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ON DISTORTION PROPERTIES OF ANALYTIC OPERATORS

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1. Integral and differential operators.

Let $\tilde{\mathcal{F}}$ denote the class of analytic functions which are holomorphic in the unit disk $E=\{|z|<1\}$. Let $\mathcal{F}$ and $\mathcal{G}$ be its subclasses consisting of f and g normalized by $f(0)=f'(0)-1=0$ and $g(0)=1$, respectively. These subclasses are connected either by $f(z)=zg(z)$ or $f'(z)=g(z)$ with $f(0)=0$ in the sense that $f\in\mathcal{F}$ is equivalent to $g\in\mathcal{G}$.

In a previous paper [4] we have dealt with an integral operator $\mathcal{L}(a)$ defined on $\mathcal{F}$, which is represented by

$$\mathcal{L}(a)f(z)=a\int_I t^{a-2}f(zt)dt$$

where $a>0$ is a parameter and I denotes the unit interval [0, 1]. It has been shown in [6] that the inverse of $\mathcal{L}(a)$ is given by the differential operator

$$\Theta(a)=\frac{1}{a}\left(\frac{d}{d\log z}+a-1\right).$$

On the other hand, Miller [8] and subsequently Altintas [1] discussed a differential operator

$$\Lambda(\alpha)=1+\alpha\frac{d}{d\log z}$$

where $\alpha>0$ is a parameter. The interrelation between Θ and Λ is obvious. In fact, we see that

$$\Lambda(\alpha)=(\alpha+1)\Theta\left(\frac{\alpha+1}{\alpha}\right),\qquad \alpha\neq 0$$

and hence

$$\Theta(a)=\frac{a-1}{a}\Lambda\left(\frac{1}{a-1}\right),\qquad a\neq 1.$$

It is readily verified that $f\in\mathcal{F}$ implies $\mathcal{L}(a)f\in\mathcal{F}$ for $a>0$ and $\Theta(a)f\in\mathcal{F}$ while $g\in\mathcal{G}$ implies $\Lambda(\alpha)g\in\mathcal{G}$. Thus, the normalization concerning $\mathcal{F}$ or $\mathcal{G}$ suits for Θ or Λ, respectively. For this reason $\mathcal{L}$ has initially been defined on $\mathcal{F}$.

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However, the defining representation of $\mathcal{L}(a)$ shows that the normalization $f'(0)=1$ is inessential and further it is applicable to any function of $\mathfrak{F}$ provided $a>1$; in particular, $\mathcal{L}(a)1=a/(a-1)$ for $a>1$. Accordingly, in view of $\Lambda(a)\mathcal{L}(a)=\mathrm{id}$ we have, for instance,

$$\Lambda(\alpha)^{-1}=\frac{1}{\alpha+1}\mathcal{L}\left(\frac{\alpha+1}{\alpha}\right), \qquad \alpha\neq 0.$$

In the following lines, we first remark the effects on the operators subject to a shift of parameter. Then, the main part of the present note is devoted to derive distortion inequalities concerning the real part of some functions which show the effect of the operators. Applications are made to give brief proofs of theorems due to Miller and Altintas and to Chichra [2] in an improved form.

2. Some properties of operators.

As shown in [4], the operator $\mathcal{L}(a)$ with $a>0$ is interpolated into a family of operators $\{\mathcal{L}(a)^{\lambda}\}_{\lambda\geq 0}$ satisfying the additivity $\mathcal{L}(a)^{\lambda}\mathcal{L}(a)^{\mu}=\mathcal{L}(a)^{\lambda+\mu}$. Every member of the family is explicitly represented by

$$\mathcal{L}(a)^{\lambda}f(z)=\frac{a^{\lambda}}{\Gamma(\lambda)}\int_I t^{a-2}\left(\log\frac{1}{t}\right)^{\lambda-1}f(zt)dt,$$

$\mathcal{L}(a)^0$ being understood to be the identity operator. This representation applies to $f\in\mathfrak{F}$ for any $a>0$. However, it applies also to $g\in\mathfrak{G}$ instead of f provided $a>1$. Moreover, we have the following theorem.

THEOREM 1. *If $a>1$, we have for $f\in\mathfrak{F}$*

$$\frac{\mathcal{L}(a-1)^{\lambda}f(z)}{z}=\left(\frac{a-1}{a}\right)^{\lambda}\mathcal{L}(a)^{\lambda}\frac{f(z)}{z},$$

and hence for $g\in\mathfrak{G}$

$$\mathcal{L}(a)^{\lambda}g(z)=\frac{a^{\lambda}}{\Gamma(\lambda)}\int_I t^{a-2}\left(\log\frac{1}{t}\right)^{\lambda-1}g(zt)dt.$$

Proof. Direct calculation yields

$$\begin{aligned}\frac{\mathcal{L}(a-1)^{\lambda}f(z)}{z}&=\frac{1}{z}\frac{(a-1)^{\lambda}}{\Gamma(\lambda)}\int_I t^{a-3}\left(\log\frac{1}{t}\right)^{\lambda-1}f(zt)dt\\&=\left(\frac{a-1}{a}\right)^{\lambda}\frac{a^{\lambda}}{\Gamma(\lambda)}\int_I t^{a-2}\left(\log\frac{1}{t}\right)^{\lambda-1}\frac{f(zt)}{zt}dt\\&=\left(\frac{a-1}{a}\right)^{\lambda}\mathcal{L}(a)^{\lambda}\frac{f(z)}{z}.\end{aligned}$$

Substituting $zg(z)$ for $f(z)$, we get

$$\mathcal{L}(a)^{\lambda} g(z)=\left(\frac{a}{a-1}\right)^{\lambda} \frac{\mathcal{L}(a-1)^{\lambda}(z g(z))}{z}$$

$$=\left(\frac{a}{a-1}\right)^{\lambda} \frac{1}{z} \frac{(a-1)^{\lambda}}{\Gamma(\lambda)} \int_{I} t^{a-3}\left(\log \frac{1}{t}\right)^{\lambda-1} z t g(z t) d t$$

$$=\frac{a^{\lambda}}{\Gamma(\lambda)} \int_{I} t^{a-2}\left(\log \frac{1}{t}\right)^{\lambda-1} g(z t) d t .$$

COROLLARY 1. *If* $a>1$ *and* $f \in \mathcal{F}$, *we have*

$$\frac{\Theta(a-1)^{\lambda} f(z)}{z}=\left(\frac{a}{a-1}\right)^{\lambda} \Theta(a)^{\lambda} \frac{f(z)}{z} .$$

Proof. Substitution of $\Theta(a-1)^{\lambda} f$ for f in the first relation of Theorem 1 yields

$$\frac{f(z)}{z}=\left(\frac{a-1}{a}\right)^{\lambda} \mathcal{L}(a)^{\lambda} \frac{\Theta(a-1)^{\lambda} f(z)}{z},$$

whence follows

$$\frac{\Theta(a-1)^{\lambda} f(z)}{z}=\left(\frac{a}{a-1}\right)^{\lambda} \mathcal{L}(a)^{-\lambda} \frac{f(z)}{z}$$

$$=\left(\frac{a}{a-1}\right)^{\lambda} \Theta(a)^{\lambda} \frac{f(z)}{z} .$$

On the other hand, we have noted in [6] that, in view of $\Theta(a)\mathcal{L}(a)=\mathrm{id}$, $\mathcal{L}(a)^{\lambda}$ is prolongable into $\{\lambda<0\}$ by means of the relation $\mathcal{L}(a)^{\lambda-1}=\Theta(a)\mathcal{L}(a)^{\lambda}$ and accordingly $\Theta(a)^{-\lambda}$ with $\lambda>0$ may be defined by

$$\Theta(a)^{-\lambda}=\mathcal{L}(a)^{\lambda} .$$

This leads us to the following Corollary.

COROLLARY 2. *If* $\alpha>0$, *we have*

$$\Lambda(\alpha)^{-\lambda}=\frac{1}{(\alpha+1)^{\lambda}} \mathcal{L}\left(\frac{\alpha+1}{\alpha}\right)^{\lambda} .$$

3. Distortion concerning the real part.

Several distortion properties with respect to $\mathcal{L}(a)^{\lambda}$ have been derived in previous papers [3] and [4]. Among others the following assertion stated as Lemma 1 is a typical one.

LEMMA 1. *If a function* $f \in \mathcal{F}$ *satisfies in* E *the inequality* $\underline{\beta}<\operatorname{Re}(f(z)/z)<\bar{\beta}$, *then*

$$\underline{\beta}+(1-\underline{\beta})\Phi(\lambda, a)<\mathrm{Re}\frac{\mathcal{L}(a)^{\lambda}f(z)}{z}<\bar{\beta}-(\bar{\beta}-1)\Phi(\lambda, a)$$

where λ and a are positive and Φ is defined by

$$\Phi(\lambda, a)=\frac{a^{\lambda}}{\Gamma(\lambda)}\int_I t^{a-1}\left(\log\frac{1}{t}\right)^{\lambda-1}\frac{1-t}{1+t}dt.$$

In Lemma 1 the bounds $\underline{\beta}$ and $\bar{\beta}$ satisfy necessarily $\underline{\beta}<1<\bar{\beta}$ and $\underline{\beta}$ or $\bar{\beta}$ may be $-\infty$ or $+\infty$, respectively.

We now state a Lemma which designates the class of functions which will be observed in the following lines.

LEMMA 2. *If $f\in\mathfrak{F}$, then any function F of the type*

$$F(z)=\int_{-\infty}^{\infty}\Theta(a)^{\kappa}f(z)d\tau(\kappa), \qquad \int_{-\infty}^{\infty}d\tau(\kappa)=1,$$

belongs to $\mathfrak{F}$, where τ may be complex-valued.

Proof. The normalization of $f\in\mathfrak{F}$ at the origin implies $\mathcal{L}(a)^{-\kappa}f\in\mathfrak{F}$ for any κ, whence follows $\Theta(a)^{\kappa}f\in\mathfrak{F}$. This yields $F\in\mathfrak{F}$.

THEOREM 2. *For any $f\in\mathfrak{F}$, the function F of the type given in Lemma 2 yields*

$$\mathcal{L}(a)^{\lambda}F(z)=\int_{-\infty}^{\infty}\Theta(a)^{\kappa}f(z)d\tau(\kappa+\lambda)$$

and the statement of Lemma 1 holds for F instead of f.

Proof. The first part follows from the relation

$$\mathcal{L}(a)^{\lambda}\Theta(a)^{\kappa}=\mathcal{L}(a)^{\lambda}\Theta(a)^{\lambda}\Theta(a)^{\kappa-\lambda}=\Theta(a)^{\kappa-\lambda},$$

while the second part is an immediate consequence of Lemma 1.

In terms of $g\in\mathfrak{G}$ and Λ, Theorem 2 is brought into the following form.

THEOREM 3. *For any $g\in\mathfrak{G}$, the function G of the type*

$$G(z)=\int_{-\infty}^{\infty}\Lambda(\alpha)^{\kappa}g(z)d\tau(\kappa), \qquad \int_{-\infty}^{\infty}d\tau(\kappa)=1,$$

yields

$$\Lambda(\alpha)^{-\lambda}G(z)=\int_{-\infty}^{\infty}\Lambda(\alpha)^{\kappa}g(z)d\tau(\kappa+\lambda)\in\mathfrak{G}.$$

If both α and λ are positive, then $\underline{\beta}<\mathrm{Re}\,G(z)<\bar{\beta}$ implies

$$\underline{\beta}+(1-\underline{\beta})\Phi\left(\lambda, \frac{1}{\alpha}\right)<\mathrm{Re}\, \Lambda(\alpha)^{-\lambda} G(z)<\bar{\beta}-(\bar{\beta}-1)\Phi\left(\lambda, \frac{1}{\alpha}\right),$$

where Φ is the quantity given in Lemma 1.

Proof. Put $g(z)=f(z)/z$ and $\alpha=1/a$. Then, based on Corollary 1 of Theorem 1, we get for F in Theorem 2

$$\begin{aligned}\frac{F(z)}{z}&=\int_{-\infty}^{\infty}\frac{\Theta(a)^{\kappa}f(z)}{z}d\tau(\kappa)\\ &=\int_{-\infty}^{\infty}\left(\frac{a+1}{a}\right)^{\kappa}\Theta(a+1)^{\kappa}g(z)d\tau(\kappa)\\ &=\int_{-\infty}^{\infty}\Lambda(\alpha)^{\kappa}g(z)d\tau(\kappa),\end{aligned}$$

while, based on Theorem 1, we get

$$\frac{\mathcal{L}(a)^{\lambda}F(z)}{z}=\left(\frac{a}{a+1}\right)^{\lambda}\mathcal{L}(a+1)^{\lambda}\frac{F(z)}{z}=\Lambda(\alpha)^{-\lambda}\frac{F(z)}{z}.$$

Consequently, by putting $F(z)=zG(z)$, the desired result follows from Theorem 2.

4. Illustrative examples.

We now illustrate particular cases where $\tau(\kappa)$ contained in Theorem 2 or 3 possesses discrete spectra laid only on $\{\kappa\}_{\kappa=k}^{K}$ with $k\geqq 1$. The integral with respect to τ then reduces to a corresponding sum, among which the special case $\lambda=1$ has been observed in a previous note [5].

If we take $\lambda=k$, then the functions referred to in Theorems 2 and 3 become

$$F(z)=\sum_{\kappa=k}^{K}A_{\kappa}\Theta(a)^{\kappa}f(z),\qquad \mathcal{L}(a)^{k}F(z)=\sum_{\kappa=0}^{K-k}A_{\kappa+K}\Theta(a)^{\kappa}f(z);$$

$$G(z)=\sum_{\kappa=k}^{K}B_{\kappa}\Lambda(\alpha)^{\kappa}g(z),\qquad \Lambda(\alpha)^{-k}G(z)=\sum_{\kappa=0}^{K-k}B_{\kappa+k}\Lambda(\alpha)^{\kappa}g(z).$$

Here, the constant factors A's and B's may take complex values, provided they satisfy

$$\sum_{\kappa=k}^{K}A_{\kappa}=1,\qquad \sum_{\kappa=k}^{K}B_{\kappa}=1.$$

The factor Φ involved in the bounds is given by

$$\Phi(\lambda, a)=\frac{a^{\lambda}}{\Gamma(\lambda)}\int_{I}t^{a-1}\left(\log\frac{1}{t}\right)^{\lambda-1}\frac{1-t}{1+t}dt,$$

which can be expanded in series form

$$\Phi(\lambda,\ a)=1-2a^{\lambda}\sum_{\nu=2}^{\infty}\frac{(-1)^{\nu}}{(\nu+a-1)^{\lambda}}.$$

By making use of the generalized Riemann zeta function $\zeta(\lambda;\ c)=\sum_{\nu=1}^{\infty}(\nu+c)^{-\lambda}$, the right-hand member of the last expression can be brought into a concrete form

$$\Phi(\lambda,\ a)=1-2\left(\frac{a}{2}\right)^{\lambda}\left(\zeta\left(\lambda;\ \frac{a+1}{2}\right)-\zeta\left(\lambda;\ \frac{a+2}{2}\right)\right)$$

for $\lambda\neq 1$, while $\Phi(1,\ a)$ is regarded as the limit of $\Phi(\lambda,\ a)$ when λ tends to 1.

As indicated in [4], $\Phi(\lambda,\ a)$ with fixed $a>0$ is a strictly increasing function of λ and $\Phi(+0,\ a)=0$, $\Phi(\infty,\ a)=1$, while $\Phi(\lambda,\ a)$ with fixed $\lambda>0$ is a strictly decreasing function of a and $\Phi(\lambda,\ +0)=1$, $\Phi(\lambda,\ \infty)=0$. For lower integral values of $\lambda,\ a$ we have, for instance,

$$\Phi(1,\ 1)=2\log 2-1, \qquad \Phi(1,\ 2)=3-4\log 2,$$

$$\Phi(1,\ k)=1-2(-1)^{k}k\left(\log 2-\sum_{\nu=1}^{k}\frac{(-1)^{\nu-1}}{\nu}\right),$$

$$\Phi(2,\ 1)=\frac{\pi^{2}}{6}-1, \qquad \Phi(4,\ 1)=\frac{7\pi^{4}}{360}-1.$$

We now observe a simple case of $K=2$ and $k=1$. Then, the generic forms of $F\in\mathfrak{F}$ and $G\in\mathfrak{G}$ are given by

$$\begin{aligned}F(z)&=(1-A)\Theta(a)f(z)+A\Theta(a)^{2}f(z)\\&=\frac{1}{a^{2}}(a-1)(a-A)f(z)\\&\qquad+(a+(a-1)A)zf'(z)+Az^{2}f''(z)),\\G(z)&=(1-B)\Lambda(\alpha)g(z)+B\Lambda(\alpha)^{2}g(z)\\&=g(z)+\alpha(1+(1+\alpha)B)zg'(z)+\alpha^{2}Bz^{2}g''(z).\end{aligned}$$

Accordingly, we have

$$\begin{aligned}\mathcal{L}(a)F(z)&=(1-A)f(z)+A\Theta(a)f(z)\\&=\frac{1}{a}((a-A)f(z)+Azf'(z)),\\\Lambda(\alpha)^{-1}G(z)&=(1-B)g(z)+B\Lambda(\alpha)g(z)\\&=g(z)+\alpha Bzg'(z).\end{aligned}$$

The assertion of Theorems 2 and 3 are, of course, applicable to these expressions where A and B are arbitrary complex constants.

For instance, by taking $B=-is/\alpha$ with real s, we have

$$\mathrm{Re}\, G(z)=\mathrm{Re}\,(g(z)+\alpha zg'(z))+s\,\mathrm{Im}\,((1+\alpha)zg'(z)+\alpha z^2g''(z))$$

and

$$\mathrm{Re}\, \Lambda(\alpha)^{-1}G(z)=\mathrm{Re}\, g(z)+s\,\mathrm{Im}\, zg'(z).$$

Thus, a result due to Miller [8] and Altintas [1] stating that $\mathrm{Re}\,(g(z)+\alpha zg'(z)) >\underline{\beta}$ with $\alpha>0$ implies $\mathrm{Re}\, g(z)>\underline{\beta}$ is an immediate consequence of Theorem 3 for this particular G with $s=0$. Moreover, the same assumption implies

$$\mathrm{Re}\, g(z)>\underline{\beta}+(1-\underline{\beta})q(\alpha)$$

where q is given by

$$q(\alpha)=\Phi\Big(1, \frac{1}{\alpha}\Big)=\int_I t^{1/\alpha-1}\frac{1-t}{1+t}\,dt.$$

Further, if we take $s\underline{\beta}$ and $s\bar{\beta}$ with $s>0$ instead of $\underline{\beta}$ and $\bar{\beta}$, respectively, in Theorem 3 and let s tend to $+\infty$, the we obtain a corresponding result. In fact, we can assert that

$$\underline{\beta}<\mathrm{Im}\,((1+\alpha)zg'(z)+\alpha z^2g''(z))<\bar{\beta}$$

where necessarily $\underline{\beta}<0<\bar{\beta}$ implies

$$\underline{\beta}(1-q(\alpha))<\mathrm{Im}\, zg'(z)<\bar{\beta}(1-q(\alpha)).$$

It is remarked that the direct verification of the last result is made briefly. In fact, in view of identical relation

$$(z^{1/\alpha+1}g'(z))'=\frac{1}{\alpha}z^{1/\alpha}((1+\alpha)g'(z)+\alpha zg''(z)),$$

we have

$$\begin{aligned} zg'(z)&=\frac{1}{\alpha z^{1/\alpha}}\int_0^z \zeta^{1/\alpha}((1+\alpha)g'(\zeta)+\alpha zg''(\zeta))d\zeta \\ &=\frac{1}{\alpha}\int_I t^{1/\alpha-1}((1+\alpha)g'(zt)+\alpha(zt)^2g''(zt))dt. \end{aligned}$$

Applying the Harnack inequality to two positive harmonic functions

$$\mathrm{Im}\,\frac{(1+\alpha)\zeta g'(\zeta)+\alpha\zeta^2g''(\zeta)-\underline{\beta}}{-\underline{\beta}}$$

and

$$\mathrm{Im}\,\frac{\bar{\beta}-(1+\alpha)\zeta g'(\zeta)-\alpha\zeta^2g''(\zeta)}{\bar{\beta}},$$

which are both equal to unity at the origin, we get

$$\underline{\beta}-\underline{\beta}\frac{1-t}{1+t}<\mathrm{Im}\,((1+\alpha)ztg'(zt)+\alpha(zt)^2g''(zt))<\bar{\beta}-\bar{\beta}\frac{1-t}{1+t}$$

for the last integrand, whence readily follows the desired result.

5. Supplement to Chichra's theorem.

In the previous section we have referred to a result due to Miller and Altintas. Since $f\in\mathcal{F}$ is equivalent to $f'\in\mathcal{G}$ with $f(0)=0$, its original form is equivalent to the statement that if $f\in\mathcal{F}$ satisfies $\mathrm{Re}\,(f'(z)+\alpha zf''(z))>\underline{\beta}$ with $\alpha>0$ then $\mathrm{Re}\,f'(z)>\underline{\beta}$. However, Chichra [2] has shown more generally that the same statement remains valid for complex values of α provided $\mathrm{Re}\,\alpha>0$.

For the case of real α with $\alpha>0$ we have given in [7] a proof in an improved form. Here we shall give a corresponding proof with respect to the result of Chichra. The assertion to be proved is stated as follows.

THEOREM 4, *If* $f\in\mathcal{F}$ *satisfies* $\underline{\beta}<\mathrm{Re}\,(f'(z)+\alpha zf''(z))<\bar{\beta}$ *with* $u=\mathrm{Re}\,\alpha\geqq 0$, *then*

$$\underline{\beta}+(1-\underline{\beta})q(u)<\mathrm{Re}\,f'(z)<\bar{\beta}-(\bar{\beta}-1)q(u),$$

where q *is given by*

$$q(0)=0, \qquad q(u)=\frac{1}{u}\int_I t^{1/u-1}\frac{1-t}{1+t}dt \qquad (u>0).$$

Proof. Based on the identity

$$f'(z)+\alpha zf''(z)=\alpha z^{1-1/\alpha}(z^{1/\alpha}f'(z))',$$

we get the relation

$$f'(z)=\frac{1}{\alpha}z^{-1/\alpha}\int_0^z \zeta^{1/\alpha-1}(f'(\zeta)+\alpha\zeta f''(\zeta))d\zeta$$

$$=\frac{1}{\alpha}\int_I t^{1/\alpha-1}(f'(zt)+\alpha ztf''(zt))dt,$$

where the power functions denote the principal branches and the integration with respect to ζ is taken along the segment. First, suppose that α is imaginary and $\mathrm{Re}\,\alpha>0$, and put $t=\tau^{\alpha}$. In view of $\mathrm{Re}\,\alpha>0$ we have $\mathrm{Re}\,\alpha^{-1}>0$ and hence $|\tau|=t^{\mathrm{Re}\,\alpha^{-1}}$ yields that $t\in I$ implies $|\tau|<1$. By putting $\alpha=u+iv$, we have

$$\log|\tau|=\frac{u}{u^2+v^2}\log t, \qquad \arg\tau=\frac{-v}{u^2+v^2}\log t.$$

When t moves from 0 to 1 along the segment I, then τ moves from 0 to 1 along an arc S of logarithmic spiral $u\arg\tau+v\log|\tau|=0$ which lies on the closed unit disk on the τ-plane. By means of this change of variable we obtain

$$f'(z)=\int_S (f'(z\tau^\alpha)+\alpha z\tau^\alpha f''(z\tau^\alpha))d\tau.$$

As $t\to+0$, τ moves on the spiral toward 0 winding around the origin in positive or negative sense according to $v>0$ or $v<0$, respectively. Though the integrand, qua function of τ, is many-valued because of the logarithmic branch point lying at $\tau=0$, it converges uniformly to a definite value $1=f'(0)$ as τ near terminal part of S tends to 0, and the length of S is finite. Consequently, we may replace the integration path S by the segment I on the τ-plane, whence follows

$$\operatorname{Re} f'(z)=\int_I \operatorname{Re}(f'(z\tau^\alpha)+\alpha z\tau^\alpha f''(z\tau^\alpha))d\tau.$$

If α is real, the last relation is evident since S then degenerates to I. Now, it follows from the Harnack inequality that the assumption yields

$$\underline{\beta}+(1-\underline{\beta})\frac{1-|\zeta|}{1+|\zeta|}<\operatorname{Re}(f'(\zeta)+\alpha\zeta f''(\zeta))$$

$$<\bar{\beta}-(\bar{\beta}-1)\frac{1-|\zeta|}{1+|\zeta|}$$

for $|\zeta|<1$. Therefore, we obtain the desired inequality with

$$q(u)=\int_I \frac{1-\tau^u}{1+\tau^u}d\tau=\frac{1}{u}\int_I t^{1/u-1}\frac{1-t}{1+t}dt.$$

Next, in case $u=0$ we have only to consider the limit process. In fact, the statement for $u=\operatorname{Re}\alpha>0$ then reduces to the desired one after $u\to+0$.

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