# ON THE NUMBER OF BRANCHES OF BIFURCATION POINTS

By Marek Izydorek And Sławomir Rybicki

## Introduction.

Let $\mathcal{O}$ be an open subset of $\boldsymbol{R}^n \times \boldsymbol{R}^k$. Consider a continuous map $f: \mathcal{O} \to \boldsymbol{R}^n$ satisfying $f(0, \lambda)=0$ for all

$$(0, \lambda) \in Q := \mathcal{O} \cap (\{0\} \times \boldsymbol{R}^k).$$

$\mathcal{O}$ is called the set of trivial zeroes of $f$. The main question in bifurcation theory concerns the existence of nontrivial solutions of the equation

(1) $$f(x, \lambda)=0$$

i. e. we ask if the set $\mathcal{Z} := \{(x, \lambda) \in \mathcal{O} - Q : f(x, \lambda)=0\}$ is nonempty.

The answer for this question one usually gets by searching connected sets of zeroes of $f$ bifurcating from $Q$ or more precisely by looking for the set of bifurcation points $\mathcal{B}(f) := cl(\mathcal{Z}) \cap Q$.

In multiparameter bifurcation problems ($k > 1$) the following questions can be raised:

1) What assumptions one should put on $f$ in order to get bifurcation for the equation (1)?

2) What is the local (and global) structure of the set of bifurcation points $\mathcal{B}(f)$?

Many authors have considered these questions (see e. g. [A], [A. A.], [A. F.], [B1], [B2], [I. M. P. V.]).

In 1956 M. A. Krasnosielski proved in [K] his famous bifurcation theorem (for one dimensional parameter space) which gives sufficient conditions to the existence of bifurcation points for the equation (1), in terms of the Brouwer topological degree.

This theorem has many generalizations proceeding in various directions.

Replacing an Euclidean space $\boldsymbol{R}^n$ by any Banach space $X$, continuous map $f$ by a compact perturbation of identity and using the Leray-Schauder degree instead of the Brouwer degree we get an infinite dimensional version of Krasnosielski theorem see [MA], [R1], [R2].

On the other hand multiparameter theorems in finite and infinite case have been proved in [A], [C], [C.H.], [P].

If $\lambda_0 \in \mathcal{B}(f) \subset \boldsymbol{R}^k$, $k > 1$, is isolated there exists a small sphere $S^{k-1}$ around $\lambda_0$ and $\varepsilon > 0$ such that for all $\lambda \in S^{k-1}$, $0 < \|x\| \leq \varepsilon$, $f(x, \lambda) \neq 0$. For isolated bifurcation points J. C. Alexander has defined in [A] a topological invariant which is an element of the homotopy group $\pi_{k-1}(GL(n, \boldsymbol{R}))$ and which nontriviality implies the global bifurcation for (1) at $(0, \lambda_0) \in Q$.

However, if $k > 1$ the existence of isolated bifurcation points is not generic. That is why our aim is to research the local structure of the set of bifurcation points (in case $k = 2, 3$) instead of looking for isolated bifurcation points.

Using the Brouwer topological degree we examine the bifurcation phenomena associated with ordinary differential equations and the classical Dirichlet problem.

Our aproach is going to be made via singularity theory.

Recently K. Aoki, T. Fukuda, Wei-Zhi Sun and T. Nishimura using the topological degree derived a formula which computes the number of analytic curves of zeroes emanating from a critical zero of some analytic germ $F: (\boldsymbol{R}^k, 0) \to (\boldsymbol{R}^{k-1}, 0)$ (in case $k = 2$ [A. F. S.], in general case [A. F. N. 1]).

This result can be succesfully applied in our considerations due to the fact that instead of searching the set of bifurcation points $\mathcal{B}(f)$ we can construct (under some additional assumptions on $f$) an analytic germ $\varPhi: U \subset \boldsymbol{R}^2 \to \boldsymbol{R}$ such that

$$\mathcal{B}(f) \cap U = \varPhi^{-1}(0).$$

After this introduction the paper is organised into three parts.

In the first part, for an analytic map $f: \mathcal{O} \subset \boldsymbol{R}^n \times \boldsymbol{R}^2 \to \boldsymbol{R}^n$ the number of branches of bifurcation points is expressed in terms of the topological degree of a map which is constructed explicitly in terms of $f$ (Th. 1.2.).

Next, we restrict our considerations to the case the partial derivative of $f$ is of the form

$$D_x f(0, \lambda) = \sum_{i=1}^{k} \eta_i(\lambda) \cdot A_i$$

where $\eta_i(\lambda)$ are homogenous polynomials and $A_i$ are real $n \times n$-matrices.

In this situation the map

$$\varPhi(\lambda) := \det(D_x f(0, \lambda))$$

is just of the type mentioned above and additionally is a polynomial map.

This in turn allows us to apply a computer program, written by Andrzej Łecki from Institute of Mathematics of Gdańsk University, in order to compute the topological degree of appropriate map and consequently (by [A. F. N. S. 1]) determine the number of branches of the set $\mathcal{B}(f)$ (locally in $U = \text{dom } \varPhi$).

Proceeding further we define a set of regular sequences of matrices $\mathcal{R}(n; \eta_1, \cdot, \eta_k)$ (Def. 2.1.) and we prove that $\mathcal{R}(n; \omega_0, \cdots, \omega_q)$ is an open and dense subset of $(M(n))^{q+1}$ (Th. 2.1.), where $\omega_i(\lambda) = \lambda_1^{q-i} \cdot \lambda_2^i$.

Finally the above results we apply to a boundary value problem

$$(2) \quad \begin{cases} \dot{x}(t) = \left( \sum_{i=1}^{k} \eta_i(\lambda) \cdot A_i \right)(x(t)) + \Psi(t, x(t), \lambda) \\ \\ x(0) = x(1) \end{cases}$$

and to the Dirichlet problem

$$(3) \quad \begin{cases} \Delta(u) = \left( \sum_{i=1}^{k} \eta_i(\lambda) \cdot T_i \right)(u) + \Psi(u, \lambda) \quad & \text{in } D \\ \\ u = 0 & \text{on } \partial D \end{cases}$$

here $T_i$ are linear operators for $i=1, \cdots, k$ and $\Psi$ is an nonlinear perturbation.

In the case $k=2$ and $\eta_1(\lambda)=\lambda_1$, $\eta_2(\lambda)=\lambda_2$ linear parts of both problems are of the form $B - \lambda_1 \cdot A_1 - \lambda_2 \cdot A_2$.

This kind of linear operators has been considered by S. N. Chow and J. K. Hale (see [C. H.]) and R. S. Cantrell (see [C]) but our methods and results are completely different.

In particular Theorem 3.1. gives a description of the set of bifurcation points of the problem (2) in an open neighbourhood of $0 \in \mathbf{R}^2$ and Theorem 3.2. gives the similar results for (3).

## 1. The structure of the set of bifurcation points.

Let $f: \mathbf{R}^n \times \mathbf{R}^2 \rightarrow \mathbf{R}^n$ be a continuous map such that $f(0, \lambda)=0$ for all $\lambda \in \mathbf{R}^2$. Define the set of nontrivial zeroes of $f$ by

$$Z_f = \{(x, \lambda) \in \mathbf{R}^n \times \mathbf{R}^2 : f(x, \lambda)=0 \text{ and } x \neq 0\}.$$

DEFINITION 1.1. Any point $\lambda_0 \in \mathbf{R}^2$ is said to be a bifurcation point of $f$ provided $(0, \lambda_0) \in cl(Z_f)$. The set of all bifurcation points of $f$ we will denote by $\mathscr{B}(f)$.

Suppose that $H: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is a continuous map such that $0 \in \mathbf{R}^2$ is an isolated point in $H^{-1}(0)$. Let $D \subset \mathbf{R}^2$ be a disc with center at the origin and let $H^{-1}(0) \cap D = \{0\}$.

To simplify the notation the topological degree of $H$ we will denote by $\deg H$ instead of $\deg(H, D, 0)$.

Define a map $\Phi: \mathbf{R}^2 \rightarrow \mathbf{R}$ by the formula

$$\Phi(\lambda) = \det(D_x f(0, \lambda)).$$

Note, that $\Phi(\lambda)=0$ for $\lambda \in \mathscr{B}(f)$.

Choose any $\lambda_0 \in \Phi^{-1}(0)$ and assume that there is a disc $D(\lambda_0) \subset \boldsymbol{R}^2$ with the center at $\lambda_0$ such that $\lambda_0$ is an isolated critical point in the set $X = \Phi^{-1}(0) \cap D(\lambda_0)$ (i.e. $\lambda_0 \in \boldsymbol{R}^2$ is an isolated point in the set $\{\lambda \in X : D\Phi(\lambda) = 0\}$). Without loss of generality we can suppose $\lambda_0 = 0$.

For sufficiently small $D = D(0)$ the set $X - \{0\}$ is empty or is a disjoint sum of the finite number of analytic curves.

Putting $\omega(\lambda) = \lambda_1^2 + \lambda_2^2$ we can formulate the following theorem.

THEOREM 1.1. *Let* $\nabla = \det\left(\dfrac{\partial(\omega, \Phi)}{\partial(\lambda_1, \lambda_2)}\right)$ *and* *let* $H = (\nabla, \Phi) : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$. *Then* $0 \in \boldsymbol{R}^2$ *is an isolated point in* $H^{-1}(0)$ *and the set* $X$ *consists of exactly* $2 \cdot \deg H$ *analytic curves emanating from* $0$. $\qquad\square$

Theorem 1.1. has been proved in more general case in [A. F. N. 1] and [A. F. S.]. But only the above version will be needed in our considerations.

Notice that if $\deg H = 0$ then the point $0 \in \boldsymbol{R}^2$ is an isolated point in the set $\Phi^{-1}(0)$. This is a case the Alexander invariant can be applied (see [A]).

We are interested in the case $\deg H > 0$. Theorem concerning this situation states as follows.

THEOREM 1.2. *If* $\deg H > 0$ *then* $X \cap D = \mathscr{B}(f) \cap D$. *In other words, there is* $2 \cdot \deg H$ *analytic curves of bifurcation points emanating from* $0 \in \boldsymbol{R}^2$.

PROOF. From Theorem 1.1. we have $2 \cdot \deg H$ analytic curves amanating from $0 \in \boldsymbol{R}^2$ and consisting of zeroes of $\Phi$ only.

Since each $\lambda \in \Phi^{-1}(0) \cap D$, $\lambda \neq 0$, is a regular point of $\Phi$ we get $\lambda \in \mathscr{B}(f)$. $\qquad\square$

This theorem will be applied later to proof Theorem 3.1. and 3.2.

## 2. Regular matrices.

In this part of paper we will consider the case in which the partial derivative of $f$ is of some special form.

Namely, assume that $D_x f(0, \lambda) = \sum_{i=1}^{k} \eta_i(\lambda) \cdot A_i$, where $\eta_i(\lambda)$ are homogenous polynomials of the same degree and $A_i$ are real $n \times n$-matrices. Adopting notations from Section 1 we have

$$\Phi(\lambda) = \det\left(\sum_{i=1}^{k} \eta_i(\lambda) \cdot A_i\right).$$

The following lemma easily follows from the standard arguments of the polynomial theory.

LEMMA 2.1. *If* $\Phi \not\equiv 0$ *then* $\Phi^{-1}(0)$ *is a sum of a finite number of* 1-*dimensional linear subpaces of* $\mathbf{R}^2$ *or is equal to* $\{0\}$.    $\square$

By $M(n)$ we shall denote the set of all real $n \times n$-matrices and consider the space $(M(n))^{q+1}$ as a metric space with the metric which is defined as follows:

$$\rho((A_0, \cdots, A_q), (B_0, \cdots, B_q)) = \max_s \{|a_{ij}^s - b_{ij}^s| : i, j = 1, \cdots, n\}.$$

DEFINITION 2.1. Any sequence $(A_1, \cdots, A_k) \in (M(n))^k$ is said to be a regular sequence of matrices (with the respect to a sequence of homogenous polynomials of the same degree $(\eta_1, \cdots, \eta_k)$) if $\Phi : \mathbf{R}^2 \to \mathbf{R}$ defined by the formula $\Phi(\lambda) = \det\left(\sum_{i=1}^{k} \eta_i(\lambda) \cdot A_i\right)$ has no critical zeroes in $\mathbf{R}^2 - \{0\}$. The set of all regular sequences will be denoted by $\mathcal{R}(n; \eta_1, \cdots, \eta_k)$.

Notice, that for any sequence $(A_1, \cdots, A_k) \in \mathcal{R}(n; \eta_1, \cdots, \eta_k)$ the map $\Phi(\lambda) = \det\left(\sum_{i=1}^{k} \eta_i(\lambda) \cdot A_i\right)$ satisfies the assumptions of Theorem 1.2., that is why it is of our interest to know whether the set of regular sequences of matrices is "big" or "small" in some sense. If a sequence $(A_1, \cdots, A_k) \in \mathcal{R}(n; \eta_1, \cdots, \eta_k)$ then $0 \in \mathbf{R}$ is a regular value of $\Phi_1 = \Phi_{|S^1}$. From the transversality theorem it follows that each map from some neighbourhood $U \subset C^1(S^1, \mathbf{R})$ of $\Phi_1$ has this property.

The question is, if for any neighbourhood $U$ of $\Phi_1$ one can find a map $\Phi_2 \in U$ which can be expressed by using a regular sequence of matrices i. e. if there exists a sequence $(X_1, \cdots, X_k) \in \mathcal{R}(n; \eta_1, \cdots, \eta_k)$ such that

$$\Phi_2(\lambda) = \det\left(\sum_{i=1}^{k} \eta_i(\lambda) \cdot X_i\right).$$

The answer for the question is given by the following theorem.

THEOREM 2.1. *For arbitrary sequence of homogeneous polynomials* $\eta_1, \cdots, \eta_k$ *of degree* $q$ *the set* $\mathcal{R}(n; \eta_1, \cdots, \eta_k)$ *is an open subset of* $(M(n))^k$. *Moreover the set* $\mathcal{R}(n; \omega_0, \cdots, \omega_q)$ *is an open and dense subset of* $(M(n))^{q+1}$, *where* $\omega_i(\lambda) = \lambda_1^{q-i} \cdot \lambda_2^i$.

*Proof.* First we prove that $\mathcal{R}(n; \eta_1, \cdots, \eta_k)$ is an open set.

Let $(A_1, \cdots, A_k) \in \mathcal{R}(n; \eta_1, \cdots, \eta_k)$, then we have $\Phi \not\equiv 0$. Consider a map $\Phi_1 : S^1 \to \mathbf{R}$ which is a restriction of the map $\Phi$ to the unit circle.

Since $\Phi_1$ is transversal to $\{0\} \in \mathbf{R}$ there exists an open neighbourhood $U$ of $\Phi_1$ in $C^1(S^1, \mathbf{R})$ such that each map $\hat{\Phi} \in U$ is also transversal to $\{0\} \in \mathbf{R}$.

Define a map $\zeta : (M(n))^k \to C^1(S^1, \mathbf{R})$ by the formula

$$\zeta(X_1, \cdots, X_k)(\lambda) = \det\left(\sum_{i=1}^{k} \eta_i(\lambda) \cdot X_i\right) \quad \text{for} \quad \lambda \in S^1.$$

Since $\zeta$ is a continuous map, $\zeta^{-1}(U)$ is a open neighbourhood of $(A_1, \cdots, A_k)$ in $M(n)^k$, moreover $\zeta^{-1}(U) \subset \mathcal{R}(n ; \eta_1, \cdots, \eta_k)$ so the openness is concluded. In particular the set $\mathcal{R}(n ; \omega_0, \cdots, \omega_q)$ is open.

Now we turn to the density of $\mathcal{R}(n ; \omega_0, \cdots, \omega_q)$.

To prove this we will show that for any positive number $\varepsilon$ there exists a regular sequence of matrices $(B_0, \cdots, B_q)$ with

$$\rho((A_0, \cdots, A_q), (B_0, \cdots, B_q)) < \varepsilon .$$

If $(A_0, \cdots, A_q) \in \mathcal{R}(n ; \omega_0, \cdots, \omega_q)$ then our statement is obvious that is why we can assume that $(A_0, \cdots, A_q) \notin \mathcal{R}(n ; \omega_0, \cdots, \omega_q)$.

There are two posibilities to be considered, $\Phi \equiv 0$ and $\Phi \not\equiv 0$.

In the first case it is easily seen that for any positive number $\varepsilon_1$ one can always find a sequence $(X_0, \cdots, X_q) \in (M(n))^{q+1}$ such that

$$\zeta(X_0, \cdots, X_q) \not\equiv 0 \quad \text{and} \quad \rho((X_0, \cdots, X_q), (A_0 \cdots, A_q)) < \varepsilon_1 .$$

So having proved the case $\Phi \not\equiv 0$ the general result will be derived.

Without loss of generality we can assume that the line $\lambda_2 = 0$ is not included in the set of zeroes of $\Phi$. Putting $\lambda_2 = 1$ we get a polynomial of one variable $\Phi(\lambda_1, 1)$.

Clearly, the set of zeroes of the polynomial $\Phi(\lambda_1, 1)$ is in one-to-one corespondence with the set of all 1-dimensional subspaces of $\mathbf{R}^2$ included in $\Phi^{-1}0$).

Let $\{t_1 \cdots, t_r, u_1, \cdots, u_s\}$ be the set of all real roots of the polynomial $\Phi(\lambda_1, 1)$. Suppose that each $t_i$ has a multiplicity $\alpha_i > 1$ and each $u_i$ has a multiplicity which is equal to 1.

Our polynomial $\Phi(\lambda_1, 1)$ can be expressed in the form $\Phi(\lambda_1, 1) = \det\left(\sum_{i=0}^{q} \omega_i(\lambda_1, 1) \cdot A_i\right) = \det(V_1(\lambda_1), \cdots, V_n(\lambda_1))$, where $V_i(\lambda_1)$ is the $i$-th column-vector of the matrix $\sum_{i=0}^{q} \omega_i(\lambda_1, 1) \cdot A_i$.

Choose any $\xi > 0$. We will find a sequence of matrices $(X_0, \cdots, X_q)$ such that the set of zeroes of the polynomial $\zeta(X_0, \cdots, X_q)(\lambda_1, 1)$ is equal to $\{t_1, \cdots, t_r, u_1, \cdots, u_s\} \cup \{u_{s+1}\}$, where the multiplicity of $t_1$ equals $\alpha_1 - 1$, the multiplicity of $u_{s+1}$ equals 1 and the rest of multiplicities are not changed. Additionally, distance between $(A_0, \cdots, A_q)$ and $(X_0, \cdots, X_q)$ will be less than $\xi$.

The method proceeds as follows. Vectors $V_1(t_1), \cdots, V_n(t_1)$ are linearly dependent, because $\Phi(t_1, 1) = 0$. Therefore, there exists a vector, say $V_n(t_1)$, which can be expressed in the form

$$V_n(t_1) = \sum_{j=1}^{n-1} \beta_j \cdot V_j(t_1), \quad \text{where} \quad \beta_j \in \mathbf{R}.$$

From this we claim

$$V_n(\lambda_1) - \sum_{j=1}^{n-1} \beta_j \cdot V_j(\lambda_1) = W(\lambda_1) \cdot (\lambda_1 - t_1).$$

Therefore $\Phi(\lambda_1, 1)=\det(V_1(\lambda_1), \cdots, V_{n-1}(\lambda_1), (\lambda_1-t_1)\cdot W(\lambda_1))=(\lambda_1-t_1)\cdot\det(V_1(\lambda_1), \cdots$ $V_{n-1}(\lambda_1), W(\lambda_1))$.

Now, for arbitrary $t$ we can consider a polynomial $\hat{\Phi}(\lambda_1, 1)=(\lambda_1-t_1-t)\cdot$ $\det(V_1(\lambda_1), \cdots, V_{n-1}(\lambda_1), W(\lambda_1))=\det(V_1(\lambda_1), \cdots, V_{n-1}(\lambda_1), V_n(\lambda_1)-t\cdot W(\lambda_1))=$ $\det\left(\sum_{i=0}^{q}\omega_i(\lambda_1, 1)\cdot(A_i-t\cdot C_i)\right)$.

For sufficiently small $t$ the polynomial $\hat{\Phi}(\lambda_1, 1)$ has all properties mentioned above.

Putting $\xi=\varepsilon\cdot(\alpha_1+\cdots+\alpha_r-r)^{-1}$ we can proceed this method as long as we get a polynomial $\zeta(B_0, \cdots, B_q)(\lambda_1, 1)$ with all roots of the multiplicity equals 1.

But this means that $(B_0, \cdots, B_q)\in\mathcal{R}(n; \omega_0, \cdots, \omega_q)$ and from the triangle inequality of metric we have

$$\rho((A_0, , A_q), (B_0, \cdots, B_q))<\varepsilon$$

so our proof is concluded.                                                    □

Although it is possible that $\mathcal{R}(n; \eta_1, \cdots, \eta_k)=\varnothing$ for some $\eta_1, \cdots, \eta_k$, nevertheless one can always express a map $\Phi(\lambda)=\det\left(\sum_{i=1}^{k}\eta_i(\lambda)\cdot A_i\right)$ in the form $\Phi(\lambda)=\det\left(\sum_{i=0}^{q}\omega_i(\lambda)\cdot B_i\right)$, where $\omega_i(\lambda)=\lambda_1^{q-i}\cdot\lambda_2^{i}$. In this sense Theorem 2.1. states that the case the origin is the only critical zero of $\Phi(\lambda)$ is a generic one.

*Example* 2.1. As an example we consider the following situation. Let $A_0, A_1\in M(n)$ be a pair of normal matrices e.g. $A_0\cdot A_1=A_1\cdot A_0$ and $A_i\cdot A_i^{*}=A_i^{*}\cdot A_i$ for $i=0, 1$.
Denote by $\sigma(A_i)$ the spectrum of the matrix $A_i$.
Assume that
1) if $z_0\in\sigma(A_0)$ and $z_1\in\sigma(A_1)$ are complex numbers then they are lineary independent over $\boldsymbol{R}$,
2) all real eigenvalues of matrix $A_i$ are different among themselves for $i=0, 1$,
3) if $a_i^1, a_i^2\in\sigma(A_i)$ are reals then vectors $(a_0^1, a_0^2)$ and $(a_1^1, a_1^2)$ are lineary independent.

By Theorem 12′ page 265 in [G] the above assumptions imply $(A_0, A_1)\in\mathcal{R}(n; \lambda_1, \lambda_2)$.

It means that $(0, 0)\in\boldsymbol{R}^2$ can be the only critical zero of the polynomial $\Phi(\lambda_1, \lambda_2)=\det(\lambda_1\cdot A_0+\lambda_2\cdot A_1)$.

## 3. Applications.

We start with ordinary differential equations.
Let $h: \boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^2\to\boldsymbol{R}^n$ be a $C^2$-map which is of the form

$$h(t, x, \lambda)=\left(\sum_{i=1}^{k}\eta_i(\lambda)\cdot A_i\right)\cdot x+\psi(t, x, \lambda),$$

such that

$$\phi(t, 0, \lambda)=0 \quad \text{and} \quad \frac{\partial \phi}{\partial x}(t, 0, \lambda)=0 \quad \text{for all} \quad (t, \lambda) \in \mathbf{R} \times \mathbf{R}^2.$$

Consider the following boundary value problem:

(1)
$$\begin{cases} \dot{x}(t)=h(t, x(t), \lambda) \\ x(0)=x(1) \end{cases}$$

The problem of the existence of solutions of (1) can be replaced by the associated bifurcation problem in the following way.

For Banach spaces $X=\{x \in C^1([0, 1], \mathbf{R}^n): x(0)=x(1)\}$ and $Y=C^0([0, 1], \mathbf{R}^n)$ we can define a family of operators $L(\lambda): X \rightarrow Y$ by the formula:

$$L(\lambda)(x(t))=\dot{x}(t)-\left(\sum_{i=1}^k \eta_i(\lambda) \cdot A_i\right) \cdot x(t).$$

We would like to notice that $L(0)$ is a Fredholm operator of the Fredholm index 0 which will be important in our further considerations.

Additionally, let $\Psi: X \times \mathbf{R}^2 \rightarrow Y$ be a map defined as follows $\Psi(x(t), \lambda)=\phi(t, x(t), \lambda)$.

Now we define an operator $F: X \times \mathbf{R}^2 \rightarrow Y$ putting

$$F(x(t), \lambda)=L(\lambda)(x(t))-\Psi(x(t), \lambda).$$

It is easily seen that the solutions of the problem (1) are in one-to-one correspondence with the solutions of the equation

$$F(x(t), \lambda)=0.$$

Let us recall that for $\nabla=\det\left(\frac{\partial(\omega, \Phi)}{\partial(\lambda_1, \lambda_2)}\right)$ we have defined a map $H=(\nabla, \Phi)$: $(\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$, where $\Phi$ as before is defined by $\Phi(\lambda)=\det\left(\sum_{i=1}^k \eta_i(\lambda) \cdot A_i\right)$.

Since the origin is an isolated zero of $H$ the topological degree can be used.

THEOREM 3.1. *If a sequence* $(A_1, \cdots, A_k) \in \mathcal{R}(n; \eta_1, \cdots, \eta_k)$ *and* $\deg H > 0$ *then there exist an open neighbourhood* $U$ *of the origin of* $\mathbf{R}^2$ *and exactly* $d(\Phi)=2 \cdot \deg H$ *intervals* $I_1, \cdots, I_{d(\Phi)}$ *emanating from 0 such that*

$$\sum_{j=1}^{d(\Phi)} I_j \cap U = \mathcal{B}(F) \cap U.$$

In particular, $0 \in \mathbf{R}^2$ is a bifurcation point of $F$.

*Proof*: Since $L(0): X \rightarrow Y$ is a Fredholm operator of the Fredholm index 0 we can express Banach spaces $X$ and $Y$ as a direct sums $\ker L(0) \oplus X_0$, $Y_0 \oplus \operatorname{im} L(0)$, respectively. In this case $\ker L(0)$ and $Y_0$ are spaces consisting

of constant maps.

Applaying the Lapunov-Schmidt reduction to the equation $F(x(t), \lambda)=0$ we obtain a map $f: \Omega \cap (\boldsymbol{R}^n \times \boldsymbol{R}^2) \to \boldsymbol{R}^n$ defined by the formula

$$f(x, \lambda) = \left( \sum_{i=1}^{k} \eta_i(\lambda) \cdot A_i \right) \cdot x + \bar{\Psi}(x, \lambda),$$

where $\Omega$ is sufficiently small open neighbourhood of $(0, 0) \in \boldsymbol{R}^n \times \boldsymbol{R}^2$ (see [RB] for more details).

The map $f$ satisfies all the assumptions of Theorem 1.2. and our proof is completed. □

*Remark* 3.1. If $A_0$, $A_1$ are normal matrices as in Example 2.1. then

$$\frac{1}{2} \cdot d(\Phi) = \deg H = \#(\sigma(A_0) \cap \boldsymbol{R}) = \#(\sigma(A_1) \cap \boldsymbol{R}). \qquad □$$

Now we turn to the partial differential equations. Namely, we want to consider Dirichlet problem on 3-dimensional disc.

Let $D \subset V$ be an open unit disc in a nontrivial 3-dimensional real representation of the group $S^1$.

For a Sobolev space $H^2(D)$ denote $X = \{u \in H^2(D) \mid u_{|\partial D} = 0\}$ and $Y = H^0(D) = L_2(D)$.

The action of the group $S^1$ on $D$ induces a $S^1$-action on the spaces $X$ and $Y$ by the formula

$$\langle g \cdot u, \varphi(x) \rangle = \langle u, \varphi(g \cdot x) \rangle.$$

Then the Laplace opertor $\Delta : X \to Y$ is a $S^1$-equivariant linear isomorphism.

Let us define linear operators $A_1, \cdots, A_k : X \to Y$ putting $A_i(u) = \alpha(g_i \cdot u)$ for some $g_i \in S^1$ and $\alpha : X \to Y$ which is the inclusion. Additionally, set $A_{k+1} = Id$.

Assume that $\eta_1, \cdots, \eta_{k+1} : \boldsymbol{R}^3 \to \boldsymbol{R}$ are homogeneous polynomials which satisfy the following conditions:
1)  $\eta_1(\lambda_1, \lambda_2, 1), \cdots, \eta_k(\lambda_1, \lambda_2, 1)$ are homogenous polynomials of the same degree,
2)  $\eta_{k+1}(\lambda_1, \lambda_2, \lambda_3) = \lambda_3$.

Suppose that $\Psi : X \times \boldsymbol{R}^3 \to Y$ is a map such that $\Psi(0, \lambda) = 0$ and $D_u \Psi(0, \lambda) = 0$ for all $\lambda \in \boldsymbol{R}^3$.

Consider the following Dirichlet problem

(2)
$$\begin{cases} \Delta(u) = \left( \sum_{i=1}^{k+1} \eta_i(\lambda) \cdot A_i \right) (u) + \Psi(u, \lambda) & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

Our aim is to research the set of bifurcation points of (2).

It is easy to see that the solutions of the problem (2) are in **one-to-one** correspondence with zeroes of an operator $F : X \times \boldsymbol{R}^3 \to Y$ which is defined as follows:

(3)
$$F(u, \lambda)=\Big(\Delta-\sum_{i=1}^{k+1}\eta_i(\lambda)\cdot A_i\Big)(u)-\Psi(u, \lambda).$$

We recall a description of eigenvalues and eigenvectors of the Laplace operator on 3-dimensional disc.

Denote by $\{J_\nu(\mu):\nu\in R\}$ the family of Bessel functions.

Let $\{\mu_j^{(\nu)}:j=1, 2, \cdots\}$ be the set of all positive zeroes of $J_\nu(\mu)$.

Then all eigenvalues of $\Delta$ are of the form $\lambda_{tj}=[\mu_j^{(t+1/2)}]^2$, $t=0, 1, \cdots$, $j=1, 2, \cdots$ and eigenvectors corresponding to them are expressed by the formulas

$$u_{tjm}(x)=c_{tjm}\cdot r^{-1/2}\cdot J_{t+1/2}(r\cdot\mu_j^{(t+1/2)})\cdot P_t^m(\cos(\theta))\cdot\exp)i\cdot m\cdot\varphi),$$

$m=0, 1, \cdots, t$, where $c_{tjm}$ are constants, $P_t^m$ denotes the Legendre polynomials and $x=(r\cdot\cos\varphi\cdot\cos\theta, r\cdot\sin\varphi\cdot\cos\theta, r\cdot\sin\theta)\in R^3$.

The representation $V$ splits into a direct sum of two summands $R^1\oplus R[n]$, where $R^1$ is a 1-dimensional trivial representation and $R[n]$ is a 2-dimensional representation which is defined by

$$S^1\ni\varphi\longrightarrow\begin{bmatrix}\cos(n\varphi) & -\sin(n\varphi)\\ \sin(n\varphi) & \cos(n\varphi)\end{bmatrix}.$$

By $V(\lambda_{tj})$ we will denote an eigenspace corresponding to the eigenvalue $\lambda_{tj}$. The space $V(\lambda_{tj})$ is an orthogonal $2\cdot t+1$-dimensional representation of $S^1$, and moreover, using the above notation we get

$$V(\lambda_{tj})=R^1\oplus R[n]\oplus R[2n]\oplus \cdots \oplus R[t\cdot n].$$

It follows from definition that $V(\lambda_{tj})$ are invariant subspaces for operators $A_1, \cdots, A_{k+1}$ for all $t$ and $j$.

We recall that $A_i(u)=\alpha(g_i\cdot u)=\alpha(\exp(\sqrt{-1}\cdot\varphi_i)\cdot u)$ for some $\varphi_i\in[0, 2\cdot\pi)$, $i=1, \cdots, k$.

Now we came to the following theorem.

THEOREM 3.2. *Consider an operator defined by* (3). *For a fixed eigenvalue* $\lambda_{tj}$ *assume that*

a) *for each* $m=1, \cdots, t$, $\lambda_1=\lambda_2=0$ *is the only solution of the equation*

$$\Big(\sum_{i=1}^{k}\eta_i(\lambda_1, \lambda_2, 1)\cdot\cos(mn\varphi_i)\Big)^2+\Big(\sum_{i=1}^{k}\eta_i(\lambda_1, \lambda_2, 1)\cdot\sin(mn\varphi_i)\Big)^2=0,$$

b) *if* $(\lambda_1, \lambda_2)$ *is a critical zero of the polynomial* $\sum_{i=1}^{k}\eta_i(\lambda_1, \lambda_2, 1)$ *then* $\lambda_1=\lambda_2=0$.
*Then*

1) $(A_1, \cdots, A_k)$ *is a regular sequence of matrices with respect to the polynomials* $\eta_1(\lambda_1, \lambda_2, 1), \cdots, \eta_k(\lambda_1, \lambda_2, 1)$,

2) *there exist an open neighbourhood* $U$ *of the poin* $P_{tj}=(0, (0, 0, \lambda_{tj}))$ *in* $X\times R^3$

*and exactly* $d(\Phi)=2 \cdot \deg H$ *intervals* $I_1, \cdots, I_{d(\Phi)}$ *emanating from* $P_{t_j}$ *such that*

$$\bigcup_{j=1}^{d(\Phi)} I_j \cap U \subset \mathcal{B}(F) \cap U .$$

*Proof.* Let us define an operator $G: X \times \boldsymbol{R}^2 \to Y$ as follows

$$G(u, \lambda_1, \lambda_2) = F(u, \lambda_1, \lambda_2, \lambda_{tj})$$

then $D_u G(0, 0, 0) = (\Delta - \lambda_{tj} \cdot Id)$ is a Fredholm operator of the Fredholm index 0.

We start with the Lyapunov-Schmidt reduction with respect to the following splittings

$$X \oplus \boldsymbol{R}^2 = (V(\lambda_{tj}) \oplus \boldsymbol{R}^2) \oplus X_1, \quad Y = \alpha(V(\lambda_{tj})) \oplus \mathrm{im}(D_u G(0, 0, 0)),$$

Let $E: Y \to Y$ denote a projection onto $\alpha(V(\lambda_{tj}))$ chosen in such a way that $(Id-E)$ maps $Y$ onto $\mathrm{im}(D_u G(0, 0, 0))$.

Then equality $G(u, \lambda_1, \lambda_2)=0$ is equivalent to the system

$$(*) \qquad \begin{cases} (E \circ G)(u, \lambda_1, \lambda_2)=0 \\ ((Id-E) \circ G)(u, \lambda_1, \lambda_2)=0 \end{cases}$$

Now, applying implicit function theorem we have a map

$$\bar{\zeta}: (V(\lambda_{tj}) \oplus \boldsymbol{R}^2) \cap U \longrightarrow \alpha(V(\lambda_{tj})),$$

where $U$ is a sufficiently small neighbourhood of the origin in $X \times \boldsymbol{R}^2$, which zeroes are locally in one to one correspondence to the solutions of system $(*)$.

Choosing appropriate basis in $V(\lambda_{1j})$ we obtain the map

$$\zeta: (\boldsymbol{R}^1 \oplus \boldsymbol{R}[n] \oplus \cdots \oplus \boldsymbol{R}[t \cdot n] \times \boldsymbol{R}^2) \cap U \longrightarrow \boldsymbol{R}^1 \oplus \boldsymbol{R}[n] \oplus \cdots \oplus \boldsymbol{R}[t \cdot n]$$

which is of the form

$$\zeta(v, \lambda_1, \lambda_2) = \left( \sum_{i=1}^{k} (\eta_i(\lambda_1, \lambda_2, \lambda_{1j}) \cdot A_i) \right)(v) + \bar{\Psi}(v, \lambda_1, \lambda_2),$$

where $\bar{\Psi}(0, \lambda_1, \lambda_2)=0$ and $D_v \bar{\Psi}(0, \lambda_1, \lambda_2)=0$.

Both assumptions a) and b) imply that

$$(A_1, \cdots, A_k) \in \mathcal{R}(2 \cdot t+1; \bar{\eta}_1, \cdots, \bar{\eta}_k).$$

where $\bar{\eta}_i(\lambda_1, \lambda_2)=\eta_i(\lambda_1, \lambda_2, \lambda_{tj})$.

The map $\zeta$ satisfies all the assumptions of Theorem 1.2. and our proof is completed.                                                                $\square$

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TECHNICAL UNIVERSITY OF GDANSK
DEPARTMENT OF MATHEMATICS
MAJAKOWSKIEGO 11/12, 80-952 GDANSK
POLAND