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ANALYSIS AND TOPOLOGY OF HYPERPLANE COMPLEMENTS: THE GENERALIZED WITT FORMULA

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Introduction.

The *classical Witt formula* which gives the dimensions of the homogeneous components of the free Lie algebra over a finite set, has a nice interpretation as a relation between the topology, i.e. cohomology and homotopy of the complement of a finite set of $\boldsymbol{C}$, and the analysis, i.e. an ordinary linear differential equation with regular singular points at this finite set of $\boldsymbol{C}$.

Such a relation remains true for complements of some hyperplane arrangements such as *complexified Coxeter arrangements* and *fiber-type arrangements*.

Namely, let $\mathcal{A}$ be a finite family of hyperplanes of $\boldsymbol{C}^n$ through the origin and let $M=\boldsymbol{C}^n \setminus \bigcup_{H\in A} H$ be the complement. The cohomology algebra $H^*(M; K)$, where $K=\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ or $\boldsymbol{C}$ is isomorphic to $\mathcal{E}/I$ where $\mathcal{E}$ is the free exterior algebra over $\mathcal{A}$ and I is the ideal defined by some dependence relations between the hyperplanes of $\mathcal{A}$. Moreover:

$$P_M(t)=\sum_{p\geq 0}(\text{rank } H^p(M))t^p=\sum_{x\in L(A)}\mu(x)(-t)^{\text{codim} x}$$

where $L(\mathcal{A})$ is the lattice of intersections hyperplanes ordered by reverse inclusion, $\mu(x)=\mu(0, x)$, μ being the Möbius function. These results are due to P. Orlik and L. Solomon [OS].

The algebra of the integrable logarithmic connections along $\mathcal{A}$ is called the *holonomy Lie algebra* of M and is denoted $\mathcal{G}_M$. T. Kohno [K1] showed that $\mathcal{G}_M=\text{Lib}(A)/\mathcal{N}$ where $|A|=|\mathcal{A}|$ and $\mathcal{N}$ is the ideal defined by some dependence relations between the hyperplanes of $\mathcal{A}$.

Let $\mathcal{L}_M$ be the *Malcev algebra* of M which is obtained (cf Sullivan [S]) from the 1-minimal model of M. Using the mixed Hodge structure on the minimal model, T. Kohno [K2] showed that:

$$\mathcal{G}_M^* \approx \mathcal{L}_M$$

where $\mathcal{G}_M^*$ is the nilpotent completion of $\mathcal{G}_M$.

Then T. Kohno [K3] proved that:

$$\varphi_j(M)=\dim(\Gamma_j\mathcal{G}_M/\Gamma_{j+1}\mathcal{G}_M)=\text{rank}(\Gamma_j\pi_1(M)/\Gamma_{j+1}\pi_1(M))$$

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and

$$\sum_{p\geqq 0}\chi(p)t^{p}=\prod_{j\geqq 1}(1-t^{j})^{-\varphi_{j-1}(M)}$$

where $\chi(p)=\dim\mathcal{E}_{p}(\mathcal{G}_{M})$, the dimension of the pth homogeneous component of the universal algebra of $\mathcal{G}_{M}$.

If $\mathcal{A}$ is a complexified Coxeter arrangement or a fiber-type arrangement, for instance, then the following relation, called *LCS formula* is satisfied ([FR1], [K4], [J]):

$$\sum_{p\geqq 0}\chi(p)t^{p}=\prod_{j\geqq 1}(1-t^{j})^{-\varphi_{j-1}(M)}=(P_{M}(-t))^{-1}$$

In this paper, we begin explaining how the *LCS* formula is a *generalized Witt formula*. However, if for the complexified Coxeter arrangements and the fiber-type arrangements, there are several methods to prove such a formula, M. Falk and R. Randell [FR2] noticed that for an arbitrary arrangement "$\cdots$ the *LCS* formula is virtually impossible to verify $\cdots$". Hence in the last section, following a suggestion of T. Kohno, we develop some method which can be useful to verify the *LCS* formula.

According to K. Aomoto [Ao], we consider the complex $(R.,\partial.)$ defined as follows:

$$R_{k}=\mathrm{Hom}_{\mathcal{E}(\mathcal{G}_{M})}(\mathcal{E}(\mathcal{G}_{M})\otimes_{\boldsymbol{Q}}H^{k}(M;\boldsymbol{Q}),\ \mathcal{E}(\mathcal{G}_{M}))$$

$$\partial_{k}(f)(x\otimes\varphi)=f(x\textstyle\sum_{H\in A}X_{H}\otimes(H\cup\varphi))$$

where X_{H} is the element of the set A defining $\mathcal{G}_{M}$ which corresponds to $H\in\mathcal{A}$ and $\varphi\in H^{k-1}(M;\boldsymbol{Q})$ and $\mathcal{E}(\mathcal{G}_{M})$ denotes the universal enveloping algebra of $\mathcal{G}_{M}$.

If $H_{j}(R.)=0$ for any $j>0$, i.e. the complex is acyclic, then the *LCS* formula is satisfied. In order to prove the acyclicity of this complex, we introduce a structure of graded algebra on $H^{*}(M)$ in such a way that the spectral sequence of the associated filtered complex satisfies:

$$E_{1}^{pq}=0 \quad \text{if } p+q\neq 0$$

As an example, we construct such a filtration for the fiber-type arrangements.

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Remark. Throughout the sections I to III we assume, for simplicity, $K=\boldsymbol{C}$ although the results can be extended to $\boldsymbol{Q}$ or $\boldsymbol{C}$.

I. Classical Witt formula.

I.1. Let $A=\{X_{1},\cdots,X_{l}\}$ and let $Lib(A)=\bigoplus_{n\geqq 1}Lib_{n}(A)$ be the *fre Lie algebra.*

The dimensions of the homogeneous component of $Lib(A)$ are given by the classical Witt formula:

$$\dim Lib_n(A) := N_n = n^{-1} \sum_{d \mid n} \mu(d) l^{nd^{-1}}$$

where μ is the *classical Möbius function* given by:

$$\mu : \boldsymbol{N}^* \longrightarrow \{-1, 0, 1\}$$
$$n \rightsquigarrow 0 \quad \text{if } p^2 \mid n \text{ where } p \text{ is prime}$$
$$n \rightsquigarrow (-1)^k \quad \text{if } n = p_1 \cdots p_k,\ p_i \neq p_j$$

The *enveloping algebra* $\mathcal{E}(Lib(A))$ is the free associative algebra $K\langle A\rangle$ and the canonical morphism

$$Lib(A) \longrightarrow K\langle A\rangle$$

is injective. Moreover, there exists a sequence $\{z_1, z_2, \cdots\}$ of homogeneous Lie elements with nodecreasing degrees such that:

$\{z_1, z_2, \cdots\}$ is a base of the space of the Lie elements

$\{z^{e_1}{}_{i_1} \cdots z^{e_k}{}_{i_k},\ 1 \leqq i_1 < \cdots < i_k,\ k \geqq 1,\ e_1, \cdots, e_k \in N\} \cup \{1\}$ is a base of $K\langle A\rangle$.

Then $\{z_1, \cdots, z_{N_1}\}$ are the degree 1 elements, $\{z_{N_1+1}, \cdots, z_{N_1+N_2}\}$ are the degree 2 elements $\cdots$. The number of possibilities of selecting n objects (repetitions allowed) out of a set of N different ones equals the coefficient of t^n in the power series expansion $(1-t)^{-N}$. On the other hand, $\dim K_n\langle A\rangle = l^n$, then:

$$\prod_{p \geqq 1} (1-t^j)^{-N_j} = \sum_{n \geqq 0} l^n t^n = (1-lt)^{-1} \quad (*)$$

Taking logarithms, differentiating with respect to t, after multiplication by t and by application of the Möbius inversion, we obtain the *classical Witt formula*. Henceforth, in the following we call the relation (*) the *classical Witt formula*.

Remark. Let F_l be the free group on l generators $\alpha_1, \cdots, \alpha_l$ and let $(\Gamma_n F_l)_{n \in N}$ be the lower central series. Then there exist natural isomorphisms as abelian groups:

$$Lib_n(A) \longrightarrow \Gamma_n F_l / \Gamma_{n+1} F_l$$

I. 2. Topological interpretation of the classical Witt formula.

Let $M = \boldsymbol{C} \setminus \{\alpha_1, \cdots, \alpha_l\}$, then:

$$\pi_1(M; *) \approx F_l, \quad \text{then } N_j = \operatorname{rank}(\Gamma_j \pi_1(M; *) / \Gamma_{j+1} \pi_1(M; *))$$
$$H^0(M; \boldsymbol{Z}) \approx \boldsymbol{Z}, \quad H^1(M; \boldsymbol{Z}) \approx \boldsymbol{Z}^l \text{ and } H^i(M; \boldsymbol{Z}) = \{0\} \quad \text{for } i > 1$$

and the *Poincaré polynomial* of M is:

$$P_M(t) = 1 + lt$$

The classical Witt formula (*) establishes a relation between the fundamental group and the cohomology of M:

$$\prod_{j\geq 1}(1-t^{j})^{-N_{j}}=(1-lt)^{-1}=(P_{M}(-t))^{-1}$$

The second term of the Witt formula (*), $\sum_{n\geq 0} l^{n}t^{n}$, is the *Poincaré series* of the enveloping algebra $K\langle A\rangle$ of the free Lie algebra $Lib(A)$.

Consider the first order linear differential equation:

$$dY=\omega Y$$

where $\omega=(\omega^{ij})_{1\leq i,j\leq m}$ and $\omega^{ij}=\sum_{k=1}^{l} a^{ij}{}_{k}d\log(t-\alpha_{k})$, $a^{ij}{}_{k}\in \boldsymbol{C}$. ω is a meromorphic $gl(m;\boldsymbol{C})$-valued 1-form on M and defines a meromorphic connection ∇ on the trivial bundle $\boldsymbol{C}^{m}\times M\to M$ by:

$$\nabla f=df-f\omega$$

where $f: M\to \boldsymbol{C}^{m}$ is a locally defined function. This connection is holomorphic on $\boldsymbol{C}$ and has regular singular points at $\{\alpha_{1}, \cdots, \alpha_{l}\}$.

The transport function:

$$T: PM \longrightarrow Gl(m;\boldsymbol{C})$$

where PM denotes the space of piecewise smooth maps $\gamma: [0, 1]\to M$ is defined as follows: let $\gamma_{t}(s)=\gamma(st)$, then $T(\gamma)$ is the solution at $t=1$ of the equation:

$$dT(\gamma_{t})=T(\gamma_{t})\gamma^{*}\omega, \qquad T(\gamma_{0})=1$$

An explicit formula for T is given in terms of ω by Picard iteration along γ where $\int\omega\omega\cdots\omega$ are *iterated integrals* [Ch]:

$$T(\gamma)=I+\int_{\gamma}\omega+\int_{\gamma}\omega\omega+\cdots$$

Moreover ω is integrable, i.e. $\omega\wedge\omega=d\omega=0$, then the value of T on the path γ depends only on its homotopy class relative to its endpoints.

Thus, T induces the *monodromy representation*:

$$\rho: \pi_{1}(M;*) \longrightarrow Gl(m;\boldsymbol{C})$$

$$\gamma \rightsquigarrow I+\int_{\gamma}\omega+\int_{\gamma}\omega\omega+\cdots$$

Notice that the series converges absolutly.

Examples. 1) $M=\boldsymbol{C}\setminus\{0\}$ and $dY=Pz^{-1}dzY$, $P\in M_{m}(\boldsymbol{C})$. Let γ: $[0, 1]\to M$ where $\gamma(t)=\exp(2i\pi t)$. Then $\gamma^{*}\omega=2i\pi Pdt$ where $\omega=Pz^{-1}dz$, $\int_{\gamma}\overbrace{\omega\cdots\omega}^{r\text{ times}}=(2i\pi P)^{r}/r!$ and $\rho(\gamma)=\exp(2i\pi P)$.

2) $M=C\setminus\{0, 1\}$ and $dY=\omega Y$ where

$$\omega=\begin{pmatrix} 0 & z^{-1}dz & 0 \\ 0 & 0 & (1-z)^{-1}dz \\ 0 & 0 & 0 \end{pmatrix}$$

Let $\Omega=C\setminus\{]-\infty, 0]\cup[1, +\infty[\}$. The matrix

$$u(z)=\begin{pmatrix} 1 & \log z & \operatorname{dilog} z \\ 0 & 1 & \log(1/1-z) \\ 0 & 0 & 0 \end{pmatrix}$$

satisfies the equation $du=\omega u$ on Ω, where we take the principal determination of the logarithms and dilogz is the analytic continuation of the series $\sum_{n\geq 1} t^n/n^2$ which converges for $|t|\leqq 1$.

Let $M=C\setminus\{\alpha_1, \cdots, \alpha_l\}$ and $dY=\omega Y$ as above. We can express ω in terms of $\omega^k=d\log(t-\alpha_k)$, $k=1, \cdots, l$

$$\omega=\sum_{k=1}^{l}\omega^k A^k$$

where each A^k is a constant matrix.

Let $A=\{X_1, \cdots, X_l\}$ and the homomorpeism:

$$\theta : \pi_1(M; *)\longrightarrow C\langle\langle A\rangle\rangle$$

$$\gamma \rightsquigarrow \sum_{1\leq i_1\leq\cdots\leq i_k\leq l}\int_\gamma \omega^{i_1}\cdots\omega^{i_k}X_{i_1}\cdots X_{i_k}$$

The monodromy representation ρ is obtained by substituting $A^{ij}\in gl(m; C)$ to X_{ij}. Finally, let us point out that $Lib(A)$ is the *primitive part* of $C\langle\langle A\rangle\rangle$, then $\sum_{n\geq 0} l^n t^n$ is the Poincaré series of the enveloping algebra of the holonomy Lie algebra $Lib(A)$ of M.

II. Witt formula for the braid groups

1. Braid groups

A *braided n-path* is a set of n paths $f_1, \cdots, f_n$ in R^3 satisfying:

i) for any $t\in[0, 1]$, $f_i(t)\neq f_j(t)$ if $i\neq j$

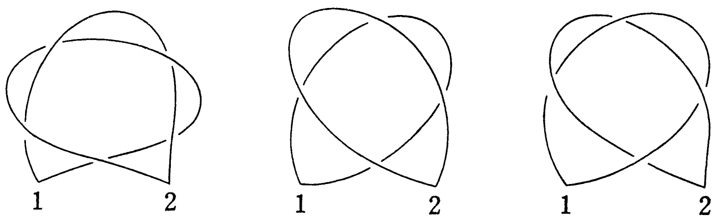
ii) $f_i(0)=i$ for $i=1, \cdots, n$

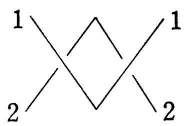
iii) $\{f_1(1), \cdots, f_n(1)\}=\{1, \cdots, n\}$.

Two braided n-paths are equivalent iff it is possible to deform one into the other respecting the three above conditions throughout the deformation.

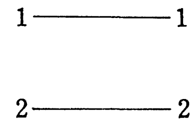
A *n-braid* is an equivalence class of braided n-paths.

Examples.





equivalent to



Two braids can be multiplied and we get the group $B(n)$.

The map $p: B(n) \to S_n$, where S_n is the symmetric group

$$f \rightsquigarrow \sigma_f = \begin{pmatrix} 1 & \cdots & n \\ f_1(1) & \cdots & f_n(1) \end{pmatrix}$$

is a homomorphism.

$\operatorname{Ker} p := C(n)$ is called the *colored (or pure) braid group.*

Let $M = \boldsymbol{C}^n \setminus \bigcup_{1 \leq i \leq j \leq n} H_{ij}$ where $H_{ij} = \{(z_1, \cdots, z_n) \in \boldsymbol{C}^n$ such that $z_i \neq z_j\}$. The set of the hyperplanes H_{ij} is the *complexified Coxeter arrangement of type* A_{n-1}

$$\boldsymbol{C}(n) \approx \pi_1(M; *)$$

2. Cohomology of M

PROPOSITION [Ar]. *Let $\mathcal{A}_n$ be the algebra of holomorphic differential forms generated on $\boldsymbol{C}$ by $\omega^{ij} = d \log (z_i - z_j)$ for $1 \leqq i < j \leqq n$ where $z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n$. Then $\mathcal{A}_n \approx H^*(M; \boldsymbol{C})$ where at ω^{ij} is associated its de Rham cohomology class $[\omega^{ij}]$. A presentation of $\mathcal{A}_n$ can be given by*

—the generators ω^{ij} for $1 \leqq i < j \leqq n$

—the relations $\omega^{ij} \wedge \omega^{jk} + \omega^{jk} \wedge \omega^{ki} + \omega^{ki} \wedge \omega^{ij} = 0$ where i, j, k are distinct and $\omega^{ij} = \omega^{ji}$. ■

COROLLARY. *The Poincaré polynomial $P_M(t)$ is*

$$P_M(t) = \prod_{k=1}^{n-1} (1 + kt). \quad ■$$

Notice that $\{1, \cdots, n-1\}$ is the set of the *exponents* of the Coxeter group of type A_{n-1}.

The proofs of these results follow from the tower of fibrations:

$$\begin{array}{ccc}
C\setminus\{n-1 \text{ points}\} & \longrightarrow & M=M_n=C^n\setminus\cup H_{ij} \\
 & & \downarrow \\
C\setminus\{n-2 \text{ points}\} & \longrightarrow & M_{n-1} \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
C\setminus\{1 \text{ point}\} & \longrightarrow & M_2 \\
 & & \downarrow \\
 & & C^*
\end{array}$$

where $M_k \to M_{k-1}$ is the projection on the last $k-1$ factors. Moreover M is an Eilenberg-MacLane space of type $K(C(n); 1)$.

3. Holonomy Lie algebra of M.

Let $\omega=\sum_{1\leq i<j\leq n} A^{ij}\omega^{ij}$ be the 1-form on M where A^{ij} is a $m\times m$ complex constant matrix and

$$dF=\omega F$$

The solutions are holomorphic $gl(m; C)$-valued functions defined in open sets of M. Let γ be a loop in M,

$$\gamma:[0, 1] \longrightarrow M$$

and let F_0 be a solution in a neighborhood of $\gamma(0)$. By analytic continuation of F_0 along γ, we get the solution F_1 in a neighborhood of $\gamma(0)=\gamma(1)$. This solution is given by the Lappo-Danilevsky formula:

$$F_1(z)=F_0(z)T(\gamma)$$

where $T(\gamma)=\sum_{p\geq 0}\int_\gamma \overbrace{\omega\cdots\omega}^{p \text{ times}}$.

As in the preceding section, let ∇ be the associated connection on the trivial bundle $C^m\times M\to M$.

LEMMA. *The connection* ∇ *is flat iff:*

$$[A^{ij}, A^{ik}+A^{jk}]=0 \quad \textit{for } i, j, k \textit{ distinct}$$

$$[A^{ij}, A^{kl}]=0 \quad \textit{for } i, j, k, l \textit{ distinct}.$$

Proof. The curvature vanishes i.e.:

$$d\omega+\omega\wedge\omega=0$$

iff $\omega\wedge\omega=0$ which is a consequence of the defining relations of $\mathcal{A}_n$ given in the proposition II. 2. ∎

Therefore the monodromy representation is given by:

$$\rho : \pi_1(M; *) \longrightarrow Gl(m; \boldsymbol{C})$$

$$\gamma \rightsquigarrow \rho(\gamma)=\sum\int_\gamma \omega \cdots \omega$$

Now, define $R:=\boldsymbol{C}\langle\langle X_{12}, \cdots, X_{ij}, \cdots, X_{n-1,n}\rangle\rangle/I$, $1\leqq i<j\leqq n$ where I is the ideal generated by the elements:

$$[X_{ij}, X_{ik}+X_{jk}] \quad \text{for } i, j, k \text{ distinct}$$

$$[X_{ij}, X_{kl}] \quad \text{for } i, j, k, l \text{ distinct.}$$

Let $\omega=\sum_{1\leqq i<j\leqq n}\omega^{ij}X_{ij}\in\mathcal{A}_n\otimes R$ which is called *universal integrable 1-form* on M and the homomorphism:

$$\theta : \pi_1(M; *) \longrightarrow R$$

$$\gamma \rightsquigarrow \sum_{k\geqq 0}\int_\gamma \omega^{i_1j_1}\cdots\omega^{i_kj_k}X_{i_1j_1}\cdots X_{i_kj_k}$$

The monodromy representation ρ is obtainod from θ by substituting $A^{ij}\in gl(m; \boldsymbol{C})$ to X_{ij}.

R is a Hopf algebra where the coproduct Δ is defined by $\Delta(X_{ij})=X_{ij}\otimes 1+1\otimes X_{ij}$, i.e. X_{ij} is a primitive element. The primitive part of R, denoted $\mathcal{G}_M$ is called the *holonomy Lie algebra* of M.

THEOREM [K3].

$$\prod_{j\geqq 1}(1-t^j)^{-\varphi_{j-1}(M)}=\sum_{p\geqq 0}\chi(p)t^p=(P_M(-t))^{-1}$$

where $\varphi_j(M)=\text{rank}$ *of* $\Gamma_jC(n)/\Gamma_{j+1}C(n)$

$\sum_{p\geqq 0}\chi(p)t^p$ *is the* Poincaré *series of the enveloping algebra of* $\mathcal{G}_M$
$P_M(t)=\prod_{k=1}^{n-1}(1+kt)$ *is the Poincaré polynomial of* M. ∎

In [K4], T. Kohno extends this result to the other complexified Coxeter arrangements. Notice that the left hand side equality is true for any complement of hypersurfaces.

III. Generalized Witt formula

Let $\mathcal{A}$ be a finite family of codimension 1 linear subspaces of $\boldsymbol{C}^n$ and let $M=\boldsymbol{C}^n\setminus\bigcup_{H\in A}H$.

1. Cohomology of M

E. Brieskorn [B] generalized the result of Arnold as follows; let $\mathcal{A}_M$ be the algebra of holomorphic differential forms on M generated by $\omega = d\log\varphi$ where $\ker\varphi = H$ for $H \in \mathcal{A}$. Then there exists a natural isomorphism:

$$\mathcal{A}_M \longrightarrow H^*(M; C)$$
$$\omega \rightsquigarrow [\omega]$$

Let $\mathcal{E} = \Lambda(\mathcal{A})$ be the free exterior algebra over $\mathcal{A}$. If $J = \{i_1, \cdots, i_p\} \subseteq \{1, \cdots, |\mathcal{A}|\}$, we write $e_J = H_{i_1} \wedge \cdots \wedge H_{i_p}$ and $\partial e_J = \sum_{k=1}^{p} (-1)^{k-1} H_{i_1} \wedge \cdots \wedge \hat{H}_{i_k} \wedge \cdots \wedge H_{i_p}$ where ^ means deletion. J is called *dependent* if $\operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_p}) < p$.

Let $\rho : \mathcal{E} \to \mathcal{A}_M$ be the algebra map which sends H_i to ω^i for any $i = 1, \cdots, |\mathcal{A}|$.

PROPOSITION [OS]. *The map $\rho : \mathcal{E} \to \mathcal{A}_M$ is surjective and $\ker\rho$ is the ideal I generated by $\{\partial e_J,\ J$ dependent$\}$.* ■

PROPOSITION [OS]. *The Poincaré polynomial is*

$$P_M(t) = \sum_{p \geq 0} (\dim H^p(M)) t^p = \sum_{x \in L(A)} \mu(x)(-t)^{\operatorname{codim} x}$$

where $L(\mathcal{A})$ is the intersection lattice ordered by reverse inclusion, $\mu(x) = \mu(0, x)$, μ being the Mobius function. ■

Then $\mathcal{A}_M \approx \mathcal{E}/I$.

Let us consider a linear order on $\mathcal{A}$: $H_1 < H_2 < \cdots$. A set $\{H_{i_1}, \cdots, H_{i_p}\}$ is called a *circuit* if $\operatorname{codim} \bigcap_{j=1}^{p} H_{i_j} = p-1$ and $\operatorname{codim}(H_{i_1} \cap \cdots \cap \hat{H}_{i_k} \cap \cdots \cap H_{i_p}) = p-1$ for any $k = 1, \cdots, p$. Suppose $H_{i_j} < H_{i_k}$ if $j < k$. Then the subset $\{H_{i_1}, \cdots, H_{i_{p-1}}\}$ is called a *broken-circuit*. We define the module $C(M) = \bigoplus_{k \geq 0} C_k(M)$ where $C_0(M) = K$ is the ground ring and $C_k(M)$ is the free module with the base $\{H_{i_1} \wedge \cdots \wedge H_{i_k}\}$ such that $\{H_{i_1}, \cdots, H_{i_k}\}$ does not countain any broken-circuit.

PROPOSITION [JT]. $\mathcal{E} = C(M) \oplus I$. ■

Then $\mathcal{A}_M \approx H^*(M) \approx \mathcal{E}/I \approx C(M)$.

2. Holonomy Lie algebra of M

Let $\omega = \sum \omega^k A_k \in \mathcal{A}_M \otimes gl(m; \boldsymbol{C})$, the summation is taken for all $\omega^k = d\log\varphi_k$ where $\ker\varphi_k = H_k \in \mathcal{A}$. As above, this 1-form defines a connection ∇ on the trivial bundle $\boldsymbol{C}^m \times M \to M$.

Notice that $d\omega = 0$, then ∇ is flat iff

$$\omega \wedge \omega = 0$$
$$\text{i.e.}\quad \sum_{j<k} \omega^j \wedge \omega^k [A^j, A^k] = 0$$

where $[A, B] = A.B - B.A$.

The exterior product of differential forms corresponds to the cup product for the cohomology classes $[\omega]$:

$$H^1(M)\times H^1(M)\longrightarrow H^2(M)$$

Let $\{\nu^1, \cdots, \nu^p\}$ be a base of $H^2(M)$, then:

$$[\omega^j\wedge\omega^k]=[\omega^j]\cup[\omega^k]=\sum_{l=1}^{p} a^{jk}{}_l\nu^l$$

Therefore $\omega\wedge\omega=0$ iff $\sum_{j<k} a^{jk}{}_l[A^j, A^k]=0$, $l=1, \cdots, p$.

Let $\delta: H_2(M)\to H_1(M)\times H_1(M)$ be the dual morphism of the cup product morphism and let $\{X_1, \cdots, X_q\}$ be the dual base of the base $\{[\omega^1], \cdots, [\omega^q]\}$ of $H^1(M)$.

Consider the algebra $R=\boldsymbol{C}\langle\langle X_1, \cdots, X_q\rangle\rangle/I$ where I is the ideal generated by the image of δ, i.e. by the elements $\sum_{j<k} a^{jk}{}_l[X_j, X_k]$, $l=1, \cdots, p$.

Let $\omega=\sum_k \omega^k X^k\in\mathcal{A}_M\otimes R$ and let be the following homomorphism:

$$\theta: \pi_1(M; *)\longrightarrow R$$

$$\gamma \rightsquigarrow \sum_{k\geq 0}\int_\gamma \omega^{i_1}\cdots\omega^{i_k}X_{i_1}\cdots X_{i_k}$$

The monodromy representation:

$$\rho: \pi_1(M; *)\longrightarrow Gl(m; \boldsymbol{C})$$

is again obtained from θ by substituting $A^i\in gl(m; \boldsymbol{C})$ to X_i.

The *holonomy Lie algebra* of M, denoted $\mathcal{G}_M$, is the primitive part of R.

Remark. ω corresponds to the identity of $(H^1(M)(\otimes(H^1(M))^*$ and is independent of the choice of the bases.

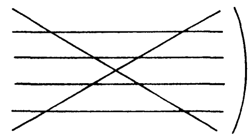
PROPOSITION [K1]. *$\mathcal{G}_M$ is isomorphic to $Lib(X_1, \cdots, X_q)/\mathcal{N}$ where $\mathcal{N}$ is the ideal generated by the elements $[\sum_{j=1}^s X_{i_j}, X_{i_s}]$ such that* codim $\bigcap_{j=1}^s H_{i_j}=2$ *and* codim $(\bigcap_{j=1}^s H_{i_j})\cap H>2$ *for any $H\notin\{H_{i_1}, \cdots, H_{i_s}\}$.* ■

3. Some examples

1. Let $M=C^l\setminus\bigcup_{i=1}^l H_i$ where $H_i=\{z=(z_1, \cdots, z_l)\in C^l$ such that $z_i=0\}$. Then $\mathcal{G}_M=H^1(M)$ and $\dim_C\mathcal{E}_p(\mathcal{G}_M)=l$ if $p=1$ and $=0$ if $p>1$, i.e. $\varphi_j(M)=0$ if $j\geqq 1$. On the other hand, $P_M(t)=(1+t)^l$ and the Witt formula is satisfied.

Remark [Ao]. If $A(t)=\sum_{p\geqq 0} a_p t^p$ and $B(t)=\sum_{p\geqq 0} b_p t^p$ are two power series with real coefficients, define $A(t)\leqq B(t)$ if $a_p\leqq b_p$ for all $p\geqq 0$. Then $(1-t)^{-l}\leqq \sum_{p\geqq 0}\chi(p)t^p\leqq(1-lt)^{-1}$ where $(1-t)^{-l}$ corresponds to the arrangement of coordinate hyperplanes of $\boldsymbol{C}^l$ thus to the holonomy Lie algebra which is abelian and $(1-lt)^{-1}$ to $\boldsymbol{C}\setminus\{l$ points$\}$ thus to the holonomy Lie algebra which is free.

2. This arrangement denoted $\mathcal{X}_1$, [FR], does not satisfy the Witt formula.



4. Fiber-type arrangements

DEFINITION. The arrangement $\mathcal{A}$ in $\boldsymbol{C}^n$ is *fiber-type* if there is a tower of bundle maps:

$$M=M_n \xrightarrow{p_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{p_2} M_1=\boldsymbol{C}^*$$

such that for each k, $2 \leqq k \leqq n$:

(i) M_k is the complement of an arrangement in $\boldsymbol{C}^k$

(ii) p_k is the restriction of a linear map $\boldsymbol{C}^k \to \boldsymbol{C}^{k-1}$

(iii) the fiber F_k of p_k is a copy of $\boldsymbol{C} \setminus \{\text{finite points}\}$.

These numbers $\{a_1, \cdots, a_n\}$ of points removed of $\boldsymbol{C}$ in each fiber are called *exponents* of $\mathcal{A}$ and

$$P_M(t)=\prod_{i=1}^{n}(1+a_i t)$$

Notice that the complexified Coxeter arrangements of type A_{n-1} are fiber-type with exponents $\{1, 2, \cdots, n-1\}$.

THEOREM [J]. *Let the bundle map* $p_n: M \to M_{n-1}=N$, *then the natural map*:

$$\varphi: Lib(A_1) \oplus \mathcal{G}_N \longrightarrow \mathcal{G}_M$$

where $A_1=\{X_1, \cdots, X_{a_1}\}$ *is a graded linear isomorphism.* ■

COROLLARY [J]. *Let* $\mathcal{A}$ *be a fiber-type arrangement of* $\boldsymbol{C}^n$; *then there exists a graded linear isomorphism*:

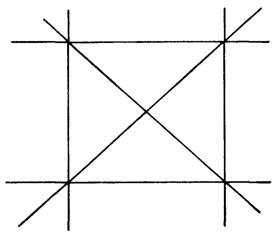
$$\bigoplus_{i=1}^{n} Lib(A_i) \longrightarrow \mathcal{G}_M$$

where $|A_i|=a_i$ *for* $i=1, \cdots, n$. ■

COROLLARY [FR], [J], [K]. *The Witt formula is satisfied for the fiber-type arrangements.* ■

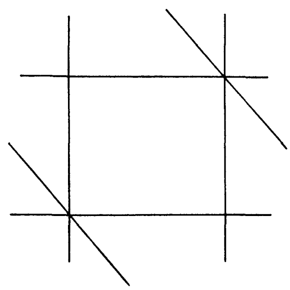
Examples. The arrangement denoted J_2 is not fiber-type does not satisfy the Witt formula [FR] and is free, $P_M(t)=(1+t)(1+3t)^2$.

J_2



The arrangement denoted $\mathcal{X}_2$ is not free, not fiber-type and satisfies the Witt formula, $P_M(t)=(1+t)(1+3t)^2$.

$\mathcal{X}_2$



IV. Aomoto's complex

1. A resolution of $\boldsymbol{Q}$

Let M be the complement of an arrangement $\mathcal{A}$ of $\boldsymbol{C}^n$. Let $\{\omega^1, \cdots, \omega^{|\mathcal{A}|}\}$ be a base of $H^1(M; \boldsymbol{Q})$, e.g. $\omega^i = d \log \varphi_i$ where $\ker \varphi_i = H_i \in \mathcal{A}$ and let $\{X_1, \cdots, X_{|\mathcal{A}|}\}$ be the dual base of $H_1(M; \boldsymbol{Q})$. Let

$$R^k = \mathcal{E}(\mathcal{G}_M) \otimes_{\boldsymbol{Q}} H^k(M; \boldsymbol{Q}), \qquad k \geqq 0$$

and define the $\mathcal{E}(\mathcal{G}_M)$-modules morphism:

$$\delta : R^{k-1} \longrightarrow R^k$$

$$1 \otimes \varphi \rightsquigarrow \textstyle\sum_{i-1} X_i \otimes (\omega^i \cup \varphi), \qquad \varphi \in H^{k-1}(M; \boldsymbol{Q})$$

Let $R_k = \mathrm{Hom}_{\mathcal{E}(\mathcal{G}_M)}(R^k, \mathcal{E}(\mathcal{G}_M))$, $k \geqq 0$ and

$$\partial_k : R_k \longrightarrow R_{k-1}$$

the dual morphism of δ^k.

Then $(R., \partial.)$ is a complex and the differential ∂_k, $k \geqq 0$, does not depend of the choice of the bases. This complex was introduced by K. Aomoto [Ao].

PROPOSITION [K3, 4]. *If the complex* $(R., \partial.)$ *satisfies* $H_j(R.)=0$ *for* $j>0$ *then*:
i) *there is a resolution of* $\boldsymbol{Q}$ *as a* $\mathcal{E}(\mathcal{G}_M)$*-module*:

$$0 \longrightarrow R_n \xrightarrow{\partial_n} R_{n-1} \longrightarrow \cdots \longrightarrow R_1 \xrightarrow{\partial_1} R_0 \xrightarrow{\varepsilon} \boldsymbol{Q} \longrightarrow 0$$

where rank $R_j=\dim H^j(M;\boldsymbol{Q})$ *and* ε *is the augmentation morphism*

ii) *M satisfies the Witt formula*

$$\Pi_{j\geq 1}(1-t^j)^{-\varphi_{j-1}(M)}=\sum_{p\geq 0}\chi(p)t^p=(P_M(-t))^{-1}. \quad \blacksquare$$

The following lemma (which is due to T. Kohno [K4]) is used to prove the next proposition.

Let $\mathcal{B}$ be a subset of $\mathcal{A}$.

LEMMA. *Let* $d^{\mathcal{B}}:\mathcal{E}(\mathcal{G}_M)^{\mathcal{B}}\to\mathcal{E}(\mathcal{G}_M)$ *be the (right)* $\mathcal{E}(\mathcal{G}_M)$*-module morphism defined by*:

$$d^{\mathcal{B}}(u)=\sum_i u_iX_i \quad \textit{for} \quad u=(u_i)\in\mathcal{E}(\mathcal{G}_N)^{\mathcal{B}} \textit{ and } H_i\in\mathcal{B}.$$

Let $\mathcal{V}$ be the degree 1 part of Ker $d^{\mathcal{B}}$ and denote $Lib(A)_{\mathcal{B}}$ the Lie subalgebra of $Lib(A)$ generated by the X_i such that $H_i\in\mathcal{B}$. Let $\varphi:\mathcal{N}^{(2)}\cap Lib(A)_{\mathcal{B}}\to\mathcal{V}$ be the linear map defined by:

$$\varphi(r)=(\partial r/\partial X_i) \quad \text{for } i \text{ such that } H_i\in\mathcal{B}.$$

Then Ker $d^{\mathcal{B}}$ is generated by $\mathcal{V}$ as a $\mathcal{E}(\mathcal{G}_M)$-module. Moreover φ is an isomorphism of vector-spaces. ■

PROPOSITION. ∂_n *is injective.*

Proof. Define a linear order $<$ on the set $\mathcal{A}$ of hyperplanes:

$$H_1<H_2<\cdots<H_{|\mathcal{A}|}$$

and denote $H:=H_{|\mathcal{A}|}$. Let us recall (II 2) that $C(M)=\bigoplus_{k\geq 0}C_k(M)$ where $C_0(M)=\boldsymbol{Q}$ and $C_k(M)$ is the $\boldsymbol{Q}$-linear space with the base $\{H_{i_1}\wedge\cdots\wedge H_{i_k}\}$ such that $\{H_{i_1},\cdots,H_{i_k}\}$ does not countain any broken-circuit. If we assume $i_1<i_2<\cdots<i_k$, then this base is called *BC-standard*. Then the BC-standard base of $C_n(M)$ is $\{\varphi\wedge H$ such that φ belongs to the BC-standard base of $C_{n-1}(M)\}$, $\{1\otimes(\varphi\wedge H)$ for all such $\varphi\}$ is a base of R^n and by duality $\{1\otimes(\varphi\wedge H)^*$ for all such $\varphi\}$ is the dual base of R_n. Therefore for φ and ϕ in the BC-standard base of $C_{n-1}(M)$:

$$\partial_n(1\otimes(\varphi\wedge H)^*)(1\otimes\phi)=(1\otimes(\varphi\wedge H)^*)(\sum_{H_i\in A}X_i\otimes(H_i\wedge\phi))$$

and

$$\partial_n(1\otimes(\varphi\wedge H)^*)(1\otimes\varphi)=\pm X+\sum_\alpha(\pm X_\alpha) \quad \text{where } X_\alpha\neq X$$

$$\partial_n(1\otimes(\varphi\wedge H)^*(1\otimes\phi)=\sum_\beta(\pm X_\beta) \quad \text{where } X_\beta\neq X \text{ and } \phi\neq\varphi.$$

Let $\sum_\varphi f_\varphi\partial_n(1\otimes(\varphi\wedge H)^*))=0$ where $f_\varphi\in\mathcal{E}(\mathcal{G}_M)$ and the sum is over φ in the BC-standard base of $C_{n-1}(M)$. Suppose there exists $f_\phi\neq 0$, then:

$$\Sigma f_{\varphi}\partial_n(1\otimes(\varphi\wedge H)^*)(1\otimes\phi)=f_{\phi}(\pm X+\Sigma_{\alpha}\pm(X_{\alpha}))+\Sigma_{\varphi\neq\phi}f_{\varphi}(\Sigma_{\beta}(\pm X_{\beta}))$$

where X_α and $X_\beta \neq X$. Let us denote $f_\phi = f'_\phi \cdot X_\phi$, then $f_\phi \cdot X = f'_\phi \cdot X_\phi \cdot X$. By the lemma, there exists ν such that $f_\nu = -f'_\phi \cdot X$.

$$f_\nu\partial_n(1\otimes(\nu\wedge H)^*)(1\otimes\nu)=f_\nu\cdot X+f_\nu\Sigma_\gamma(\pm X_\gamma)$$
$$=-f'_\phi\cdot X^2-f'_\phi X\Sigma_\gamma(\pm X_\gamma)$$
$$f_\varphi\partial_n(1\otimes(\varphi\wedge H)^*)(1\otimes\nu)=f_\mu\Sigma_\alpha(\pm X_\alpha).$$

Using lemma, we get:

$$\Sigma_\varphi f_\varphi\partial_n(1\otimes(\varphi\wedge H)^*)(1\otimes\nu)\neq 0$$

and the result follows. ■

THEOREM [K4]. *The complex $(R_\cdot, \partial_\cdot)$ associated with the Coxeter arrangements of type A_l, C_l, D_l are acyclic.* ■

N.B.: The main difficulty of the proof is the injectivity of ∂_n.

IV.2. Acyclicity of the Aomoto's complex

Suppose there exists a structure of graded algebra on $H^*(M; \boldsymbol{Q})$ and let $K.H^*(M; \boldsymbol{Q})$ be the associated decreasing filtration:

$$K_{-p}H^*(M;\boldsymbol{Q})=\{x\in H^*(M;\boldsymbol{Q}),\text{ such that }\deg x\leqq p\}$$

Let:

$$Gr_{-p}H^*(M;\boldsymbol{Q})=K_{-p}H^*(M;\boldsymbol{Q})/K_{-p+1}H^*(M;\boldsymbol{Q})$$

The filtration $K.$ on $H^*(M; \boldsymbol{Q})$ induces a filtration on the complex $(R., \partial.)$ by

$$K_{-p}R_k=\mathrm{Hom}_{\mathcal{E}(\mathcal{G}_M)}(\mathcal{E}(\mathcal{G}_M)\otimes_Q K_{-p}H^*(M;\boldsymbol{Q});\mathcal{E}(\mathcal{G}_M))$$

The natural projection map:

$$\pi_p: K_{-p}H^k(M;\boldsymbol{Q})\longrightarrow K_{-p+1}H^k(M;\boldsymbol{Q})$$

induces an injective morphism of $\mathcal{E}(\mathcal{G}_M)$-modules:

$$K_{-p+1}R_k\longrightarrow K_{-p}R_k$$

LEMMA. *∂ is compatible with the filtration, i.e.*

$$\partial_k(K_{-p}R_k)\subseteqq K_{-p}R_{k-1}$$

Proof. Consider the map:

$$\delta^k:\mathcal{E}(\mathcal{G}_M)\otimes_Q K_{-p}H^{k-1}(M;\boldsymbol{Q})\longrightarrow\mathcal{E}(\mathcal{G}_M)\otimes_Q K_{-p}H^k(M;\boldsymbol{Q})$$
$$1\otimes\varphi\rightsquigarrow\Sigma_{i-1}X_i\otimes([\omega^i]\cup\varphi)$$

and the commutative diagram:

$$\begin{array}{ccc} \mathcal{E}(\mathcal{G}_M)\otimes_Q K_{-p}H^{k-1}(M;\boldsymbol{Q}) & \searrow & \\ \downarrow & & \mathcal{E}(\mathcal{G}_M) \\ \mathcal{E}(\mathcal{G}_M)\otimes_Q K_{-p}H^{k}(M;\boldsymbol{Q}) & \nearrow & \end{array}$$

The result follows. ■

Thus we obtain a structure of filtered complex on $(R., \partial.)$.
Consider the spectral sequence of this filtered complex. We put:

$$Z^{pq}{}_r=\{x\in K_pR_{-p-q},\ \partial x\in K_{p+r}R_{-p-q+1}\}$$

$$B^{pq}{}_r=\{x\in K_pR_{-p-q},\ \text{there exists } y\in K_{-p-r}R_{-p-q-1} \text{ such that } x=\partial y\}$$

$$E^{pq}{}_r=Z^{pq}{}_r/(B^{pq}{}_{r-1}+Z^{p+1,q-1}{}_{r-1})$$

$$Gr_pR_{-p-q}=K_pR_{-p-q}/K_{p+1}R_{-p-q}$$

Then $E^{pq}{}_0=Gr_rR_{-p-q}\approx\mathrm{Hom}_{\mathcal{E}(\mathcal{G}_M)}(\mathcal{E}(\mathcal{G}_M)\otimes_Q Gr_pH^{-p-q}(M;\boldsymbol{Q}),\ \mathcal{E}(\mathcal{G}_M))$.

PROPOSITION. *Suppose that $E^{pq}{}_1=0$ if $p+q\neq 0$. Then the complex $(R.,\partial.)$ is acyclic, i.e. $H_j(R.)=0$ for $j>0$.*

Proof. The differential $d_1\colon E^{pq}{}_1\to E^{p+1,q}{}_1$ is the zero map. Hence we have $E^{pq}{}_2=E^{pq}{}_1$. By induction, we prove that the differential:

$$d_r\colon E^{pq}{}_r \longrightarrow E^{p+r,q-r+1}{}_r$$

is the zero map for $r\geqq 1$. Then

$$E^{pq}{}_1=E^{pq}{}_2=\cdots=E^{pq}{}_\infty$$

Since $E^{pq}{}_\infty=K_pH_{-p-q}(R.)/K_{p+1}H_{-p-q}(R.)$, the result follows. ■

COROLLARY. *If $E^{pq}{}_1=0$ for $p+q\neq 0$, then M satisfies the Witt formula.* ■

IV. 3. Filtration of the Aomoto's complex

Let us consider a chain of $L(\mathcal{A})$ of lenght $r\leqq r(L(\mathcal{A}))$ such that:

$$0=x_0<x_1<\cdots<x_{r-1}<x_r=1$$

Let $\mathcal{A}=\bigcup_{i=1}^r\mathcal{A}_i$ be the disjoint union where $\bigcup_{j=1}^p\mathcal{A}_j=\{H\in\mathcal{A},\ H\not\leqq x_{r-p}\}$. Let us define a linear order $\prec$ on $\mathcal{A}$ such that $H_i\prec H_j$ if $H_i\in\mathcal{A}_i$, $H_j\in\mathcal{A}_j$ and $i<j$. We begin to define a decreasing filtration on the algebra $\mathcal{E}$: $K_{-p}\mathcal{E}=\{H_{i_1}\wedge\cdots\wedge H_{i_q}$, such that there exists $j\in\{1,\cdots,q\}$ where $H_{i_j}\in\mathcal{A}_{r-p+1}\cup\cdots\cup\mathcal{A}_r\}$.

Therefore: $\quad\cdots\mathcal{E}=K_{-r}\mathcal{E}\supset K_{-r+1}\mathcal{E}\supset\cdots\supset K_{-1}\mathcal{E}\supset K_0\mathcal{E}=\boldsymbol{Q}=\cdots$

and

$$K_{-p}\mathcal{E}\wedge K_{-q}\mathcal{E}\subset K_{-(p+q)}\mathcal{E}.$$

Then $\mathcal{E}$ is a filtered algebra with the following gradation:

$$\begin{aligned} Gr_{-p}\mathcal{E}&=K_{-p}\mathcal{E}/K_{-p+1}\mathcal{E} \\ &=\{H_{i_1}\wedge\cdots\wedge H_{i_q} \text{ such that there exists } j\in\{1,\cdots,q\} \\ &\quad \text{where } H_{i_j}\in\mathcal{A}_{r-p+1} \text{ and } H_{i_k}\in\mathcal{A}_1\cup\cdots\cup\mathcal{A}_{r-p+1} \text{ for} \\ &\quad k=1,\cdots,q\}. \end{aligned}$$

In fact, $K_{-p}\mathcal{E}=\oplus_{q\geq 0}(K_{-p}\mathcal{E})_q$ where $(K_{-p}\mathcal{E})_q$ is the $\boldsymbol{Q}$-vector space with the base $H_{i_1}\wedge\cdots\wedge H_{i_q}$ such that $H_{i_1}\prec\cdots\prec H_{i_q}$ and $H_{i_q}\in\mathcal{A}_{r-p+1}\cup\cdots\cup\mathcal{A}_r$ and $(Gr_{-p}\mathcal{E})_q$ is the $\boldsymbol{Q}$-vector space with the base $H_{i_1}\wedge\cdots\wedge H_{i_q}$ such that $H_{i_1}\prec\cdots\prec H_{i_q}$ and $H_{i_q}\in\mathcal{A}_{r-p+1}$.

Let us recall that $\mathcal{E}=C(M)\oplus I$, then $C(M)\approx\mathcal{E}/I$.

PROPOSITION. *The above filtration $K.$ on $\mathcal{E}$ induces a decreasing filtration $K.$ on $C(M)$ (as algebra).*

Proof. Let $a=\wedge_{k=1}^{s}H_{i_k}$ and $b=\wedge_{k=1}^{t}H_{j_k}$ be two elements of the standard BC-base of $K_{-p}C(M)$ (resp. $K_{-q}C(M)$); then $a\in K_{-p}\mathcal{E}$ and $b\in K_{-q}\mathcal{E}$, then $a\wedge b\in K_{-(p+q)}\mathcal{E}$. Moreover $a\wedge b=c+d$ where $c\in C(M)$ and $d\in I$. For simplicity, we denote $a=\wedge_{k=1}^{s}H_k$ and $b=\wedge_{k=s+1}^{s+t}H_k$. Suppose $\{H_{i_1},\cdots,H_{i_l}\}$ is a broken-circuit included in $\{H_i,\ i=1,\cdots,s\}\cup\{H_i,\ i=s+1,\cdots,s+t\}$. Then there exists $H_{i_{l+1}}\succ H_i$ for any $i=1,\cdots,s+t$ such that $\{H_{i_1},\cdots,H_{i_l},H_{i_{l+1}}\}$ is a circuit. Therefore $H_{i_j}\in\mathcal{A}_{k_j}$ and $H_{i_{l+1}}\in\mathcal{A}_{k_{l+1}}$ where $k_{l+1}\geq k_j$ for any $j=1,\cdots,l$. Repeat ing this operation, we finally get a sum of terms without countaining any broken-circuit, i.e. a sum of terms which belong to the BC-base and which is the element c. Moreover $c\in K_{-(p+q)}C(M)$ and $C(M)$ is a filtered algebra. ■

The main application of this result is the following and the proof is straighforward:

PROPOSITION. *Let $\mathcal{A}$ be a fiber-type arrangement of $\boldsymbol{C}^l$ and $K.$ the filtration on $C(M)$ associated with a maximal modular chain of $L(\mathcal{A})$. Then $(Gr_{-p}C(M))_q$ is the $\boldsymbol{Q}$-vector space with the standard BC-base $H_{i_1}\wedge\cdots\wedge H_{i_q}$ such that $H_{i_q}\in\mathcal{A}_{l-p+1}$ and each $H_{i_k}\in\mathcal{A}_{j_k}$ where $k=1,\cdots,q-1$ and $j_k<l-p+1$, j_k pairwise distinct.* ■

COROLLARY. *The spectral sequence of the associated filtered complex $(R.\partial.)$ satisfies $E^{pq}{}_1=0$ for $p+q\neq 0$ and this complex is acyclic.* ■

It is another way to prove that a fiber-type arrangement satisfies the LCS property, i.e. the generalized Witt formula [FR], [J], [K].

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