$$

It is easy to see

(11.14) $$I_{10}(\varepsilon\,;\varphi_j)=(\varepsilon/2)\varphi_j(\tilde{w})+O(\varepsilon^2).$$

Thus, we have

(11.15) $$|I_7(\varepsilon\,;\varphi_j)+I_{10}(\varepsilon\,;\varphi_j)|\leqq C\varepsilon^2|\log\varepsilon|.$$

Summing up (11.3), (11.4), (11.5), (11.6), (11.8), (11.9) and (11.15), we have

(11.16) $$\left|J_\varepsilon(x\,;\varphi_j)-k\varepsilon^\sigma\frac{\partial}{\partial x_1}J_\varepsilon(x\,;\varphi_j)|_{x=(\varepsilon,0)}\right|$$
$$\leqq C(\varepsilon+\varepsilon^{2+\sigma}|\log\varepsilon|).$$

By (11.16) and Lemma 3.1, we have

(11.17) $$\|J_\varepsilon(\cdot\,;\varphi_j)\|_{2,\varepsilon}\leqq C\varepsilon^{1-\sigma}(\varepsilon+\varepsilon^{2+\sigma}|\log\varepsilon|)=C\cdot H(\varepsilon).$$

Therefore we get Theorem 5.

Furthermore we want to estimate $\|J_\varepsilon(\cdot\,;\psi(\varepsilon))\|_{2,\varepsilon}$. By (11.17), Lemmas 10.2 and 11.1, we have

$$\begin{aligned}\|J_\varepsilon(\cdot\,;\psi(\varepsilon))\|_{2,\varepsilon}&\leqq\|J_\varepsilon(\cdot\,;\varphi_j)\|_{2,\varepsilon}+|\bar{g}(\varepsilon)|\,\|J_\varepsilon(\cdot\,;\psi_1)\|_{2,\varepsilon}\\&\quad+|h(\varepsilon)\|J_\varepsilon(\cdot\,;\psi_2)\|_{2,\varepsilon}+|i(\varepsilon)|\,\|J_\varepsilon(\cdot\,;\psi_3)\|_{2,\varepsilon}\\&\leqq C(\varepsilon^{2-\sigma}+\varepsilon^3|\log\varepsilon|+\varepsilon^{3-\sigma-2/p}+\varepsilon^3+\varepsilon^{4-2/p})\\&\leqq C(\varepsilon^{2-\sigma}+\varepsilon^3|\log\varepsilon|)=C\cdot H(\varepsilon)\qquad\text{for } p>2.\end{aligned}$$

Therefore we get the following.

PROPOSITION 11.2. *There exists a constant C independent of ε such that*

$$\|(\boldsymbol{P}_\varepsilon\chi_\varepsilon-\chi_\varepsilon\overline{\boldsymbol{P}}_\varepsilon)\psi(\varepsilon)\|_{2,\varepsilon}\leqq C\cdot H(\varepsilon)$$

holds.

12. Proof of Theorem 2.

Now we are in a position to prove Theorem 2. By Propositions 10.3, 10.4 and 11.2, we have

$$\|(\boldsymbol{G}_\varepsilon-\lambda(\varepsilon))(\chi_\varepsilon\psi(\varepsilon))\|_{2,\varepsilon}\leqq C\cdot H(\varepsilon).$$

Notice that $\|\psi(\varepsilon)\|_{2,\varepsilon}\in(1/2,\,2)$ for small ε.

Therefore, there exists at least one eigenvalue $\lambda^*(\varepsilon)$ of $\boldsymbol{G}_\varepsilon$ satisfying

(12.1) $$|\lambda^*(\varepsilon)-\lambda(\varepsilon)|\leqq C\cdot H(\varepsilon).$$

We here represent λ_1, λ_2, λ_3 as follows:

(12.2) $$\lambda_1=|(G\phi_0(\tilde{w})|^2=\mu_j^{-2}\varphi_j(\tilde{w})^2$$

(12.3) $$\lambda_2=\langle\nabla_w(G\xi_\varepsilon\phi_0)(\tilde{w}),\ \nabla_w(G\xi_\varepsilon\phi_0)(\tilde{w})\rangle$$
$$=\sum_{n=1}^{2}\left(\frac{\partial}{\partial w_n}\int_\Omega G(w,\ y)\xi_\varepsilon(y)\varphi_j(y)dy\right)^2|_{w=\tilde{w}}$$

(12.4) $$\lambda_3=\langle H_w(G\xi_\varepsilon\phi_0)(\tilde{w}),\ H_w(G\xi_\varepsilon\phi_0)(\tilde{w})\rangle$$
$$=\sum_{m,n=1}^{2}\left(\frac{\partial^2}{\partial w_m\partial w_n}\int_\Omega G(w,\ y)\xi_\varepsilon(y)\varphi_j(y)dy\right)^2|_{w=\tilde{w}}$$

We see that

$$\left|\frac{\partial^2}{\partial w_m\partial w_n}\int_\Omega G(w,\ y)\xi_\varepsilon(y)\varphi_j(y)dy\,|_{w=\tilde{w}}\right|$$
$$\leqq C\int_{\Omega\setminus B_{\varepsilon/2}}|y-\tilde{w}|^{-2}dy\leqq C|\log\varepsilon| \qquad (1\leqq m,\ n\leqq 2).$$

Thus, we have

(12.5) $$\lambda_3=O(|\log\varepsilon|^2).$$

Also,

(12.6) $$\frac{\partial}{\partial w_n}\int_\Omega G(w,\ y)\xi_\varepsilon(y)\varphi_j(y)dy\,|_{w=\tilde{w}}$$
$$=\mu_j^{-1}\frac{\partial}{\partial w_n}\varphi_j(\tilde{w})+I_{11}^{(n)}(\varepsilon)+I_{12}^{(n)}(\varepsilon),$$

where

$$I_{11}^{(n)}(\varepsilon)=-\frac{\partial}{\partial w_n}\int_\Omega S(w,\ y)(1-\xi_\varepsilon(y))\varphi_j(y)dy\,|_{w=\tilde{w}}$$
$$I_{12}^{(n)}(\varepsilon)=-\frac{\partial}{\partial w_n}\int_\Omega L(w,\ y)(1-\xi_\varepsilon(y))\varphi_j(y)dy\,|_{w=\tilde{w}}$$
for $n=1,\ 2$.

Here, we put

$$L(w,\ y)=G(w,\ y)-S(w,\ y)=-(2\pi)^{-1}\log|w-y|.$$

We see that

(12.7) $$|I_{11}^{(n)}(\varepsilon)|\leqq C\int_{B_\varepsilon}1dy\leqq C'\varepsilon^2 \qquad (n=1,\ 2).$$

Furthermore, we have the following formula (12.8) in p. 271 of Ozawa [7].

(12.8) $$|I_{12}^{(n)}(\varepsilon)|\leqq C\varepsilon^2|\log\varepsilon| \qquad (n=1,\ 2).$$

Summing up (12.3), (12.6), (12.7) and (12.8), we have

(12.9) $$\lambda_2=\mu_j^{-2}|\mathrm{grad}\,\varphi_j(\tilde{w})|^2+O(\varepsilon^2|\log\varepsilon|).$$

By (12.2), (12.5) and (12.9), we see that

$$(12.10)\qquad \lambda(\varepsilon)=\mu_j^{-1}+\bar{g}(\varepsilon)\lambda_1+h(\varepsilon)\lambda_2+i(\varepsilon)\lambda_3$$
$$=\mu_j^{-1}-\mu_j^{-2}Q_j\varepsilon^{1-\sigma}-\mu_j^{-2}R_j\varepsilon^2+O(\varepsilon^4|\log\varepsilon|^2)+O(\varepsilon^{2-2\sigma}|\log\varepsilon|),$$

where Q_j and R_j are as mentioned before.

By (12.1), (12.10) and the fact (9.7), we see that $\lambda^*(\varepsilon)$ must be $\mu_j(\varepsilon)^{-1}$. Then. we have

$$|\mu_j(\varepsilon)^{-1}-(\mu_j^{-1}-\mu_j^{-2}Q_j\varepsilon^{1-\sigma}-\mu_j^{-2}R_j\varepsilon^2)|$$
$$\leqq C\cdot H(\varepsilon)+C(\varepsilon^4|\log\varepsilon|^2+\varepsilon^{2-2\sigma}|\log\varepsilon|)$$
$$=C(\varepsilon^{2-\sigma}+\varepsilon^3|\log\varepsilon|+\varepsilon^4|\log\varepsilon|^2+\varepsilon^{2-2\sigma}|\log\varepsilon|)$$
$$\leqq C(\varepsilon^{2-\sigma}+\varepsilon^3|\log\varepsilon|).$$

Therefore, we get the desired Theorem 2.

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