T. WATANABE
KODAI MATH. J.
15 (1992), 50—64

ON YAMAZATO'S PROPERTY OF UNIMODAL ONE-SIDED LÉVY PROCESSES

BY TOSHIRO WATANABE

1. Introduction and results.

Let $R=(-\infty, \infty)$ and $R_+=[0, \infty)$. A measure μ on R is said to be *unimodal* with mode a if $\mu(dx) = c\delta_a(dx)+f(x)dx$, where $-\infty<a<\infty$, $c\geqq 0$, $\delta_a(dx)$ is the delta measure at a, and $f(x)$ is non-decreasing for $x<a$ and non-increasing for $x>a$. We say that a unimodal probability measure μ on R_+ has *Yamazato's property, or property Y*, if one of the following conditions holds: (i) μ is unimodal with mode 0; (ii) μ is unimodal with mode $a>0$ and $\mu(dx)=f(x)dx$ with $f(x)$ being such that $f(x)>0$ for $0<x<a$, $f(a-)\geqq f(a+)$, and $\log f(x)$ is concave on $(0, a)$. Let $Z=\{0, \pm 1, \pm 2, \cdots\}$ and $Z_+=\{0, 1, 2, \cdots\}$. A measure $\eta(dx)=\sum_{n=-\infty}^{\infty} p_n\delta_n(dx)$ on Z is said to be *discrete unimodal* with mode $a(a\in Z)$ if p_n is non-decreasing for $n\leqq a$ and non-increasing for $n\geqq a$. We say that a discrete unimodal probability measure $\eta(dx)=\sum_{n=0}^{\infty} p_n\delta_n(dx)$ on Z_+ has *discrete property Y* if one of the following conditions holds: (i) η is discrete unimodal with mode 0; (ii) η is discrete unimodal with mode a, $p_n>0$ for $0\leqq n\leqq a$ and $p_n^2\geqq p_{n+1}p_{n-1}$ for $1\leqq n\leqq a$. A probability measure μ_1 on R (resp. η_1 on Z) is said to be *strongly unimodal* (resp. *discrete strongly unimodal*) if, for every unimodal (resp. discrete unimodal) probability measure μ_2 on R (resp. η_2 on Z), the convolution $\mu_1*\mu_2$ (resp. $\eta_1*\eta_2$) is unimodal (resp. discrete unimodal). Let $\{X_t, t\in[0, \infty)\}$ (resp. $\{Y_t, t\in[0, \infty)\}$) be a Lévy process (that is, a stochastically continuous process with stationary independent increments starting at the origin) not identically zero on R (resp. on Z) and let μ_t (resp. η_t) be the distribution of X_t (resp. Y_t). The process $\{X_t\}$ (resp. $\{Y_t\}$) is said to be unimodal (resp. discrete unimodal) if μ_t (resp. η_t) is unimodal (resp. discrete unimodal) for every t. A unimodal (resp. discrete unimodal) one-sided Lévy process $\{X_t\}$ on R_+ (resp. $\{Y_t\}$ on Z_+) is said to have property Y (resp. discrete property Y) if μ_t (resp. η_t) has property Y (resp. discrete property Y) for every t.

Yamazato [16] proved the unimodality of infinitely divisible distributions of class L, which had been an open problem for a long time. The unimodality had already been shown for one-sided distributions of class L comparatively easily. However, difficulty lay in showing the unimodality for two-sided distributions, because the convolution of two unimodal distributions is not neces-

Received April 1, 1991.

sarily unimodal. Yamazato [16] introduced a notion corresponding to "property Y" for the first time and, using it, gave a sufficient condition (equivalent to Lemma 3.4) that the convolution of two unimodal distributions becomes unimodal again. It was a key for the solution of the unimodality problem. It was proved by him that every one-sided Lévy process of class L has property Y. If the distribution of a Lévy process $\{X_t\}$ is of class L for some $t>0$, then it is of class L for every t and hence unimodal, that is, $\{X_t\}$ is a unimodal Lévy process. There exist one-sided unimodal Lévy processes with the distributions that are not of class L (Watanabe [12]). It is also proved in [12] that some of these processes have property Y. Thus it is a natural problem whether, in general, every one-sided unimodal Lévy process $\{X_t\}$ has property Y. The purpose of this paper is to answer this problem in the affirmative. Owing to this, we can get many two-sided unimodal Lévy processes that are not of class L. In order to prove this, we use an approximation by Lévy processes with discrete distributions. So, we first prove an analogous result for one-sided Lévy processes with discrete distributions.

Our main results are the following two theorems.

THEOREM 1.1. *Every discrete unimodal one-sided Lévy process $\{Y_t\}$ on Z_+ has discrete property Y.*

THEOREM 1.2. *Every unimodal one-sided Lévy process $\{X_t\}$ on R_+ without drift has property Y.*

We obtain the following corollaries from the theorems above combined with Yamazato's lemma [16] or its discrete version (see Lemmas 2.6 and 3.4).

COROLLARY 1.1. *Let $\{Y_t^{(1)}\}$ and $\{Y_t^{(2)}\}$ be independent discrete unimodal one-sided Lévy processes on Z_+. Then $Y_t=Y_t^{(1)}-Y_t^{(2)}$ is a discrete unimodal Lévy process.*

COROLLARY 1.2. *Let $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$ be independent unimodal one-sided Lévy processes on R_+. Let $\{B(t)\}$ be a Brownian motion independent of $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$, and $\sigma\geqq 0$, $\gamma\in R$. Then $X_t=X_t^{(1)}-X_t^{(2)}+\sigma B(t)+\gamma t$ is a unimodal Lévy process.*

We add that many results on the unimodality of Lévy processes are observed in Medgyessy [6], Sato [7], Sato-Yamazato [8], Steutel-van Harn [10], Watanabe [13], Wolfe [14, 15], and Yamazato [17].

We prove Theorem 1.1 and Corollary 1.1 in Section 2. Results of Katti [5] and Watanabe [13] are employed. In Section 3, we prove Theorem 1.2 and Corollary 1.2 using Theorem 1.1. Argument of Forst [2] plays an essential role in our proof. In Section 4, we give two examples related to the property Y of one-sided unimodal infinitely divisible distributions on R_+.

2. Proof of Theorem 1.1.

Let $\{Y_t\}$ be a one-sided Lévy process on Z_+ not identically zero and let η_t be the distribution of Y_t. Then we have

$$\int_0^\infty e^{-zx}\eta_t(dx)=\exp(t\psi(z)), \tag{2.1}$$

$$\psi(z)=\sum_{n=1}^{\infty}(e^{-zn}-1)g_n$$

for every $z\geqq 0$, where $\nu(dx)=\sum_{n=1}^{\infty}g_n\delta_n(dx)$ is a measure with $\sum_{n=1}^{\infty}g_n<\infty$, called the Lévy measure of $\{Y_t\}$. Let $\eta_t(dx)=\sum_{n=0}^{\infty}p_n(t)\delta_n(dx)$. By Katti [5] or Steutel [9], we have a relation:

$$nP_n(t)=t\sum_{j=1}^{n}k_jP_{n-j}(t) \tag{2.2}$$

for $n\geqq 1$, where $k_n=ng_n$ for $n\geqq 1$ and $P_n(t)=p_n(t)/p_0(t)$ for $n\geqq 0$. Define $P_{-1}(t)=0$ and $Q_n(t)=P_n(t)-P_{n-1}(t)$ for $n\geqq 0$. Then we obtain from (2.2) that

$$nQ_n(t)=\sum_{j=1}^{n}(k_jt-1)Q_{n-j}(t) \tag{2.3}$$

for $n\geqq 1$. If $g_1>0$, then $P_n(t)$ and $Q_n(t)$ are polynomials of degree n and the highest coefficients are positive. If $g_1=0$, then $P_1(t)=0$ for every $t>0$ and hence $\{Y_t\}$ is not discrete unimodal. Therefore, hereafter we assume $g_1>0$.

LEMMA 2.1. (Watanabe [13]) *A one-sided Lévy process $\{Y_t\}$ on Z_+ is discrete unimodal if and only if $Q_n(t)$ has a unique positive zero α_n of odd order for every $n\geqq 1$ and α_n is non-decreasing in n.*

Remark 2.1. If a one-sided Lévy process $\{Y_t\}$ on Z_+ is discrete unimodal, then $k_1\geqq k_2$ and $\alpha_1=k_1^{-1}$. (see Corollary 2.1 of Watanabe [13])

Remark 2.2. Let $a(t)$ be the largest mode of the distribution η_t of a discrete unimodal one-sided Lévy process $\{Y_t\}$. Then the proof of Theorem 2.1 of Watanabe [13] (Lemma 2.1) shows that $a(t)=0$ for $0<t<\alpha_1$ and $a(t)=n$ for $\alpha_n\leqq t<\alpha_{n+1}$ for every $n\geqq 1$. This means that $a(t)\geqq n$ is equivalent to $\alpha_n\leqq t$ for every $n\geqq 1$.

Define $A_n(t)=P_n(t)^2-P_{n+1}(t)P_{n-1}(t)$. Then $A_n(t)$ is a polynomial of degree $2n$ and the highest coefficient is positive.

LEMMA 2.2. *Let $\{Y_t\}$ be a one-sided Lévy process on Z_+. Then we have*

$$\frac{d}{dt}P_n(t)=\sum_{j=1}^{n}g_jP_{n-j}(t) \tag{2.4}$$

and

(2.5) $$\frac{d}{dt}A_n(t)=I_1(t)+I_2(t)$$

for every $n\geqq 1$, *where*

$$I_1(t)=P_{n-1}(t)^{-1}A_n(t)\sum_{j=1}^{n-1}g_jP_{n-j-1}(t)$$

and

$$I_2(t)=P_{n-1}(t)^{-1}\sum_{j=2}^{n+1}(g_{j-1}P_n(t)-g_jP_{n-1}(t))C_{nj}(t)$$

with $C_{nj}(t)=P_{n+1-j}(t)P_{n-1}(t)-P_{n-j}(t)P_n(t)$.

Proof. We shall first prove (2.4). Letting $z\to\infty$ in (2.1), we find that $p_0(t)=\exp(-t\sum_{n=1}^{\infty}g_n)$. Hence we get by (2.1) that

(2.6) $$\sum_{n=0}^{\infty}e^{-zn}P_n(t)=\exp\left(t\sum_{n=1}^{\infty}e^{-zn}g_n\right)$$

for every $z\geqq 0$. Differentiating both side of (2.6) in t, we have

(2.7) $$\sum_{n=0}^{\infty}e^{-zn}\frac{d}{dt}P_n(t)=\exp\left(t\sum_{n=1}^{\infty}e^{-zn}g_n\right)\sum_{n=1}^{\infty}e^{-zn}g_n,$$

which implies (2.4). Next we shall show (2.5). Here, abusing the notation, we write simply P_n and A_n for $P_n(t)$ and $A_n(t)$, respectively. Using (2.4), we get

(2.8) $$I_1(t)=P_{n-1}^{-1}A_n\frac{d}{dt}P_{n-1}=P_{n-1}^{-1}P_n^2\frac{d}{dt}P_{n-1}-P_{n+1}\frac{d}{dt}P_{n-1}$$

and

(2.9) $$\begin{aligned}I_2(t)&=P_{n-1}^{-1}\sum_{j=2}^{n+1}(P_{n-1}P_nP_{n+1-j}g_{j-1}-P_{n-1}^2P_{n+1-j}g_j\\&\quad -P_n^2P_{n-j}g_{j-1}+P_{n-1}P_nP_{n-j}g_j)\\&=P_{n-1}^{-1}\left\{P_{n-1}P_n\frac{d}{dt}P_n-P_{n-1}^2\left(\frac{d}{dt}P_{n+1}-g_1P_n\right)\right.\\&\quad\left.-P_n^2\frac{d}{dt}P_{n-1}+P_{n-1}P_n\left(\frac{d}{dt}P_n-g_1P_{n-1}\right)\right\}\\&=2P_n\frac{d}{dt}P_n-P_{n-1}\frac{d}{dt}P_{n+1}-P_{n-1}^{-1}P_n^2\frac{d}{dt}P_{n-1}.\end{aligned}$$

Hence we obtain from (2.8) and (2.9) that

(2.5) $$\frac{d}{dt}A_n=2P_n\frac{d}{dt}P_n-P_{n+1}\frac{d}{dt}P_{n-1}-P_{n-1}\frac{d}{dt}P_{n+1}=I_1+I_2.$$

The proof of Lemma 2.2 is complete.

LEMMA 2.3. (Wolfe [15]) *Let $\{Y_t\}$ be a discrete unimodal one-sided Lévy process on Z_+. Then g_n is non-increasing for $n \geqq 1$.*

Proof of Theorem 1.1. A discrete unimodal one-sided Lévy process $\{Y_t\}$ has discrete property Y if and only if $A_n(t) \geqq 0$ for $1 \leqq n \leqq a(t)$, where $a(t)$ is the largest mode of the distribution of Y_t. From Remark 2.2, this is equivalent to $A_n(t) \geqq 0$ for each $t \geqq \alpha_n$ for every $n \geqq 1$. Hence, in order to prove the discrete property Y of $\{Y_t\}$, it is sufficient to show that

(a) $A_n(t) > 0$ for each $t > \alpha_n$ for every $n \geqq 1$.

We shall prove the assertion (a) by induction in n.

(I) Suppose that $n=1$. We obtain from (2.2) that

$$2A_1(t) = k_1^2 t^2 - k_2 t. \tag{2.10}$$

Hence $A_1(t) > 0$ if and only if $t > k_1^{-2} k_2$. We find from Remark 2.1 that $A_1(t) > 0$ for each $t > \alpha_1 = k_1^{-1} \geqq k_1^{-2} k_2$.

(II) Let $m \geqq 1$. Assume that the assertion (a) is true for $1 \leqq n \leqq m$. We shall prove the following two assertions:

(b) If $A_{m+1}(t_0) = 0$ for some $t_0 > \alpha_{m+1}$, then $(d/dt)A_{m+1}(t_0) > 0$.

(c) If $A_{m+1}(\alpha_{m+1}) = 0$, then there exists $\varepsilon > 0$ such that $(d/dt)A_{m+1}(t) > 0$ for $\alpha_{m+1} < t < \alpha_{m+1} + \varepsilon$.

Our proof of the assertions (b) and (c) is based on the equation (2.5) of Lemma 2.2. First let us prove (b). Let $n = m+1$. Suppose that $A_n(t_0) = 0$ for some $t_0 > \alpha_n$. Then we get

$$I_1(t_0) = 0. \tag{2.11}$$

Since $P_n(t_0) \geqq P_{n-1}(t_0)$ by Lemma 2.1 and $g_{j-1} \geqq g_j$ for every $j \geqq 2$ by Lemma 2.3, we have

$$g_{j-1} P_n(t_0) - g_j P_{n-1}(t_0) \geqq 0 \tag{2.12}$$

for every j $(2 \leqq j \leqq n+1)$. We find from Lemma 2.1 that $\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n < t_0$ and hence, by the assumption, that

$$P_1(t_0)/P_0(t_0) > P_2(t_0)/P_1(t_0) > \cdots > P_n(t_0)/P_{n-1}(t_0). \tag{2.13}$$

This implies that

$$C_{nj}(t_0) > 0 \tag{2.14}$$

for every j $(2 \leqq j \leqq n+1)$. Since $k_1 \geqq k_2$ and $k_1 > 0$ by Remark 2.1, we get

$$g_1 - g_2 = k_1 - 2^{-1} k_2 \geqq 2^{-1} k_1 > 0. \tag{2.15}$$

Hence we obtain from (2.12), (2.14), and (2.15) that

$$\begin{aligned} I_2(t_0) &\geqq P_{n-1}(t_0)^{-1} C_{n2}(t_0)(g_1 P_n(t_0) - g_2 P_{n-1}(t_0)) \\ &\geqq (g_1 - g_2) C_{n2}(t_0) > 0. \end{aligned} \tag{2.16}$$

It follows from (2.11) and (2.16) that

$$\frac{d}{dt}A_n(t_0)>0. \tag{2.17}$$

The proof of the assertion (b) is complete. Next we shall prove the assertion (c). By argument similar to the proof of (b), we find that

$$I_2(t)>0 \tag{2.18}$$

for each $t>\alpha_n$. Let r be the order of the zero α_n of $A_n(t)$. Then we can write

$$A_n(t)=(t-\alpha_n)^r B_r(t), \tag{2.19}$$

where $B_r(t)$ is a polynomial of degree $2n-r$ with $B_r(\alpha_n)\neq 0$. Hence we have

$$\lim_{t\to\alpha_n}(t-\alpha_n)^{1-r}I_1(t)=0. \tag{2.20}$$

We obtain from (2.18), (2.19), and (2.20) that

$$\begin{aligned} 0 &\leqq \lim_{t\to\alpha_n}(t-\alpha_n)^{1-r}(I_1(t)+I_2(t)) \\ &=\lim_{t\to\alpha_n}(t-\alpha_n)^{1-r}\frac{d}{dt}A_n(t) \\ &=rB_r(\alpha_n)>0. \end{aligned} \tag{2.21}$$

This proves the assertion (c).

Now we prove the assertion (a) when $n=m+1$. We get

$$\begin{aligned} A_n(\alpha_n) &= P_n(\alpha_n)Q_n(\alpha_n)-Q_{n+1}(\alpha_n)P_{n-1}(\alpha_n) \\ &= -Q_{n+1}(\alpha_n)P_{n-1}(\alpha_n)\geqq 0, \end{aligned} \tag{2.22}$$

noting that $Q_n(\alpha_n)=0$ and $Q_{n+1}(\alpha_n)\leqq 0$ by Lemma 2.1. Let θ_n be the smallest zero of the polynomial $A_n(t)$ satisfying $\theta_n>\alpha_n$. If such zero does not exist, then the assertion (a) with $n=m+1$ holds trivially from (2.22) and from $A_n(t)\to\infty$ as $t\to\infty$. We shall show that existence of θ_n leads to a contradiction. There are two cases.

(i) Suppose that $A_n(\alpha_n)>0$. Then we have $(d/dt)A_n(\theta_n)\leqq 0$, which contradicts the assertion (b).

(ii) Suppose that $A_n(\alpha_n)=0$. Then we find from the assertion (c) that $(d/dt)A_n(\theta_n)\leqq 0$, which contradicts the assertion (b).

Thus we have proved Theorem 1.1. For the proof of Corollary 1.1, we need several lemmas.

LEMMA 2.4. *Let I be an interval on Z. And let f_n and g_n be non-negative numbers for $n\in I$ such that $\sum_{n\in I} g_n=N<\infty$ and $\sum_{n\in I} f_n g_n=M<\infty$. Then there exists integers j_1 and j_2 in I such that $M\leqq f_{j_1}N$, and $M\geqq f_{j_2}N$.*

Proof. If $N=0$, then $M=0$ and the assertion is trivial. Suppose that $N>0$. Then there exists $j_0 \in I$ such that $g_{j_0}>0$. We shall first show the existence of j_1. Assume that $M/N>f_j$ for every $j\in I$. We find that

$$M=\sum_{j\in I}(M/N)g_j>\sum_{j\in I}f_jg_j=M, \tag{2.23}$$

which is a contradiction. Hence there exists $j_1\in I$ such that $M\leqq f_{j_1}N$. We can prove the existence of j_2 by similar argument. The proof of Lemma 2.4 is complete.

LEMMA 2.5. *Let* $\eta(dx)=\sum_{n=0}^{\infty}p_n\delta_n(dx)$ *be a discrete unimodal probability measure on* Z_+ *with mode* a *and support* Z_+ *having discrete property* Y. *Let* $D_n=p_n/p_{n+1}$. *Then* D_n *is non-decreasing for* $0\leqq n\leqq a$, $D_n\leqq 1$ *for* $0\leqq n\leqq a-1$, *and* $D_n\geqq 1$ *for* $n\geqq a$.

Proof is easy from the definitions of discrete unimodality and discrete property Y.

LEMMA 2.6. *Let* η_1 *and* η_2 *be discrete unimodal probability measures on* Z_+ *having discrete property* Y. *Let* $\tilde{\eta}_2(dx)=\eta_2(-dx)$. *Then the convolution* $\eta_1*\tilde{\eta}_2$ *is discrete unimodal.*

Proof. Let $\eta_1(dx)=\sum_{n=0}^{\infty}p_n\delta_n(dx)$ and $\eta_2(dx)=\sum_{n=0}^{\infty}q_n\delta_n(dx)$. Let a and b be modes of η_1 and η_2, respectively.

First step. Suppose that both η_1 and η_2 have support Z_+. Let $\eta_1*\tilde{\eta}_2(dx)=\sum_{n=-\infty}^{\infty}g_n\delta_n(dx)$. Since

$$g_n=\sum_{j=0}^{\infty}q_jp_{n+j}=\sum_{j=-n}^{\infty}q_jp_{n+j} \quad \text{and} \quad g_{n-1}=\sum_{j=0}^{\infty}q_jp_{n+j-1}=\sum_{j=-n}^{\infty}q_{j+1}p_{n+j},$$

we have

$$\begin{aligned} g_n-g_{n-1}&=\sum_{j=0}^{\infty}q_j(p_{n+j}-p_{n+j-1}) \\ &=\sum_{j=-n}^{\infty}(q_j-q_{j+1})p_{n+j}. \end{aligned} \tag{2.24}$$

The identity (2.24) implies that $g_n-g_{n-1}\leqq 0$ for every $n\geqq a+1$ and $g_n-g_{n-1}\geqq 0$ for every $n\leqq -b$. We shall prove the following assertions which will show discrete unimodality of $\eta_1*\tilde{\eta}_2$.

(i) If $g_m\geqq g_{m-1}$ for some m $(-b+1\leqq m\leqq a-b)$, then $g_n\geqq g_{n-1}$ for every n $(-b\leqq n\leqq m)$.

(ii) If $g_m\leqq g_{m-1}$ for some m $(a-b\leqq m\leqq a)$, then $g_n\leqq g_{n-1}$ for every n $(m\leqq n\leqq a)$.

We shall prove only the assertion (i). The proof of the assertion (ii) is similar. We obtain from (2.24) and Lemma 2.4 that

(2.25)
$$g_{m-1}-g_{m-2}=\sum_{j=-m+1}^{\infty}(q_j-q_{j+1})D_{m+j-1}p_{m+j}$$
$$\geqq D_{m+j_1-1}\sum_{j=-m+1}^{b-1}(q_j-q_{j+1})p_{m+j}$$
$$+D_{m+j_2-1}\sum_{j=b}^{\infty}(q_j-q_{j+1})p_{m+j},$$

where $-m+1\leqq j_1\leqq b-1$ and $b\leqq j_2<\infty$. We shall prove that

(2.26)
$$D_{m+j_2-1}\geqq D_{m+j_1-1}.$$

If $m+j_2-1\leqq a$, then (2.26) follows from Lemma 2.5. Since $m+j_1-1\leqq m+b-2\leqq a-2$, we have $D_{m+j_1-1}\leqq 1$ by Lemma 2.5. Hence, if $m+j_2-1\geqq a+1$, then $D_{m+j_2-1}\geqq 1\geqq D_{m+j_1-1}$ by Lemma 2.5. Thus we have proved (2.26). We obtain from (2.25) and (2.26) that

(2.27)
$$g_{m-1}-g_{m-2}\geqq D_{m+j_2-1}\sum_{j=-m+1}^{\infty}(q_j-q_{j+1})p_{m+j}$$
$$\geqq D_{m+j_2-1}\sum_{j=-m}^{\infty}(q_j-q_{j+1})p_{m+j}$$
$$=D_{m+j_2-1}(g_m-g_{m-1})\geqq 0,$$

noting that $q_{-m}-q_{-m+1}\leqq 0$ since $-m\leqq b-1$. Using this argument repeatedly, we can prove the assertion (i).

Second step. Suppose that η_1 or η_2 has support not equal to Z_+. Then we can find sequences $\eta_n^{(1)}$ and $\eta_n^{(2)}$ of discrete unimodal probability measures on Z_+ such that they have discrete property Y, their supports are Z_+, and $\eta_n^{(1)}$ and $\eta_n^{(2)}$ converge weakly to η_1 and η_2, respectively, as $n\to\infty$. Then $\eta_n^{(1)}*\tilde{\eta}_n^{(2)}$ is discrete unimodal by first step and covergent weakly to $\eta_1*\tilde{\eta}_2$. Hence $\eta_1*\tilde{\eta}_2$ is discrete unimodal. The proof of Lemma 2.6 is complete.

Proof of Corollary 1.1. Let η_t, $\eta_t^{(1)}$, and $\eta_t^{(2)}$ be the distributions of Y_t, $Y_t^{(1)}$, and $-Y_t^{(2)}$, respectively. Since $\{Y_t^{(1)}\}$ and $\{Y_t^{(2)}\}$ are discrete unimodal, they have discrete property Y by Theorem 1.1. Hence $\eta_t=\eta_t^{(1)}*\eta_t^{(2)}$ is discrete unimodal for every $t>0$ by Lemma 2.6. Thus we have proved Corollary 1.1.

3. Proof of Theorem 1.2.

Let μ be a measure on R_+ for which the Laplace transform $L\mu(s)=\int_0^{\infty}e^{-sx}\mu(dx)$ exists for every $s>0$. Define the measure $\eta^{(s)}(\mu, dx)$ on Z_+ for $s>0$ by

(3.1) $$\eta^{(s)}(\mu, dx)=\sum_{n=0}^{\infty} p_n^{(s)}(\mu)\delta_n(dx),$$

where

$$p_n^{(s)}(\mu)=(n\,!)^{-1}\int_0^\infty e^{-sx}(sx)^n\mu(dx).$$

Note that if μ is a probability measure on R_+, then $\eta^{(s)}(\mu, dx)$ is a probability measure on Z_+ for every $s>0$. Let $\{X_t\}$ be a one-sided Lévy process on R_+ not identically zero and without drift. Let μ_t and ν be the distribution and the Lévy measure of $\{X_t\}$, respectively. For $s>0$, define a one-sided Lévy process $\{Y_t^{(s)}\}$ on Z_+ by $Y_t^{(s)}=N(sX_t)$, where $\{N(t)\}$ is a Poisson process with mean t independent of $\{X_t\}$. Then $Y_t^{(s)}$ has the distribution $\eta_t^{(s)}(dx)=\eta^{(s)}(\mu_t, dx)$ and the Lévy measure $\nu^{(s)}$ is given by $\nu^{(s)}(dx)=\sum_{n=1}^\infty p_n^{(s)}(\nu)\delta_n(dx)$, where $p_n^{(s)}(\nu)=(n\,!)^{-1}\int_0^\infty e^{-sx}(sx)^n\nu(dx)$ for $n\geqq 1$.

LEMMA 3.1. (Watanabe [13]) *Let μ be a measure on R_+ for which the Laplace transform exists. Then μ is unimodal on R_+ if and only if $\eta^{(s)}(\mu, dx)$ is discrete unimodal for every $s>0$.*

Let $f(t, n)$ be a positive function of $t\geqq 0$ and $n\in Z_+$. Then $f(t, n)$ is said to satisfy TP_2 condition if, for $0\leqq t_1\leqq t_2$ and $0\leqq n_1\leqq n_2$, $f(t_1, n_1)f(t_2, n_2)\geqq f(t_1, n_2)f(t_2, n_1)$. Let $g(x)$ be a function on R_+ such that $\int_0^\infty f(t, n)|g(t)|dt<\infty$ for every $n\in Z_+$. Define $p_n=\int_0^\infty f(t, n)g(t)dt$. Under the assumption that $f(t, n)$ satisfies TP_2 condition, p_n changes the sign at most once as n increases from 0 to ∞ if $g(x)$ changes the sign at most once as x increases from 0 to ∞ (see Karlin [4] or Dharmadhikari & Joag-dev [1]). Note that $f(t, n)=(n\,!)^{-1}e^{-t}t^n$ satisfies TP_2 condition. This plays an essential role in the following lemma.

LEMMA 3.2. *Let μ be an absolutely continuous probobility measure on R_+ with support containing* 0. *Then μ is unimodal and has property Y if and only if $\eta^{(s)}(\mu, dx)$ is discrete unimodal and has discrete property Y for every $s>0$.*

Proof of the "only if" part of Lemma 3.2. We shall use the method in the proof of Theorem 9.7 of Dharmadhikari & Joag-dev [1]. Let a and $f(x)$ be mode and density function of μ, respectively. If $a=0$, then, by Forst [2], $\eta^{(s)}(\mu, dx)$ is discrete unimodal with mode 0 for every $s>0$. Hence $\eta^{(s)}(\mu, dx)$ has discrete property Y for every $s>0$ trivially. Therefore we assume $a>0$.

First step. Suppose that $f(x)\in C^1((0, \infty))$ and $\log f(x)$ is concave on $(0, a)$. Define $R(n, s, \delta)=1+\delta-p_{n-1}^{(s)}(\mu)/p_n^{(s)}(\mu)$ for $n\geqq 1$, $s>0$, and $\delta<0$. Using integration by parts, we have

(3.2)
$$(n!)^{-1}\int_0^\infty e^{-sx}(sx)^n\{f'(x)+s\delta f(x)\}dx$$
$$=sp_n^{(s)}(\mu)-sp_{n-1}^{(s)}(\mu)+s\delta p_n^{(s)}(\mu)=sp_n^{(s)}(\mu)R(n,\ s,\ \delta).$$

Since $f'(x)\geqq 0$ for $0<x<a$, $f'(x)\leqq 0$ for $x>a$, and $f'(x)/f(x)$ is non-negative and non-increasing for $0<x<a$, $f'(x)+s\delta f(x)=f(x)\{f'(x)/f(x)+s\delta\}$ changes the sign at most once as x increases from 0 to ∞. Hence, for every $\delta<0$ and for every $s>0$, we find from (3.2) that $R(n, s, \delta)$ changes the sign at most once as n increases from 0 to ∞. Let $b(s)$ be the largest mode of $\eta^{(s)}(\mu, dx)$, which is discrete unimodal by Lemma 3.1. Note that $p_{n-1}^{(s)}(\mu)/p_n^{(s)}(\mu)\geqq 1$ for $n\geqq b(s)+1$. Hence, if $R(n, s, \delta)$ changes the sign as n increases from 0 to ∞, it is from positive to negative. It follows that

(3.3)
$$p_{n-1}^{(s)}(\mu)/p_n^{(s)}(\mu)\leqq p_n^{(s)}(\mu)/p_{n+1}^{(s)}(\mu)$$

for $1\leqq n\leqq b(s)$ for every $s>0$, which means the discrete property Y of $\eta^{(s)}(\mu, dx)$. In fact, suppose that $p_{m-1}^{(s)}(\mu)/p_m^{(s)}(\mu)>p_m^{(s)}(\mu)/p_{m+1}^{(s)}(\mu)$ for some m $(1\leqq m\leqq b(s))$ for some $s>0$. Then we can find $\delta<0$ such that $p_{m-1}^{(s)}(\mu)/p_m^{(s)}(\mu)>1+\delta>p_m^{(s)}(\mu)/p_{m+1}^{(s)}(\mu)$. But this implies that $R(n, s, \delta)$ changes the sign from negative to positive as n increases from 0 to ∞. This is a contradiction.

Second step. In general case, we can choose a sequence of probability measures μ_n such that each μ_n satisfies the conditions in first step and μ_n converges weakly to μ as $n\to\infty$. This procedure is made possible by the condition $f(a+)\leqq f(a-)$. Then $\eta^{(s)}(\mu_n, dx)$ is discrete unimodal, has discrete property Y, and converges weakly to $\eta^{(s)}(\mu, dx)$ as $n\to\infty$ for every $s>0$. Hence $\eta^{(s)}(\mu, dx)$ is discrete unimodal and has discrete property Y for every $s>0$.

Proof of the "if" part of Lemma 3.2. Suppose that $\eta^{(s)}(\mu, dx)$ is discrete unimodal with the largest mode $b(s)$ and has discrete property Y for every $s>0$. Define

(3.4)
$$\zeta^{(s)}(dx)=\sum_{n=0}^{\infty}p_n^{(s)}(\mu)\delta_{n/s}(dx)$$

and

(3.5)
$$g_s(x)=s\sum_{n=0}^{\infty}\{p_{n+1}^{(s)}(\mu)\}^{sx-n}\{p_n^{(s)}(\mu)\}^{n+1-sx}I_{E(n)}(x),$$

where $I_{E(n)}(x)$ is the indicator function of the interval $E(n)=[n/s, (n+1)/s)$. Let $f_s(x)=c_s^{-1}g_s(x)$ with $c_s=\int_0^\infty g_s(x)dx$. Then $\mu_s(dx)=f_s(x)dx$ is a unimodal probability measure on R_+ with mode $a(s)=b(s)/s$ and has property Y. Since $\zeta^{(s)}$ converges weakly to μ as $s\to\infty$ by Forst [2], μ_s is convergent weakly to μ as $s\to\infty$. Let $a=\liminf_{s\to\infty} a(s)$. We see that $\mu(dx)=f(x)dx$ is unimodal with mode a and $\log f(x)$ is concave on $(0, a)$ when $a>0$. Because $f_s(x)$ has maximum at $x=a(s)$ and $\log f_s(x)$ is concave on $(0, a(s)]$, we have $f(a+)\leqq f(a-)$ when $a>0$. In fact, by Ibragimov's lemma [3], we can choose a sequence $s(n)$

such that $a(s(n))\to a$ and $f_{s(n)}(x)\to f(x)$ for a.e. $x\in R_+$ as $n\to\infty$. Hence we can find $\varepsilon>0$ such that $a-3\varepsilon>0$, $a\leqq a(s(n))<a+\varepsilon$, $f_{s(n)}(a+\varepsilon)\to f(a+\varepsilon)$, $f_{s(n)}(a-\varepsilon)\to f(a-\varepsilon)$, $f_{s(n)}(a-3\varepsilon)\to f(a-3\varepsilon)$ as $n\to\infty$, and

$$
\begin{aligned}
(3.6)\qquad & f_{s(n)}(a+\varepsilon)f_{s(n)}(a-3\varepsilon)\\
&\leqq f_{s(n)}(a(s(n)))f_{s(n)}(-a(s(n))+2a-2\varepsilon)\\
&\leqq \{f_{s(n)}(a-\varepsilon)\}^2,
\end{aligned}
$$

noting that $0<a-3\varepsilon<-a(s(n))+2a-2\varepsilon<a$ and using the concavity of $\log f_s(x)$ on $(0, a(s)]$. Letting $n\to\infty$ and then $\varepsilon\to 0$ in (3.6), we get $f(a+)\leqq f(a-)$. Thus μ is unimodal and has property Y. The proof of Lemma 3.2 is complete.

LEMMA 3.3. (Watanabe [13]) *A one-sided Lévy process $\{X_t\}$ on R_+ not identically zero and without drift is unimodal if and only if $\{Y_t^{(s)}\}$ is discrete unimodal on Z_+ for every $s>0$.*

Proof of Theorem 1.2. The distribution μ_t of any unimodal one-sided Lévy process $\{X_t\}$ on R_+ not identically zero and without drift does not have a point mass. In fact, suppose that μ_t has a point mass. Then the mode $a(t)$ of μ_t is 0 for every $t>0$, because μ_t has a point mass at 0. But this is a contradiction since $a(t)\to\infty$ as $t\to\infty$ by Theorem 2.1 of Sato [7]. Therefore, Theorem 1.2 follows from Theorem 1.1 and Lemmas 3.2 and 3.3.

LEMMA 3.4. (Yamazato [16]) *Let μ_1 and μ_2 be unimodal probability measures on R_+ which have property Y. Then $\mu_1 * \tilde{\mu}_2$ is unimodal on R, where $\tilde{\mu}_2(dx)=\mu_2(-dx)$.*

Proof of Corollary 1.2. As in the proof of Corollary 1.1, we find from Theorem 1.2 and Lemma 3.4 that $X_t^{(1)}-X_t^{(2)}$ is a unimodal process on R. Since the distribution of $\sigma B(t)+\gamma t$ is Gaussian, it is strongly unimodal for every $t>0$ by Ibragimov [3]. Hence $X_t=X_t^{(1)}-X_t^{(2)}+\sigma B(t)+\gamma t$ is a unimodal process.

4. Examples.

Natural questions arise. Does every unimodal infinitely divisible distribution with support R_+ have property Y? Can every unimodal infinitely divisible distribution on R_+ with property Y be embedded in the distributions of a unimodal one-sided Lévy process? Answers to the both questions are negative, as the following examples show.

Example 4.1. Let μ be an infinitely divisible distribution on R_+ such that

$$
(4.1)\qquad \int_0^\infty e^{-zx}\mu(dx)=\exp(\psi(z)),
$$

$$\phi(z)=\int_0^\infty (e^{-zu}-1)u^{-1}k(u)du$$

for $z\geqq 0$, where

$$\begin{aligned} k(u)&=1 && \text{if } 0<u\leqq 1,\\ &=1+m(u-1) && \text{if } 1\leqq u\leqq 1+\delta,\\ &=0 && \text{if } 1+\delta<u. \end{aligned}$$

Suppose that $0<\delta<1$, $m>0$, and $m^2\delta^3<1$. Then μ is unimodal but does not have property Y.

Proof. Since $\int_0^1 u^{-1}k(u)du=\infty$, μ is absolutely continuous by Tucker [11]. Let $\mu(dx)=f(x)dx$. Then we have a relation by Steutel [9]:

$$\begin{aligned} xf(x)&=\int_0^x f(x-u)k(u)du \\ &=F(x)-F(x-1-\delta)(1+m\delta)+m\int_1^{1+\delta} F(x-u)du \end{aligned} \tag{4.2}$$

for $x>0$, where $F(x)=\int_{-\infty}^x f(u)du$. Hence we find that $f(x)=0$ for $x<0$, $f(x)>0$ for $x>0$, and $f(x)$ is continuous for $x>0$. Differentiating both side of (4.2), we get

$$\begin{aligned} xf'(x)&=-f(x-1-\delta)(1+m\delta)+m\int_1^{1+\delta} f(x-u)du \\ &=-f(x-1-\delta)(1+m\delta)+m(F(x-1)-F(x-1-\delta)) \end{aligned} \tag{4.3}$$

for $x\neq 0$, $1+\delta$. We shall show that μ is unimodal with mode $1+\delta$.

(i) We obtain from (4.3) that

$$xf'(x)=0 \tag{4.4}$$

for $0<x<1$, which means that $f(x)=C$ for $0<x\leqq 1$ with a positive constant C.

(ii) For $1<x<1+\delta$ we have by (4.3)

$$xf'(x)=F(x-1)=mC(x-1)>0, \tag{4.5}$$

which implies that $f'(x)=mC(x-1)x^{-1}\leqq mC\delta$. Hence

$$C<f(x)\leqq f(1)+mC\delta(x-1)\leqq C(1+m\delta^2) \tag{4.6}$$

for $1<x\leqq 1+\delta$.

(iii) For $1+\delta<x\leqq 2+\delta$ we get, by (4.3), (4.6), and the assumption, that

$$xf'(x)\leqq -C(1+m\delta)+mC(1+m\delta^2)\delta=-C(1-m^2\delta^3)<0, \tag{4.7}$$

since $0<x-1-\delta\leqq 1$ and $\delta<x-1\leqq 1+\delta$.

(iv) For $2+\delta<x\leqq 2+2\delta$ we obtain, from (4.3), (4.6), (4.7), and the assumption, that

$$xf'(x)\leqq -C(1+m\delta)+mC(1+m\delta^2)\delta=-C(1-m^2\delta^3)<0, \tag{4.8}$$

since $1<x-1-\delta\leqq 1+\delta$ and $1+\delta<x-1\leqq 1+2\delta<2+\delta$.

(v) Let us prove that $f'(x)<0$ for every $x>2+2\delta$. Suppose that there exists $x_0>2+2\delta$ such that $f'(x_0)=0$. Define

$$s=\inf\{x : f'(x)=0,\ x>2+2\delta\}. \tag{4.9}$$

We find from (4.7) and (4.8) that $f(x)$ is decreasing for $1+\delta<x<s$. Since $1+\delta<s-1-\delta$ and $1+2\delta<s-1$, we obtain from (4.3) that

$$\begin{aligned} 0=sf'(s) &= -f(s-1-\delta)(1+m\delta)+m\int_1^{1+\delta} f(s-u)du \\ &< -f(s-1-\delta)(1+m\delta)+m\delta f(s-1-\delta) \\ &= -f(s-1-\delta)<0, \end{aligned} \tag{4.10}$$

which is a contradiction. Hence $f'(x)<0$ for $x>2+2\delta$.

Thus the proof of the unimodality of μ with mode $1+\delta$ is complete. Also we have proved that $\{f(1)\}^2<f(1+\varepsilon)f(1-\varepsilon)$ for $0<\varepsilon\leqq\delta$ because $f(1)=f(1-\varepsilon)=C$ and $f(1+\varepsilon)>C$ by (i) and (ii). Therefore, μ does not have property Y.

Example 4.2. Let $\{X_t\}$ be a one-sided Lévy process on R_+ with the distribution μ_t such that

$$\int_0^\infty e^{-zx}\mu_t(dx)=\exp(t\psi(z)), \tag{4.11}$$

$$\psi(z)=\int_0^\infty (e^{-zx}-1)x^{-1}k(x)dx$$

for $z\geqq 0$ with $\int_0^\infty (1+x)^{-1}k(x)dx<\infty$ and $\{x : k(x)>0\}=(0, c)$ $(0<c\leqq\infty)$. Assume that $\log k(x)$ is concave on $(0, c)$, $k(0+)=1$, $0<k^*(0+)\leqq\infty$ ($k^*(x)$ is the Radon-Nikodym derivative of $k(x)$).

(i) The distribution μ_t is strongly unimodal if and only if $t\geqq 1$. Hence it has log-concave density for every $t\geqq 1$ by Ibragimov's theorem [3]. But $\{X_t\}$ is not a unimodal process.

(ii) The distribution μ_t is unimodal with mode 0 for $0<t<1-m\beta$ if $m\beta<1$, where $m=\sup_{0<x<c} k^*(x)$ and $\beta=\inf\{x>0 : k(x)<1\}$.

Proof of (i). The first statement in (i) is a direct consequence of Yamazato's theorem [17]. The process $\{X_t\}$ is not unimodal by Corollary 4.2 of Watanabe [13], because $k(x)dx$ is unimodal but $k(x)$ is not non-increasing.

Proof of (ii). We assume for simplicity that $k^*(x)\geqq 0$ for $0<x<c$. In this

case, we find that $0<c<\infty$ and $\beta=c$. General case can be proved by similar argument. Assume $m\beta<1$ and let $0<t<1-m\beta$. The distribution μ_t is absolutely continuous by Tucker [11], since $\int_0^1 x^{-1}k(x)dx=\infty$. Let $\mu_t(dx)=f(x)dx$, where $f(x)$ depends on t. Then we have as in (4.2)

$$xf(x)=t\int_0^c f(x-u)k(u)du \tag{4.12}$$

$$=tF(x)-tF(x-c)k(c-)+t\int_0^c F(x-u)k^*(u)du\,,$$

where $F(x)=\int_{-\infty}^x f(u)du$. Hence we find that $f(x)=0$ for $x<0$, $f(x)>0$ for $x>0$, and $f(x)$ is continuous for $x>0$. By argument similar to Lemma 2.2 of Yamazato [17], we obtain from (4.12) that

$$xf'(x)=(t-1)f(x)-tf(x-c)k(c-)+t\int_0^c f(x-u)k^*(u)du \tag{4.13}$$

except at $x=0$ and c, noting that $m<\infty$. We get by (4.12) that

$$xf(x)=t\int_0^x f(x-u)k(u)du>t\int_0^{x\wedge c} f(x-u)du \tag{4.14}$$

for $x>0$. We find from (4.13), (4.14), and from $k^*(u)\leqq m$ for $0<u<c$ that

$$xf'(x)\leqq(t-1)f(x)+tm\int_0^{x\wedge c} f(x-u)du \tag{4.15}$$

$$<f(x)(t-1+mx)$$

for $0<x<c$ and $c<x$. Noting that $t-1+mx<t-1+mc<0$ for $0<x<c$, we obtain from (4.15) that

(a) $f'(x)<0$ for $0<x<c$.

Let us show that

(b) $f'(x)<0$ for $x>c$.

Suppose, on the contrary, that $f'(x)\geqq 0$ for some $x>c$. Let s be the infimum of such x. Then $f(x)$ is decreasing for $0<x<s$. There are two possible cases.

Case 1. Suppose that $s=c$. Then we have by (4.15)

$$0\leqq cf'(c+)<f(c)(t-1+mc)<0\,. \tag{4.16}$$

This is a contradiction.

Case 2. Suppose that $s>c$. Then we get by (4.13) that

$$0\leqq sf'(s)=(t-1)f(s)-tf(s-c)k(c-)+t\int_0^c f(s-u)k^*(u)du \tag{4.17}$$

$$<-tf(s-c)<0\,,$$

noting that $t-1<0$ and $f(s-u)<f(s-c)$ for $0<u<c$. This is a contradiction.

Thus we have proved the assertion (b). The assertions (a) and (b) imply the unimodality of μ_t with mode 0 for $0<t<1-mc$. The proof of (ii) is complete.

We remark that Lemmas 3.1 and 3.2 show the existence of discrete versions of Examples 4.1 and 4.2.

REFERENCES

[1] DHARMADHIKARI, S. AND JOAG-DEV, K., Unimodality, convexity, and applications, Academic Press, San Diego, (1988).

[2] FORST, G., A characterization of self-decomposable probabilities on the half line, Z. Wahrsch. verw. Gebiete, **49** (1979), 349-352.

[3] IBRAGIMOV, I. A., On the composition of unimodal distributions, Theor. Probability Appl., **1** (1956), 255-260.

[4] KARLIN, S., Total positivity, Vol. 1. Stanford Univ. Press, (1968).

[5] KATTI, S. K., Infinite divisibility of integer valued random variables, Ann. Math. Statist., **38** (1967), 1306-1308.

[6] MEDGYESSY, P., On a new class of unimodal infinitely divisible distribution functions and related topics, Studia Sci. Math. Hungar., **2** (1967), 441-446.

[7] SATO, K., Behavior of modes of a class of processes with independent increments, J. Math. Soc. Japan, **38** (1986), 679-695.

[8] SATO, K. AND YAMAZATO, M., On distribution functions of class L, Z. Wahrsch. verw. Gebiete, **43** (1978), 273-308.

[9] STEUTEL, F. W., On the zeros of infinitely divisible densities, Ann. Math. Statist., **42** (1971), 812-815.

[10] STEUTEL, F. W. AND VAN HARN, K., Discrete analogues of self-decomposability and stability, Ann. Probability, **7** (1979), 893-899.

[11] TUCKER, H. G., Absolute continuity of infinitely divisible distributions, Pacific J. Math., **12** (1962), 1125-1129.

[12] WATANABE, T., Non-symmetric unimodal Lévy processes that are not of class L. Japan. J. Math., **15** (1989), 191-203.

[13] WATANABE, T., On unimodal Lévy processes on the nonnegative integers, Preprint.

[14] WOLFE, S. J., On the unimodality of L functions, Ann. Math. Statist., **42** (1971), 912-918.

[15] WOLFE, S. J., On the unimodality of infinitely divisible distribution functions, Z. Wahrsch. verw. Gebiete, **45** (1978), 329-335.

[16] YAMAZATO, M., Unimodality of infinitely divisible distribution functions of class L, Ann. Probability, **6** (1978), 523-531.

[17] YAMAZATO, M., On strongly unimodal infinitely divisible distributions, Ann. Probability, **10** (1982), 589-601.

NAKAMANDOKORO
AIZUBANGEMACHI
FUKUSHIMAKEN, 969-65,
JAPAN