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ON THE COMPLEX OSCILLATION OF NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

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Abstract

In this paper, we investigate the complex oscillation of

$$f^{(k)}+b_{k-1}f^{(k-1)}+\cdots+b_0f=B(z),$$

where $b_{k-j}(j=1,\cdots,k)$ are rational functions, $B(z)$ is a meromorphic funciton, and obtain general estimates of the exponent of convergence of the zero-sequence and the pole-sequence of solutions for the above equation.

Key words: Non-homogeneous Linear differential equation, Meromorphic function, zero-sequence, Pole-sequence, Exponent of convergence.

§ 1. introduction and results.

For convenience in our statement, we first explain the notations used in this paper, we will use resperctively the notations $\lambda(f)$ and $\lambda(1/f)$ to denote the exponent of convergence of the zero-sequence and the pole-sequence of a meromorphic function $f(z)$, $\bar{\lambda}(f)$ and $\bar{\lambda}(1/f)$ to denote the exponent of convergence of the sequences of distinct zeros and distinct poles of $f(z)$, $\sigma(f)$ to denote the order of growth of $f(z)$, $\gamma_f(r)$ to denote the centralindex of entire function $f(z)$. By the Wiman-Valiron theory, we have $\sigma(f)=\varlimsup_{r\to\infty}\frac{\log\gamma_f(r)}{\log r}$.

In addition, other notations of the Nevanlinna theory are standard (e.g. see [3]), the individual ones will be shown when they appear.

We also need the following Definition.

DEFINITION. If the meromorphic function $f(z)$ has infinitely many zeros, we call $f(z)$ is oscillatory.

The complex oscillatory problems of the non-homogeneous linear differential equation

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$$f^{(k)}+a_{k-1}f^{(k-1)}+\cdots+a_0f=F$$

are a very important aspect in the complex oscillation theory of differential equations which has been an active research area recently. Just lately, Gao Shi-an proved in [2].

THEOREM A. *Let F be a transcedental entire function with $\sigma(F)<\infty$, a_{k-j} $(j=1,\cdots,k)$ polynomials. Then for every solution f of*

$$f^{(k)}+a_{k-1}f^{(k-1)}+\cdots+a_0f=F \qquad (k\geqq 1) \tag{1.1}$$

(a) *If F is oscillatory, then f is also oscillatory.*
(b) $\lambda(f)\geqq\lambda(F)$.
(c) *If $\sigma(F)$ is not a positive integer, then*

$$\lambda(f)=\sigma(f)\geqq\sigma(F)=\lambda(F).$$

(d) *If $\sigma(f)>\sigma(F)$, then $\lambda(f)=\sigma(f)>\sigma(F)$.*

THEOREM B. *For the equation*

$$f''+a_0f=P_1e^{p_0} \tag{1.2}$$

where a_0, p_0, p_1 are polynomials, $\deg a_0=n$, $\deg p_0<1+(n/2)$.

(a) *If $n>1$ and* $\deg P_1<n$, *then every solution f of* (1.2) *satisfies*

$$\lambda(f)=\bar{\lambda}(f)=\sigma(f)=1+\frac{n}{2}>\deg P_0.$$

(b) *If* $\deg p_1\geqq n\geqq 0$, *then the solution f of* (1.2) *either satisfies* $\lambda(f)=\bar{\lambda}(f)=\sigma(f)=1+(n/2)>\deg P_0$, *or is of the form $f=Qe^{P_0}$, where Q is a polynomial. And if* (1.2) *has a solution of the form Qe^{P_0} with Q polynomial, then* (1.2) *must have solutions which satisfy* $\lambda(f)=\bar{\lambda}(f)=\sigma(f)=1+(n/2)>\deg P_0$.

In this paper, we investigate the complex oscillatin of non-homogeneous linear differential equations with meromorphic coefficients, and obtain general estimates of the exponent of convergence of the zero-sequence and the pole-sequence of solutions for the considered equations.

In fact, we will prove the following theorems in this paper.

THEOREM 1. *Let $A\not\equiv 0$, $b_{k-j}(j=1,\cdots,k)$ be rational functions, b_{k-j} have a pole at ∞ of order $n_{k-j}\geqq 0$, $k\geqq 1$, P be a polynomial* $\deg P=\beta$ *satisfies*

$$1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}<\beta<\infty. \tag{1.3}$$

If the differential equation

$$f^{(k)}+b_{k-1}f^{(k-1)}+\cdots+b_0f=Ae^P \tag{1.4}$$

has a meromorphic solution f, then

(a) $\sigma(f)=\beta$, *f has only finitely many poles.*

(b) *suppose that A has a pole at* ∞ *of order* n_A. *If* $n_A<k(\beta-1)$, *then* $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\beta$, *if* $n_A\geqq k(\beta-1)$, *then all meromorphic solutions of* (1.4) *satisfy* $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\beta$, *except at most a possible one. The possible exceptional one is of the form* $f_0=A_0e^P$ (A_0 *is rational*).

THEOREM 2. *Let* $A\not\equiv 0$, $b_{k-j}(j=1, \cdots, k)$ *be rational functions,* b_{k-j} *have a pole at* ∞ *order* $n_{k-j}\geqq 0$, $k\geqq 1$, *P be a polynomial,* $\deg P=\beta\leqq 1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}$. *If* (1.4) *has a meromorphic solution f, then*

(a) $\beta\leqq\sigma(f)\leqq 1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}$, *f has only finitely many poles.*

(b) *If* $\sigma(f)>\beta$, *then* $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$.

THEOREM 3. *Let* $b_{k-j}(j=1, \cdots, k)$ *be rational functions having a pole at* ∞ *of order* $n_{k-j}\geqq 0$, $k\geqq 1$, $B(z)$ *be a transcendental meromorphic function,* $\sigma(B)=\beta$ *satisfying* (1.3). *If all solutions of the differential equation*

$$f^{(k)}+b_{k-j}f^{(k-1)}+\cdots+b_0f=B(z) \tag{1.5}$$

are meromorphic functions, then

(a) $\sigma(f)=\beta$.

(b) $\lambda(1/f)=\lambda(1/B)$, $\bar{\lambda}(1/f)=\bar{\lambda}(1/B)$. *If* $\lambda(B)>\lambda(1/B)$, *then* $\lambda(f)\geqq\lambda(B)$.

(c) *If* $\beta>\max\{\lambda(B), \lambda(1/B)\}$, *then all solutions of* (1.5) *satisfy* $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\beta$, *except at most a possible one. The possible exceptional one* f_0 *satisfies* $\lambda(f_0)<\beta$.

THEOREM 4. *Let* $b_{k-j}(j=1, \cdots, k)$ *be rational functions having a pole at* ∞ *of order* $n_{k-j}\geqq 0$, $k\geqq 1$, $B(z)\not\equiv 0$ *be a meromorphic function satisfying* $\sigma(B)=\beta\leqq 1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}$. *If all solutions of* (1.5) *are meromorphic functions, then*

$$\text{(a)}\quad \beta\leqq\sigma(f)\leqq 1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}. \tag{1.6}$$

(b) $\lambda(1/f)=\lambda(1/B)$, $\bar{\lambda}(1/f)=\bar{\lambda}(1/B)$. *If* $\lambda(B)>\lambda(1/B)$, *then* $\lambda(f)\geqq\lambda(B)$.

(c) *If* $\sigma(f)>\beta$, *then* $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$.

§2. Lemmas.

LEMMA 1. *Let the set* $E\subseteqq[0, +\infty)$ *have finite logarithmic measure,* $\Upsilon(r)$ *be a nondecreasing function on* $[0, \infty)$. *Then*

$$\varlimsup_{\substack{r\to\infty \\ r\in[0,+\infty)}}\frac{\log\Upsilon(r)}{\log r}=\varlimsup_{\substack{r\to\infty \\ r\in[0,+\infty)-E}}\frac{\log\Upsilon(r)}{\log r}.$$

Proof. We clearly have

$$\overline{\lim_{\substack{r\to\infty \\ r\in[0,+\infty)}}} \frac{\log \Upsilon(r)}{\log r} \geqq \overline{\lim_{\substack{r\to\infty \\ r\in[0,+\infty)-E}}} \frac{\log \Upsilon(r)}{\log r}.$$

On the other hand, setting $\int_E \frac{dr}{r} = \log \delta < \infty$, for a given $\{r'_n\}$, $r'_n \in [0, \infty)$, $r'_n \to \infty$, there exists a point $r_n \in [r'_n, (\delta+1)r'_n] - E$. From

$$\frac{\log \Upsilon(r'_n)}{\log r'_n} \leqq \frac{\log \Upsilon(r_n)}{\log r'_n} \leqq \frac{\log \Upsilon(r_n)}{\log r_n + \log(1/\delta+1)} = \frac{\log \Upsilon(r_n)}{\log r_n(1+o(1))},$$

it follows that

$$\overline{\lim_{r'_n\to\infty}} \frac{\log \Upsilon(r'_n)}{\log r'_n} \leqq \overline{\lim_{r_n\to\infty}} \frac{\log \Upsilon(r_n)}{\log r_n} \leqq \overline{\lim_{\substack{r\to\infty \\ r\in[0,+\infty)-E}}} \frac{\log \Upsilon(r)}{\log r}.$$

Since $\{r'_n\}$ is arbitrary, we have

$$\overline{\lim_{\substack{r\to\infty \\ r\in[0,+\infty)}}} \frac{\log \Upsilon(r)}{\log r} \leqq \overline{\lim_{\substack{r\to\infty \\ r\in[0,+\infty)-E}}} \frac{\log \Upsilon(r)}{\log r}.$$

This proves Lemma 1.

LEMMA 2. *Let f be a solution of the differential equation*

$$f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f = 0 \qquad (k \geqq 1) \tag{2.1}$$

with $a_0, \cdots, a_{k-1}$ polynomials. Then f is entire of order

$$\sigma(f) \leqq 1 + \max_{1\leq j\leq k} \frac{\deg a_{k-j}}{j}.$$

Proof. see [1].

LEMMA 3. *Let b_{k-j} $(j=1, \cdots, k)$ be rational functions having a pole at ∞ of order $n_{k-j} \geqq 0$, $k \geqq 1$, f be a meromorphic solution of the differential equation*

$$f^{(k)} + b_{k-1} f^{(k-1)} + \cdots + b_0 f = 0 \tag{2.2}$$

Then, $\sigma(f) \leqq 1 + \max_{1\leq j\leq k} \frac{n_{k-j}}{j}$.

Proof. If f is a rational function, then Lemma 3 holds. Thus, we can now suppose f is a transcendental meromorphic function. If f has a pole at z_0 of order α, and $b_{k-1}, \cdots, b_0$ are all analytic at z_0, then $f^{(k)} + b_{k-1} f^{(k-1)} + \cdots + b_0 f$ must has a pole at z_0 of order $a+k$. This contradicts (2.2) and implies that the poles of f can only occur at the poles of b_{k-j} $(j=1, \cdots, k)$. Hence f has only finitely many poles. Now let f_1 denote the sum of the principal parts of all poles of f, then f_1 is a rational function with $|f_1| = O(r^{-1})$, and $f_2 = f - f_1$ is a transcendental entire function. Now substituting $f = f_1 + f_2$ into (2.2), we obtain

$$f_2^{(k)}+b_{k-1}f_2^{(k-1)}+\cdots+b_0f_2=-(f_1^{(k)}+b_{k-1}f_1^{(k-1)}+\cdots+b_0f_1) \tag{2.3}$$

For sufficiently large $|z|$. we have $b_{k-j}=B_{k-j}z^{n_{k-j}}(1+o(1))$ ($B_{k-j}\neq 0$ are constants). Now let z be a point with $|z|=r$ at which $|f_2(z)|=M(r, f_2)$. Since f_1 is rational, we get

$$\lim_{r\to\infty}\frac{f_1^{(k)}+b_{k-1}f_1^{(k-1)}+\cdots+b_0f_1}{f_2(z)}=\lim_{r\to\infty}\frac{f_1^{(k)}+b_{k-1}f_1^{(k-1)}+\cdots+b_0f_1}{M(r, f_2)}=0. \tag{2.4}$$

From the Wiman-Valiron theory (see [4], [6], [7]), we have basic formulas

$$\frac{f_2^{(j)}(z)}{f_2(z)}=\left(\frac{\Upsilon_{f_2}(r)}{z}\right)^j(1+o(1)), \qquad j=1, \cdots, k, \tag{2.5}$$

where $|z|=r$, $|f_2(z)|=M(r, f_2)$, $r\notin E, \int_E \frac{dr}{r}<\infty, \Upsilon_{f_2}(r)$ denotes the centralindex of $f_2(z)$. Substituting (2.4), (2.5) into (2.3), we have

$$\left(\frac{\Upsilon_{f_2}(r)}{z}\right)^k(1+o(1))+B_{k-1}z^{n_{k-1}}\left(\frac{\Upsilon_{f_2}(r)}{z}\right)^{k-1}(1+o(1))+\cdots+B_0z^{n_0}(1+o(1))=o(1). \tag{2.6}$$

Since the solutions of an algebraic equation are continuous functions in its coefficients (see [4, P. 228]). As $r\to\infty$, the solutions of (2.6) are asymptotically equal to the solutions of the algebraic equation

$$\left(\frac{\Upsilon_{f_2}(r)}{z}\right)^k+B_{k-1}z^{n_{k-1}}\left(\frac{\Upsilon_{f_2}(r)}{z}\right)^{k-1}+\cdots+B_0z^{n_0}=0. \tag{2.7}$$

The solution $\Upsilon_{f_2}(r)$ of (2.7) is the centralindex of the solution g of the differential equation with polynomial coefficients

$$g^{(k)}+B_{k-1}z^{n_{k-1}}g^{(k-1)}+\cdots+B_0z^{n_0}g=0.$$

So by Lemma 2 we have $\sigma(g)\leqq 1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}$, and by the Wiman-Valiron theory, we obtain

$$\sigma(g)=\overline{\lim_{r\to\infty}}\frac{\log\Upsilon_{f_2}(r)}{\log r}=\sigma(f_2),$$

$$\sigma(f)=\sigma(f_1+f_2)=\sigma(f_2)\leqq 1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}.$$

LEMMA 4. *Let $A\not\equiv 0$, b_{k-j} $(j=1, \cdots, k)$ be retional functions, b_{k-j} have a pole at ∞ order $n_{k-j}\geqq 0$, $k\geqq 1$, P be a polynomial such that* $\deg P=\beta$, *and f be a meromorphic solution of equation* (1.4). *Then f has only finitely many poles. If $\sigma(f)>\beta$, then $\sigma(f)\leqq 1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}$.*

Proof. By the same proof as in the proof of Lemma 3, we see that poles of f can only occur at poles of b_{k-j} $(j=1, \cdots, k)$. Hence f has only fimitely many poles. If $\sigma(f)>\beta$, we let f_1, f_2 denote the same as in the proof of lemma 3. Now substituting $f=f_1+f_2$ into (1.4), we obtain

$$f_2^{(k)}+b_{k-1}f_2^{(k-1)}+\cdots+b_0f_2=Ae^p-(f_1^{(k)}+b_{k-1}f_1^{(k-1)}+\cdots+b_0f_1)Ae^p-C, \quad (2.8)$$

where $C=f_1^{(k)}+b_{k-1}f_1^{(k-1)}+\cdots+b_0f_1$. Now let z be a point with $|z|=r$ at which $|f_2(z)|=M(r, f_2)$. From the Wiman-Valiron theory (see [4], [6], [7]), (2.5) holds. Now for a given ε, $0<3\varepsilon<\sigma(f)-\beta$, there exists $\{r'_n\}$ $(r'_n\to\infty)$ such that $M(r'_n, f_2)>\exp\{r_n'^{\sigma(f)-\varepsilon}\}$. Setting $\int_E \frac{dr}{r}=\log\delta<\infty$, there exists a point $r_n \in[r'_n, (\delta+1)r'_n]-E$. At such points r_n, we have

$$M(r_n, f_2)\geqq M(r'_n, f_2)>\exp\{r_n'^{\sigma(f)-\varepsilon}\}>\exp\left\{\frac{r_n^{\sigma(f)-\varepsilon}}{(\delta+1)^{\sigma(f)}}\right\}>\exp\{r_n^{\sigma(f)-2\varepsilon}\}.$$

In addition for sufficiently large r_n we have

$$|Ae^p-C|\leqq\exp\{r_n^{\beta+\varepsilon}\}.$$

So

$$\left|\frac{Ae^p-C}{M(r_n, f_2)}\right|\leqq\exp\{r_n^{\beta+\varepsilon}-r_n^{\sigma(f)-2\varepsilon}\}\longrightarrow 0 \qquad (r_n\to\infty).$$

Therefore, at such points $|z_n|=r_n$ $(r_n\in E, |f_2(z_n)|=M(r_n, f_2))$, substituting (2.5) into (2.8), we have

$$\left(\frac{\Upsilon_{f_2}(r_n)}{z_n}\right)^k(1+o(1))+B_{k-1}z_n^{n_{k-1}}\left(\frac{\Upsilon_{f_2}(r_n)}{z_n}\right)^{k-1}(1+o(1))+\cdots+B_0z_n^{n_0}(1+o(1))=o(1). \quad (2.9)$$

As $r_n\to\infty$, at the points r_n, a solution $\Upsilon_{f_2}(r_n)$ of (2.9) is asymptotically equal to a solution of an algebraic equation

$$\left(\frac{\Upsilon_{f_2}(r_n)}{z_n}\right)^k+B_{k-1}z_n^{n_{k-1}}\left(\frac{\Upsilon_{f_2}(r_n)}{z_n}\right)^{k-1}+\cdots+B_0z_n^{n_0}=0. \quad (2.10)$$

Thus

$$\Upsilon_{f_2}(r_n)\sim c_1r_n^{\alpha_1} \quad (2.11)$$

where $c_1\neq0$ is a constant, the possible values of α_1 should coincide with the possible orders of growth of transcendental solutions of equation

$$f_2^{(k)}+B_{k-1}z^{n_{k-1}}f_2^{(k-1)}+\cdots+B_0z^{n_0}f_2=0.$$

But Lemma 3 gives $\alpha_1\leqq1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}$.

On the other hand, differentiating (2.8) gives

$$f_2^{(k+1)}+b_{k-1}f_2^{(k)}+(b'_{k-1}+b_{k-2})f_2^{(k-1)}+\cdots+b'_0f_2=A'e^p+Ap'e^p-C'$$

$$=\left(\frac{A'}{A}+p'\right)(Ae^p-C)+C\left(\frac{A'}{A}+p'\right)-C'$$

$$=\left(\frac{A'}{A}+p'\right)(f_2^{(k)}+b_{k-1}f_2^{(k-1)}+\cdots+b_0f_2)+C\left(\frac{A'}{A}+p'\right)-C',$$

$$f_2^{(k+1)}+\left[b_{k-1}-\left(\frac{A'}{A}+p'\right)\right]f_2^{(k)}+\left[b'_{k-1}+b_{k-2}-\left(\frac{A'}{A}+p'\right)b_{k-1}\right]f_2^{(k-1)}+$$

$$\cdots+\left[b'_0-\left(\frac{A'}{A}+p'\right)b_0\right]f_2=C\left(\frac{A'}{A}+p'\right)-C',$$

i. e. f_2 also solves a linear differential equation with rational coefficients, since f_2 is a transcendental entire function, by the reasoning in [6, P. 106-108], for sufficiently larger r we have $\Upsilon_{f_2}(r)\sim c_2r^{\alpha_2}(r\overline{\in}E)$, with c_2 a constant, α_2 a rational number. But by (2.11), we have $c_1r_n^{\alpha_1}\sim c_2r_n^{\alpha_2}$. So $c_1=c_2$, $\alpha_1=\alpha_2$ must hold. And by Lemma 1 we get $\sigma(f_2)=\alpha_2\leqq 1+\max_{1\leqq j\leqq k}\frac{n_{k-j}}{j}$. Therefore, $\sigma(f)=\sigma(f_2)\leqq 1+\max_{1\leqq j\leqq k}\frac{n_{k-j}}{j}$.

LEMMA 5. *Let b_{k-j} ($j=1, \cdots, k$) be rational functions having a pole at ∞ of order $n_{k-j}\geqq 0$, $k\geqq 1$, $B(z)$ be a meromorphic function with $\sigma(B)=\beta$ satisfying* (1.3). *If all solutions of the differential equation* (1.5) *are meromorphic functions, then $\sigma(f)=\beta$.*

Proof. It is easy to see that $\sigma(f)\geqq\sigma(B)=\beta$ from (1.5). On the other hand, all solutions of (2.2) that is the corresponding homogeneous differential equation of (1.5) are meromorphic functions, we assume that $\{f_1, \cdots, f_k\}$ is a fundamental solution set of (2.2). By Lemma 3 we have $\sigma(f_i)\leqq 1+\max_{1\leqq j\leqq k}\frac{n_{k-j}}{j}$ ($i=1, \cdots, k$).

By variation of parameters, we can write

$$f=A_1(z)f_1+\cdots+A_k(z)f_k, \tag{2.12}$$

where $A_1(z), \cdots, A_k(z)$ are determined by

$$A'_1f_1+\cdots+A'_kf_k=0$$
$$A'_1f'_1+\cdots+A'_kf'_k=0$$
$$\cdots\cdots\cdots\cdots$$
$$A'_1f_1^{(k-1)}+\cdots+A'_kf_k^{(k-1)}=B.$$

Noting that the Wronskian $W(f_1, \cdots, f_k)$ is a differential polynomial in $f_1, \cdots, f_k$ with constant coefficients, it is easy to know that $\sigma(w)\leqq 1+\max_{1\leqq j\leqq k}\frac{n_{k-1}}{j}$. Set

$$w_i=\begin{vmatrix} f_1, & \cdots\cdots, & \overset{(j)}{0}, & \cdots, & f_k \\ \vdots & & & & \vdots \\ f_1^{(k-1)}, & \cdots, & B, & \cdots, & f_k^{(k-1)} \end{vmatrix}=B\cdot g_i, \qquad i=1, \cdots, k,$$

where g_i are differential polynomials in $f_1, \cdots, f_k$ with constant coefficients. So $\sigma(g_i)\leqq 1+\max\limits_{1\leqq j\leqq k}\frac{n_{k-j}}{j}$ also hold. Since $A_i'=\frac{W_i}{W}=\frac{Bg_i}{W}$ and $\sigma(B)=\beta>1+\max\limits_{1\leqq j\leqq k}\frac{n_{k-j}}{j}$, we have $\sigma(A_j')\leqq\beta$, $\sigma(A_i)=\sigma(A_i')\leqq\beta$. And from (2.12) we get $\sigma(f)\leqq\beta$ Hence $\sigma(f)=\beta$ must hold. (It is not difficult to see that we can suppose all A_i are meromorphic functions here.)

LEMMA 6. *Let $U\not\equiv 0$ be a meromorphic function with $\sigma(U)<\infty$, b_{k-j} ($j=1, \cdots, k$) be rational functions. If f is a meromorphic solution of the differential equation*

$$f^{(k)}+b_{k-1}f^{(k-1)}+\cdots+b_0f=U \tag{2.13}$$

such that $\sigma(U)<\sigma(f)<\infty$, then $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$.

Proof. We can write from (2.13)

$$\frac{1}{f}=\frac{1}{U}\left(\frac{f^{(k)}}{f}+b_{k-1}\frac{f^{(k-1)}}{f}+\cdots+b_0\right). \tag{2.14}$$

Since $\sigma(f)<\infty$, we have $m(r, (f^{(j)}/f))=O(\log r)$ ($j=1, \cdots, k$), thus,

$$m\left(r, \frac{1}{f}\right)\leqq m\left(r, \frac{1}{U}\right)+m\left(r, \frac{f^{(k)}}{f}\right)+\cdots+m\left(r, \frac{f'}{f}\right)+O(\log r)$$

$$=m\left(r, \frac{1}{U}\right)+O(\log r). \tag{2.15}$$

Because $b_{k-1}, \cdots, b_0$ are rational functions, $b_{k-1}, \cdots, b_0$ must be analytic at z_0 as $|z_0|$ is sufficiently large. If f has a zero at z_0 of order β ($>k$), then U must have a zero at z_0 of order $\beta-k$. Hence,

$$n\left(r, \frac{1}{f}\right)\leqq k\bar{n}\left(r, \frac{1}{f}\right)+n\left(r, \frac{1}{U}\right)+O(1),$$

and

$$N\left(r, \frac{1}{f}\right)\leqq k\bar{N}\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{U}\right)+O(\log r). \tag{2.16}$$

(2.15) and (2.16) give

$$T(r, f)=T\left(r, \frac{1}{f}\right)+O(1)\leqq k\bar{N}\left(r, \frac{1}{f}\right)+T\left(r, \frac{1}{U}\right)+O(\log r)$$

$$=k\bar{N}\left(r, \frac{1}{f}\right)+T(r, U)+O(\log r). \tag{2.17}$$

Setting $\sigma(f)=\alpha>\sigma(U)$, there exists $\{r_n\}$ ($r_n\to\infty$) such that

$$\lim_{r_n\to\infty}\frac{\log T(r_n, f)}{\log r}=\alpha .$$

For a given ε with $0<2\varepsilon<\alpha-\sigma(U)$, as r_n is sufficiently large, we have

$$T(r_n, f)>r_n^{\alpha-\varepsilon}, \qquad T(r_n, U)<r_n^{\sigma(U)+\varepsilon}.$$

Therefore

$$\frac{T(r_n, U)}{T(r_n, f)}<r_n^{2\varepsilon-(\alpha-\sigma(U))}\longrightarrow 0 \qquad (r_n\to\infty)$$

and

$$T(r_n, U)\leqq\frac{1}{2}T(r_n, f)$$

holds for sufficiently r_n. From (2.17) we obtain

$$T(r_n, f)\leqq 2k\bar{N}\left(r_n, \frac{1}{f}\right)+O(\log r_n)$$

for such r_n. Thus,

$$\sigma(f)=\alpha=\lim_{r_n\to\infty}\frac{\log T(r_n, f)}{\log r_n}\leqq\overline{\lim_{r_n\to\infty}}\frac{\log \bar{N}(r_n, (1/f))}{\log r_n}\leqq\bar{\lambda}(f).$$

So we get $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$.

LEMMA 7. *Let b_{k-j} $(j=1, \cdots, k)$ be rational functions having a pole at ∞ of order $n_{k-j}\geqq 0$, $k\geqq 1$, $B\not\equiv 0$ be a meromorphic function with $\sigma(B)=\beta<\infty$. If all solutions of the differential equation* (1.5) *are meromorphic functions, then* $\lambda(1/f)=\lambda(1/B)$, $\bar{\lambda}(1/f)=\bar{\lambda}(1/B)$,

$$\max\left\{\lambda(f),\ \lambda\left(\frac{1}{f}\right)\right\}\geqq\max\left\{\lambda(B),\ \lambda\left(\frac{1}{B}\right)\right\}. \tag{2.18}$$

Proof. Since b_{k-j} $(j=1, \cdots, k)$ have only finitely many poles, and as $b_{k-1}, \cdots, b_0$ are all analytic at z_0, f has a pole at z_0 of order α if and only if B has a pole at z_0 of order $\alpha+k$, we have $\bar{\lambda}(1/f)=\bar{\lambda}(1/B)$. From $(\alpha+k/2k)=(\alpha/2k)+(1/2)\leqq\alpha$, it follows that

$$\frac{1}{2k}n(r, B)+O(1)\leqq n(r, f)\leqq n(r, B)+O(1),$$

and

$$\frac{1}{2k}N(r, B)+O(\log r)\leqq N(r, f)\leqq N(r, B)+O(\log r).$$

Therefore, $\lambda(1/f)=\lambda(1/B)$,

By the proof of Lemma 5 we know that $\sigma(f)<\infty$. So we can write

$$f=z^{m_1}\frac{H_1}{Q_1}e^{P_1}, \qquad B=z^{m_2}\frac{H_2}{Q_2}e^{P_2}, \tag{2.19}$$

where m_1, m_2 are integers, H_1 and H_2 are canonical products formed respectively with the nonzero zeros of f and B, Q_1 and Q_2 are canonical products formed respectively with the nonzero poles of f and B, P_1 and P_2 are polynomials with $\deg P_1 \leqq \sigma(f)$. $\deg P_2 \leqq \sigma(B)$. Substituting (2.19) into (1.5) we have

$$F(H_1, Q_1) = z^{m_2} \frac{H_2}{Q_2} e^{P_2 - P_1}, \tag{2.20}$$

where F is a rational function in H_1, Q_1 and $H_1^{(j)}$, $Q_1^{(j)}$ $(j=1, \cdots, k)$ with polynomial coefficients. From (2.20), we get

$$\max\{\sigma(H_1), \sigma(Q_1)\} \geqq \sigma(F) = \sigma\left(z^{m_2} \frac{H_2}{Q_2} e^{P_2 - P_1}\right) \geqq \max\{\sigma(H_2), \sigma(Q_2)\}.$$

So (2.18) holds.

LEMMA 8. *Let β be a positive integer and $\beta > 1$, b_{k-j} $(j=1, \cdots, k)$ be rational functions having a pole at ∞ of order $j(\beta-1)$, $k \geqq 1$, g be a meromorphic solution of the homogeneous differential equation*

$$g^{(k)} + b_{k-1} g^{(k-1)} + \cdots + b_0 g = 0. \tag{2.21}$$

Then $\sigma(g) = \beta$.

Proof. Using the same proof as in the proof of Lemma 3, we see that g has only finitely many poles. Now let g_1 denote the sum of the principal parts of all poles of g. Then g_1 is a rational function with $|g_1| = o(r^{-1})$, and $g_2 = g - g_1$ is an entire function. Substituting $g = g_1 + g_2$ into (2.21), we obtain

$$g_2^{(k)} + b_{k-1} g_2^{(k-1)} + \cdots + b_0 g_2 = -(g_1^{(k)} + b_{k-1} g_1^{(k-1)} + \cdots + b_0 g_1). \tag{2.22}$$

If g_2 is a polynomial, then there is only one term $b_0 g_2$ with degree $k(\beta-1) + \deg g_2$ being the highest one in (2.22). This is impossible. So g_2 must be a transcendental entire function. Now, we use the same proof as in the proof of Lemma 3. Let z be a point with $|z| = r$ at which $|g_2(z)| = M(r, g_2)$, $\Upsilon_{g_2}(r)$ denote the centralindex of $f_2(z)$, $E \subset [0, \infty)$ be a set such that $\int_E \frac{dr}{r} < \infty$. Similarly to (2.6), as $r \overline{\in} E$, we have

$$\left(\frac{\Upsilon_{g_2}(r)}{z}\right)^k (1+o(1)) + B_{k-1} z^{\beta-1} \left(\frac{\Upsilon_{g_2}(r)}{z}\right)^{k-1} (1+o(1)) + \cdots + B_0 z^{k(\beta-1)} (1+o(1)) = o(1). \tag{2.24}$$

Set $\sigma(g_2) = \alpha$. Then by the reasoning in [6, P. 106-108] we have $\Upsilon_{g_2}(r) \sim c z^\alpha$ $(|z| = r \overline{\in} E,\ c \neq 0$ is a constant) as $r \to \infty$. Substituting it into (2.24), it is easy to see that the degrees of all terms of (2.24) are respectively

$$k(\alpha-1),\ j(\beta-1) + (k-j)(\alpha-1) \quad (j=1, \cdots, k-1),\ k(\beta-1).$$

From the Wilman-Valiron theory we see that $\alpha = \beta$ is the only possible value.

Therefore, by Lemma 1 we get $\sigma(g_2)=\beta$, and $\sigma(g)=\sigma(g_2)=\beta$.

LEMMA 9. *Let β, b_{k-j} $(j=1, \cdots, k)$ be the same as Lemma 8, $A\not\equiv 0$ be a rational function having a pole at ∞ of order n_A consider the differential equation*

$$g^{(k)}+b_{k-1}g^{(k-1)}+\cdots+b_0 g=A. \tag{2.25}$$

If $n_A<k(\beta-1)$, then every meromorphic solution g of (2.25) is of $\sigma(g)=\beta$. If $n_A\geqq k(\beta-1)$, then all meromorphic solutions of (2.25) satisfy $\sigma(g)=\beta$ except at most one possible. The possible exceptional one g_0 is a rational function.

Proof. Assume g is a meromorphic solution of (2.25). Clearly, g has only finitely many poles. Let g_1 denote the sum of the principal parts of all poles of g. Then g_1 is a rational function, and $g_2=g-g_1$ is an entire function. Now substituting $g=g_1+g_2$ into (2.25), we get

$$g_2^{(k)}+b_{k-1}g_2^{(k-1)}+\cdots+b_0 g_2=A-(g_1^{(k)}+b_{k-1}g_1^{(k-1)}+\cdots+b_0 g_1) \tag{2.26}$$

Divide the discussion into two cases. Case I. $n_A\geqq k(\beta-1)$. In this case, if g_2 is a polynomial solution, thus $g_0=g_1+g_2$ is a rational solution of (2.25). If g_2 is a transcendental entitre function, we can use the same proof as in Lemma 8 to get $\sigma(g)=\sigma(g_1+g_2)=\sigma(g_2)=\beta$.

Case II. $n_A<k(\beta-1)$. In this case, if g_2 is a polynomial, then there is only one term $b_0 g_2$ with degree $k(\beta-1)+\deg g_2$ being the highest one in (2.26). This is impossible. Therefore, g_2 is a transcendental entitre function. Using the same proof as in Lemma 8, we can get $\sigma(g)=\sigma(g_1+g_2)=\sigma(g_2)=\beta$.

We affirm that equation (2.25) can only possess at most one exceptional rational solution g_0. In fact, if $\bar{g}_0$ is the other one, then $\sigma(g_0-\bar{g}_0)<\beta$. But $g_0-\bar{g}_0$ is a solution of the corresponding homogeneous equation (2.21) of (2.25). This contradicts Lemma 8.

LEMMA 10. *Let β, b_{k-j} $(j=1, \cdots, k)$ be the same as Lemma 8, $U\not\equiv 0$ be a meromorphic function with $\sigma(U)<\beta$, If all solutions of the equation*

$$g^{(k)}+b_{k-1}g^{(k-1)}+\cdots+b_0 g=U \tag{2.27}$$

are meromorphic functions, then all solutions of (2.27) satisfy $\sigma(g)=\beta$ except at most one possible. The possible exceptional one $\bar{g}$ is of $\sigma(\bar{g})<\beta$.

Proof. Assume that $\{g_1, \cdots, g_k\}$ is a fundamental solution set of (2.21) that is the corresponding homogeneous differential equation of (2.27). By Lemma 8, we have $\sigma(g_j)=\beta$ $(j=1, \cdots, k)$.

Using the method of variation of parameters as in Lemma 5, we can prove that all solutions of (2.27) are of $\sigma(g)\leqq\beta$.

Using the same proof as in the proof of Lemma 9, we see that (2.27) possesses at most one exceptional solution $\bar{g}$ of order $\sigma(\bar{g})<\beta$, the other solutions

g are all of order $\sigma(g)=\beta$.

§3. Proof of Theorems.

Proof of Theorem 1. (a) By (1.4) we have $\sigma(f)\geqq\sigma(Ae^p)=\beta$. If $\sigma(f)>\beta$, from Lemma 4 we have $\sigma(f)\leqq 1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}$. This is a contradiction. Therefore $\sigma(f)=\beta$. And f has only finitely many poles from Lemma 4.

(b) Set $f=ge^p$. Then $\lambda(f)=\lambda(g)$, $\bar{\lambda}(f)=\bar{\lambda}(g)$. Substituting $f=ge^p$ into (1.4), we have

$$g^{(k)}d_{k-1}g^{(k-1)}+\cdots+d_0 g=A, \tag{3.1}$$

where $d_{k-1},\cdots,d_0$ are rational functions. To work out $\sigma(g)$, we need d_{k-j} $(j=1,\cdots,k)$ in more detailed form. It is easy to check by induction that we have for $m\geqq 2$ (see [5])

$$f^{(m)}=\left\{g^{(m)}+mp'g^{(m-1)}+\sum_{i=2}^{m}[c_m^i(p')^i+H_{i-1}(p')]g^{(m-i)}\right\}e^p, \tag{3.2}$$

where $H_{i-1}(p')$ are differential polynomials in p' and its derivatives of total degree $i-1$ with constant coefficients. It is easy to see that the derivatives of $H_{i-1}(p')$ as to z are of the same form $H_{i-1}(p')\cdot C_m^i$ is the usual notation for the binomial coefficients. (1.4) and (3.2) give

$$\begin{cases} d_{k-j}=b_{k-j}+(k-j-i)b_{k-j+1}p'+\sum_{i=2}^{j}b_{k-j+i}(C_{k-j+i}^i(p')^i+H_{i-1}(p')) \\ \qquad\qquad\qquad\qquad\qquad\qquad (j=2,\cdots,k,\ b_k\equiv 1), \\ d_{k-1}=b_{k-1}+kp'. \end{cases} \tag{3.3}$$

Since $\beta>1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}$, the degree $j(\beta-1)$ of the term $b_kC_k^j(p')^j=C_k^j(p')^j$ $(i=j)$ is the highest one in the first equality of (3.3). Hence d_{k-j} must have a pole at ∞ of order $j(\beta-1)$. If $n_A\geqq k(\beta-1)$, then from Lemma 9, (3.1) may have one exceptional rational solution A_0 the other meromorphic solutions are all of $\sigma(g)=\beta$. By Lemma 6 we have $\bar{\lambda}(g)=\lambda(g)=\sigma(g)=\beta$. Therefore (1.4) may have one exceptional solution $f_0=A_0e^p$ (A_0 is a rational function), the other meromorphic solutions $f=ge^p$ are all of $\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\bar{\lambda}(g)=\beta$. If $n_A<k(\beta-1)$, then from Lemma 9 and Lemma 6, all meromorphic solutions g of (3.1) are of $\bar{\lambda}(g)=\lambda(g)=\sigma(g)=\beta$. Therefore all meromorphic solutions $f=ge^p$ of (1.4) are of

$$\bar{\lambda}(f)=\lambda(f)=\sigma(f)=\beta.$$

Proof of Theorem 2. (a) It is easy to see that $\sigma(f)\geqq\beta$ from (1.4). If $\sigma(f)>\beta$, then $\sigma(f)\leqq 1+\max_{1\leq j\leq k}\frac{n_{k-j}}{j}$ from Lemma 4. And by Lemma 4, f has only finitely many poles.

(b) If $\sigma(f)>\beta$, then $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$ from Lemma 6.

Proof of Theorem 3. (a) From Lemma 5, we have $\sigma\{f\}=\beta$.

(b) By Lemma 7, we see that $\lambda(1/f)=\lambda(1/B)$, $\bar{\lambda}(1/f)=\bar{\lambda}(1/B)$. If $\lambda(B)>\lambda(1/B)$, we have $\lambda(f)\geqq\lambda(B)$ by Lemma 7.

(c) If $\beta>\max\{\lambda(B), \lambda(1/B)\}$ we can write

$$B=z^m\frac{H}{Q}e^{p(z)}=Ue^{p(z)},$$

where m is an integer, H and Q are canonical products formed respectively with the nonzero zeros and nonzero poles of B, $U=z^m(H/Q)$, $\sigma(U)<\beta$, then β must be an integer, $p(z)$ is a polynomial with deg $P=\beta$.

We set $f=ge^p$. Then $\lambda(g)=\lambda(f)$, $\bar{\lambda}(g)=\bar{\lambda}(f)$. Substituting $f=ge^p$ into (1.5), we have

$$g^{(k)}+d_{k-1}g^{(k-1)}+\cdots+d_0g=U \tag{3.4}$$

where $d_{k-1}, \cdots, d_0$ are retional functions.

Using the same proof as in the proof of Theorem 1 (b), we see that d_{k-j} must have a pole at ∞ of order $j(\beta-1)$. Hence from Lemma 10, we see that all meromorphic solutions of (3.4) satisfy $\sigma(g)=\beta$ except at most one possible. The possible exceptional one $\bar{g}$ is of $\sigma(\bar{g})<\beta$. If $\sigma(\bar{g})<\beta$, then $\lambda(\bar{g})<\beta$. If $\sigma(g)=\beta$, by $\sigma(U)<\beta$ and Lemma 6, we have $\bar{\lambda}(g)=\lambda(g)=\sigma(g)=\beta$. Therefore the equation (1.5) may have at most one exceptional solution $f_0=\bar{g}e^p$ with $\lambda(f_0)=\lambda(\bar{g})<\beta$, the other meromorphic solutions $f=ge^p$ of (1.5) are all of $\bar{\lambda}(f)=\lambda(f)=\bar{\lambda}(g)=\beta$.

Proof of Theorem 4. (a) By (1.5), we have $\sigma(f)\geqq\beta$. On the other hand, since all solutions of (1.5) are meromorphic functions, all solutions of (2.2) that is the corresponding homogeneous equation of (1.5) are meromorphic functions. Assume $\{f_1, \cdots, f_k\}$ is fundamental solution set of (2.2). By Lemma 3 we have $\sigma(f_i)\leqq 1+\max_{1\leqq j\leqq k}\frac{n_{k-j}}{j}$ $(i=1, \cdots, k)$.

By variation of parametes, for a solution f of (1.5), we can write

$$f=A_1(z)f_1+\cdots+A_k(z)f_k.$$

Using the same proof as in the proof of Lemma 5 and noting that $\beta\leqq 1+\max_{1\leqq j\leqq k}\frac{n_{k-j}}{j}$, we have $\sigma(A_j)=\sigma(A_j')\leqq 1+\max_{1\leqq j\leqq k}\frac{n_{k-j}}{j}$. Therefore $\beta\leqq\sigma(f)\leqq 1+\max_{1\leqq j\leqq k}\frac{n_{k-j}}{j}$.

(b) By Lemma 7 we have $\lambda(1/f)=\lambda(1/B)$, $\bar{\lambda}(1/f)=\bar{\lambda}(1/B)$, $\max\{\lambda(f), \lambda(1/f)\}\geqq\max\{\lambda(B), \lambda(1/B)\}$. Therefore, $\lambda(f)\geqq\lambda(B)$, if $\lambda(B)=\lambda(1/B)$.

(c) If $\sigma(f)>\sigma(B)$, then by Lemma 6 we have $\bar{\lambda}(f)=\lambda(f)=\sigma(f)$.

§ 4. Examples for the exceptional solution.

Example 1. (concerning the exceptional solution in Theorem 1)

$f_0=ef^{z^3}$ solves $f''+zf'+z^2f=(9z^4+3z^3+z^2+6z)e^{z^3}$. There $1+\max\limits_{1\leq j\leq k}\frac{n_{k-j}}{j}=2$, $\beta=3$, $\deg A=4\geqq k(\beta-1)$. And f_0 satisfies that $\lambda(f_0)=0<\sigma(f_0)=3=\sigma(Ae^P)$.

Example 2. (concerning the exceptional solution in Theorem 3)

$f_0=\sin z\cdot e^{z^2}$ is an exceptional solution of $f''-f=(4z\cos z+4z^2\sin z)e^{z^2}$. There $1+\max\limits_{1\leq j\leq k}\frac{n_{k-j}}{j}=1$, $\beta=\sigma(B)=2$, $\beta>\max\{\lambda(B), \lambda(1/B)\}$. And f_0 satisfies $\sigma(f_0)=2$, $\lambda(f_0)=1<\beta$.

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References

[1] G. Frank and S. Hellerstein, On the meromorphic solutions of non-homogeneous linear differential equations with polynomial coefficients, Proc London Math Soc (3), **53** (1986), 407–428.

[2] Gao Shi-an, On the complex oscillation of solutions of non-homogeneous linear differential equations with polynomial coefficients, Comment. Math. Univ. Sancti Pauli, **38** (1989), No. 1, 11–20.

[3] W. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[4] He Yu-zan and Xiao Xiu-zhi, Algebroid Functions and Ordinary Differential Equations, Science Press, 1988 (in Chinese).

[5] I. Laine, A note on the complex oscillation theory of non-homogeneous linear differential equations, to appear in Comment. Math. Univ. Sancti Pauli.

[6] G. Valiron, Lectures on the general theory of integral functions, Clelsea, New York, 1949.

[7] G. Valiron, Functions analytiques, Presses Universitaires de France, Paris, 1954.

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