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INNER RADII OF TEICHMÜLLER SPACES OF FINITELY GENERATED FUCHSIAN GROUPS

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1. Introduction

Let Γ be a Fuchsian group keeping the lower half plane L invariant. The Teichmüller space $T(\Gamma)$ of Γ is a bounded domain of the Banach space $B(L, \Gamma)$ of bounded quadratic differentials for Γ. The inner radius $i(\Gamma)$ of $T(\Gamma)$ is the radius of the maximal ball in $B(L, \Gamma)$ centered at the origin which is included in $T(\Gamma)$. If $T(\Gamma)$ is not a single point, then by a theorem of Ahlfors-Weill [3] it holds that $i(\Gamma) \geqq 2$. In particular, if Γ is finitely generated of the first kind and if $T(\Gamma)$ is not a single point, then the strict inequality $i(\Gamma) > 2$ holds (cf. [10]). Denote by $I(\Gamma)$ $\inf i(W\Gamma W^{-1})$, where the infimum is taken over for all quasiconformal automorphisms W of the upper half plane compatible with Γ. Recently T. Nakanishi [10] proved the following.

THEOREM 1 (T. Nakanishi). *Let Γ qe a finitely generated Fuchsian group of the first kind such that $T(\Gamma)$ is not a single point. Then $I(\Gamma)$ is equal to* 2.

The purpose of this note is to prove the following generalization to Theorem 1.

THEOREM 2. *Let Γ be a finitely generated Fuchsian group such that $T(\Gamma)$ is not a single point. Then $I(\Gamma)$ is equal to* 2.

The proof of Theorem 2 is immediate from Theorem 1 and the following.

THEOREM 3. *Let Γ be a finitely generated Fuchsian group of the second kind. Then $i(\Gamma)$ is equal to* 2.

A careful reading of the proof of Theorem 3 shows the readers an alternative proof of Theorem 1, though we omit it. Our proof of Theorem 3 depends on results on B-groups [1], [4] and Koebe groups [9].

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2. Preliminaries

2.1. Let $PSL(2, \boldsymbol{C})$ be the group of all conformal automorphisms of the extended complex plane $\boldsymbol{C}\cup\{\infty\}$. Denote by $PSL(2, \boldsymbol{R})$ the subgroup of $PSL(2, \boldsymbol{C})$ which consists of all conformal automorphisms of the upper half plane $U=\{z ; \operatorname{Im} z>0\}$. A Fuchsian group is a discrete subgroup of $PSL(2, \boldsymbol{R})$. A Fuchsian group is of the first kind (resp. the second kind) if it acts discontinuously at no point (resp. some point) of the real axis.

2.2. We define a hyperbolic metric $\rho_U(z)|dz|$ in U as $(2\operatorname{Im} z)^{-1}|dz|$. Let f be a holomorphic function of U onto a domain $D\subset\boldsymbol{C}$ with more than two boundary points. Then the hyperbolic metric $\rho_D(z)|dz|$ is defined by $\rho_D(f(z))\cdot|f'(z)|=\rho_U(z)$. Assume moreover that D is a connected and simply connected domain of $\boldsymbol{C}$. Then $(4X(z))^{-1}\leqq\rho_D(z)$, where $X(z)$ is the Euclidean distance between a point z of D and the boundary of D. In particular, if $D=\{z ; |\operatorname{Im} z|<\pi/2\}$, then $1/(2\pi)\leqq\rho_D(z)$. If $D_1\subset D_2$, then by Schwarz's lemma we see that $\rho_{D_2}(z)\leqq\rho_{D_1}(z)$ [5; p. 45].

2.3. A holomorphic function $\phi(z)$ in the lower half plane $L=\{z ; \operatorname{Im} z<0\}$ is a bounded quadratic differential for a Fuchsian group Γ if

$$\|\phi\|=\sup_{z\in L}\rho_L(z)^{-2}|\phi(z)|<\infty$$

and

$$\phi(\gamma(z))\gamma'(z)^2=\phi(z) \qquad \text{for all } \gamma\in\Gamma \text{ and all } z\in L .$$

The space $B(L, \Gamma)$ of all bounded quadratic differentials for Γ can be regarded as a Banach space with the norm $\| \ \|$ defined above.

2.4. An element γ of Γ is primitive if $j^n=\gamma$ has no solution in Γ for $n\neq\pm1$. The following lemma is well known but the author has never seen what is stated in this form.

LEMMA 1. *Let Γ be a Fuchsian group keeping the upper half plane invariant which contains a primitive parabolic element $p(z)=z+1$. Then for each $\phi\in B(L, \Gamma)$ it holds that*

$$\sup_{\{\operatorname{Im} z\leqq-1\}}\rho_L(z)^{-2}|\phi(z)|=\sup_{\{\operatorname{Im} z=-1\}}\rho_L(z)^{-2}|\phi(z)| .$$

Proof. Recall that $\phi(z)$ has a Fourier expansion $\sum_{n=1}^{\infty}\exp(-2\pi inz)$ [5; p. 111]. Note that

$$4y^2|\phi(z)|=4y^2\exp(2\pi y)|\textstyle\sum_{n=1}^{\infty}\exp(-2\pi i(n-1)z)| .$$

where $y=\operatorname{Im} z$. Then by the principle of the maximal absolute value and $d(y^2\exp 2\pi y)/dy\geqq0$ for $y\leqq-1/\pi$, we have the desired conclusion. □

2.5. Let $Q(\Gamma)$ be the set of all conformal homeomorphisms f of L admitting quasiconformal extensions $\hat{f}$ to the extended complex plane which are compatible with Γ, that is, $\hat{f}\Gamma\hat{f}^{-1}\subset PSL(2, \boldsymbol{C})$. For each $f\in Q(\Gamma)$, its Schwarzian derivative $[f]=(f''/f')'-(f''/f')^2/2$ belongs to $B(L, \Gamma)$. The Teichmüller space $T(\Gamma)$ of Γ is the image of $Q(\Gamma)$ under the mapping $f\mapsto[f]$. The inner radius $i(\Gamma)$ of $T(\Gamma)$ is $\inf_{\phi\in B(L,\Gamma)-T(\Gamma)}\|\phi\|$. If $g_1, g_2\in PSL(2, \boldsymbol{C})$, then $[g_2\circ f\circ g_1]=([f]\circ g_1)g_1'^2$ and $\|[g_2\circ f\circ g_1]\|=\|[f]\|$. In particular, if $g\in PSL(2, \boldsymbol{R})$, then $f\circ g^{-1}\in Q(g\Gamma g^{-1})$ and $i(g\Gamma g^{-1})=i(\Gamma)$.

2.6. A component of the region of discontinuity of a Kleinian group G is called a component of G. An invariant component of G is a component of G which is invariant under G. A Kleinian group G is a B-group if G has exactly one simply connected invariant component. An Euclidean disc (including a half plane) D is a horodisc of a primitive parabolic element g of G if $j(D)=D$ for each $j\in\langle g\rangle$, the cyclic group generated by g and $j(D)\cap D=\emptyset$ for each $j\in G-\langle g\rangle$. A B-group G is regular if for each primitive parabolic element g of G there exist two mutually disjoint horodiscs of g (Abikoff [1]). A regular B-group is a Koebe group if each noninvariant component of G is an Euclidean disc. Note that our definition of a Koebe group is stronger than Maskit's original one [9].

3. Proof of theorem 3

3.1. Let Γ be a finitely generated Fuchsian group of the second kind such that L/Γ is a compact Riemann surface with finitely many points and $m\geqq 1$ discs removed. Then classical is the existence of a hyperbolically convex fundamental region ω for Γ in L satisfying the following: There exist $2m$ sides $S_1, \cdots, S_{2m}$ of ω consisting of hyperbolic half lines and primitive hyperbolic elements $\alpha_1, \cdots, \alpha_m$ of Γ such that $\alpha_k(S_k)=S_{k+m}$ and such that a component of $\boldsymbol{R}\cup\{\infty\}$ minus the fixed points of α_k is included in the region of discontinuity of Γ, $k=1, \cdots, m$.

Let E_k be the geodesic included in ω tangent to S_k and S_{k+m}, $k=2, \cdots, m$. Let H_n, H_n' and $E_{1,n}$ be geodesics included in ω such that S_1, H_n, $E_{1,n}$, H_n' and S_{1+m} lie in this order and such that the hyperbolically convex domain ω_n surrounded by all sides of ω together with H_n, $E_{1,n}$, H_n' and $E_2, \cdots, E_m$ is of a finite hyperbolic area. Let $\varepsilon_k\in PSL(2, \boldsymbol{R})$ (resp. $\varepsilon_{1,n}\in PSL(2, \boldsymbol{R})$) be an elliptic transformation of order 2 keeping E_k (resp. $E_{1,n}$) and the middle point of E_k (resp. $E_{1,n}$) invariant, $k=2, \cdots, m$. Let γ_n be a hyperbolic transformation with $\gamma_n(H_n)=H_n'$ and $\gamma_n(\omega_n)\cap\omega_n=\emptyset$. Then Γ and γ_n and $\varepsilon_{1,n}, \varepsilon_2, \cdots, \varepsilon_m$ generate a finitely generated Fuchsian group Γ_n of the first kind with the fundamental region ω_n. We assume that $\{\gamma_n\}_{n=1}^{\infty}$ converges to a parabolic transformation. Then $\{E_{1,n}\}_{n=1}^{\infty}$ necessarily degenerates to a point.

3.2. Let $p_{1,n}, p_{2,n}, \cdots, p_{t,n}$ be a maximal list of primitive parabolic elements of Γ_n whose fixed points lie on the boundary of ω_n such that $p_{r,n}\neq p_{s,n}^{\pm 1}$, $1\leqq r<s\leqq t$. Let $D_{s,n}=\xi_{s,n}(\{z; \operatorname{Im} z<-1\})$ be the horodisc of the primitive parabolic element $p_{s,n}$, where $\xi_{s,n}$ is the element of $PSL(2, \boldsymbol{R})$ such that $\xi_{s,n}^{-1}\circ p_{s,n}\circ\xi_{s,n}$ is of the form $z\mapsto z+1$. The existence of such a horodisc is immediate from Shimizu's lemma [5; p. 58]. For our later use, we prove a preliminary lemma.

LEMMA 2. *Let* u_n *be a point of* $\omega_n-\bigcup_{s=1}^{t}D_{s,n}$. *Then* $\{d_L(u_n, \gamma_n(u_n))\}_{n=1}^{\infty}$ *is bounded, where* $d_L(u_n, \gamma_n(u_n))$ *is the hyperbolic distance between* u_n *and* $\gamma_n(u_n)$ *measured by* $\rho_U(z)|dz|$.

Proof. The axis A_n of γ_n divides ω_n into ω_n^1 and ω_n^2 whose boundary includes $E_{1,n}$. Let v_n be a point of the closure of $\omega_n-\bigcup_{s=1}^{t}D_{s,n}$ such that $d_L(v_n, A_n)\geqq d_L(z, A_n)$ for all $z\in\omega_n-\bigcup_{s=1}^{t}D_{s,n}$. Note the existence of a compact subset of L containing all $v_n\in\omega_n^1$. Then $d_L(v_n, \gamma_n(v_n))$ is less than a constant for all $v_n\in\omega_n^1$. Let τ_n be the element of $PSL(2, \boldsymbol{R})$ such that $\tau_n(z_n^*)=-i$ and $\tau_n'(z_n^*)>0$, where z_n^* is the fixed point of $\varepsilon_{1,n}$ in ω_n. Then $\{\tau_n\circ\gamma_n\circ\tau_n^{-1}\}_{n=1}^{\infty}$ converges to a parabolic transformation and a compact subset of L contains all $\tau_n(v_n)$ for all $v_n\in\omega_n^2$. By the same reasoning as above we see that $d_L(v_n, \gamma_n(v_n))=d_L(\tau_n(v_n), \tau_n\circ\gamma_n\circ\tau_n^{-1}(\tau_n(v_n))$ is less than a constant for all $v_n\in\omega_n^2$. Note that $d_L(u_n, A_n)\leqq d_L(v_n, A_n)$. Then $d_L(u_n, \gamma_n(u_n))\leqq d_L(v_n, \gamma_n(v_n))$. Now our assertion is obvious. □

3.3. Now we begin to make a proof of Theorem 3. Let χ_n be the isomorphism of Γ_n onto a regular B-group $\chi_n(\Gamma_n)$ on the boundary of $T(\Gamma_n)$ such that an element $\chi_n(\gamma)$ of $\chi_n(\Gamma_n)$ is parabolic if and only if γ is either parabolic or conjugate to γ_n in Γ_n. Let w_n be a conformal homeomorphism of L onto the invariant component of $\chi_n(\Gamma_n)$ such that $\chi_n(\gamma)\circ w_n(z)=w_n\circ\gamma(z)$ for all $z\in L$ and all $\gamma\in\Gamma$.

The existence of such a χ_n and a w_n is shown in Bers [4] and Abikoff [1]. Maskit [9] proved that there exist a Koebe group G_n and a conformal homeomorphism j_n of the invariant component of $\chi_n(\Gamma_n)$ onto that Δ_n of G_n such that $j_n\chi_n(\Gamma_n)j_n^{-1}=G_n$ and such that $j_n\circ\chi_n(\gamma_n)\circ j_n^{-1}$ is parabolic if and only if so is $\chi_n(\gamma)$. Set $f_n=j_n\circ w_n$. Then $\zeta=f_n(z)$ is a conformal homeomorphism of L onto Δ_n and $f_n\circ\gamma_n\circ f_n^{-1}$ is parabolic, so that $[f_n]$ does not belong to $T(\Gamma_n)$. Since $\|[\eta\circ f_n]\|=\|[f_n]\|$ for all $\eta\in PSL(2, \boldsymbol{R})$, without loss of generality we may assume that $g_n=f_n\circ\gamma_n\circ f_n^{-1}$ is of the form $\zeta\mapsto\zeta+b_n$, $b_n>0$, and that two non-invariant components D_n^+ and D_n^- of G_n invariant under g_n are $\{\xi;\operatorname{Im}\zeta>\pi/2\}$ and $\{\zeta;\operatorname{Im}\zeta<-\pi/2\}$, respectively. Let z_n be a point of both the axis of γ_n and the fundamental region ω_n constructed in No. 3.1. Then by the same reasoning as above, we may also assume that $\operatorname{Re} f_n(z_n)=0$. From basic properties of the hyperbolic metric stated in No. 2.2 we have

$$d_L(z_n, \gamma_n(z_n))=d_{\Delta_n}(f_n(z_n), f_n(\gamma_n(z_n)))$$
$$\geqq d_{\{\zeta;|\operatorname{Im}\zeta|<\pi/2\}}(f_n(z_n), g_n(f_n(z_n)))\geqq b_n/2\pi\,.$$

Since $\{\gamma_n\}_{n=1}^{\infty}$ converges to a parabolic transformation, the first term in the above inequalties converges to zero. Now we have the first assertion in the following.

LEMMA 3. (i) *The sequence* $\{b_n\}_{n=1}^{\infty}$ *of positive numbers converges to zero.*
(ii) *The invariant component* Δ_n *of* G_n *includes the region* $\{\zeta;\ |\mathrm{Im}\,\zeta|<(\pi/2)-b_n\}$.

Proof. We have only to prove (ii). By the assumptions on χ_n we see that G_n is constructed from Fuchsian groups $H_n^+=\{g\in G_n;\ g(D_n^+)=D_n^+\}$ and $H_n^-=\{g\in G_n;\ g(D_n^-)=D_n^-\}$ with the amalgamated parabolic cyclic subgroup generated by g_n via Maskit's combination theorem I. For terminologies see [6], [7] and [8].

For a Möbius tronsformation h of the form $z\mapsto(az+b)/(cz+d)$ with $c\neq0$, that is, $h^{-1}(\infty)=-d/c\neq\infty$, we define the isometric circle $I(h)$ of h as $\{z;\ |z-h^{-1}(\infty)|=1/|c|\}$. Denote by $\mathrm{ext}\,I(h)$ the unbounded component of $\boldsymbol{C}-I(h)$. The region $\omega_n^+=\{\zeta;\ 0<\mathrm{Re}\,\zeta<b_n\}\cap(\cap^+\mathrm{ext}\,I(h))$ (resp. $\omega_n^-=\{\zeta;\ 0<\mathrm{Re}\,\zeta<b_n\}\cap(\cap^-\mathrm{ext}\,I(h))$ is a fundamental region for H_n^+ (resp. H_n^-), where the intersection $\cap^+$ (resp. $\cap^-$) is taken over for all elements of $J_n^+=\{h\in H_n^+;\ h(\infty)\neq\infty\}$ (resp. $J_n^-=\{h\in H_n^-;\ h(\infty)\neq\infty\}$). Maskit's combination theorem I shows that $\omega_n^+\cap\omega_n^-$ is a fundamental region for G_n. Note that centers $h^{-1}(\infty)$ of the isometric circles of $h_n\in J_n^+$ (resp. J_n^-) lie on the line $\{\zeta;\ \mathrm{Im}\,\zeta=\pi/2\}$ (resp. $\{\zeta;\ \mathrm{Im}\,\zeta=-\pi/2\}$). Since G_n contains the element $g_n(z)=z+b_n$ the radius of the isometric circle of each element of $J_n^+\cup J_n^-$ is less than or equal to b_n by Shimizu's lemma. Therefore Δ_n includes the region $(\bigcup_{n=-\infty}^{\infty}g_n^s(\omega_n^+\cap\omega_n^-))\cap\{\zeta;\ |\mathrm{Im}\,\zeta|<\pi/2\}$, which also does the region $\{\zeta;\ |\mathrm{Im}\,\zeta|<(\pi/2)-b_n\}$. □

3.4. Denote by A_n the axis of γ_n.

LEMMA 4. *There exists a sequence* $\{t_n\}_{n=1}^{\infty}$ *of positive numbers converging to zero such that* $f_n(A_n)$ *is included in* $\{\zeta;\ |\mathrm{Im}\,\zeta|<t_n\}$.

Proof. Assume that our assertion is false. Let a_n be the subarc of A_n bounded by z_n and $\gamma_n(z_n)$. Let ζ_n be a point of $f_n(a_n)$ such that $|\mathrm{Im}\,\zeta_n|=\max_{\zeta\in a_n}|\mathrm{Im}\,\zeta|$. Then without loss of generality we may assume the existence of a subsequence, again denoted by $\{\zeta_n\}_{n=1}^{\infty}$, of $\{\zeta_n\}_{n=1}^{\infty}$ such that $\{\mathrm{Im}\,\zeta_n\}_{n=1}^{\infty}$ converges to a positive number v_0. By means of basic properties of the hyperbolic metric stated in No. 2.2, we have

$$\int_{a_n}\rho_L(z)|dz|=\int_{f_n(a_n)}\rho_{\Delta_n}(\zeta)|d\zeta|$$
$$\geqq\int_{f_n(a_n)}\rho_{\{\zeta;|\mathrm{Im}\,\zeta|<\pi/2\}}(\zeta)|d\zeta|\geqq(1/2\pi)\int_{f_n(a_n)}|d\zeta|.$$

Since the first term converges to zero, so does the Euclidean length $\int_{f_n(a_n)}|d\zeta|$ of $f_n(a_n)$. Therefore for a sufficiently large n on, the arc $f_n(a_n)$ is included in $\{\zeta;\operatorname{Im}\zeta>v_0/2\}$, and so is $f_n(A_n)=\bigcup_{s=-\infty}^{\infty}g_n{}^s(f_n(a_n))$. The geodesic $f_n(A_n)$ in Δ_n divides Δ_n into the upper half $\Delta_n{}^+$ and the lower half $\Delta_n{}^-$, both of which are invariant under $\langle g_n\rangle$. The region $\Delta_n{}^+$ is included in $\Pi_n{}^+=\{\zeta;v_0/2<\operatorname{Im}\zeta<\pi/2\}$ and by Lemma 2 $\Delta_n{}^-$ includes $\Pi_n{}^-=\{\zeta;(-\pi/2)+b_n<\operatorname{Im}\zeta<v_0/2\}$. Let $S_{1,n}$, $S_{2,n}$, $S_{3,n}$ and $S_{4,n}$ be sets of all loops separating two boundary components of $\Delta_n{}^+/\langle g_n\rangle$, $\Pi_n{}^+/\langle g_n\rangle$, $\Pi_n{}^-/\langle g_n\rangle$ and $\Delta_n{}^-/\langle g_n\rangle$, respectively. Denote by $\lambda_{k,n}$ the extremal length of $S_{k,n}$. Then $\lambda_{1,n}{}^{-1}\geqq\lambda_{2,n}{}^{-1}>\lambda_{3,n}{}^{-1}\geqq\lambda_{4,n}{}^{-1}$ if n is large enough so that $v_0/2>b_n$ [2; p. 15]. On the other hand, the Moebius transformation r_n of the form $z\mapsto -\imath z$ maps $f_n{}^{-1}(\Delta_n{}^+)=\{z;-\pi/2<\arg z<0\}$ onto $f_n{}^{-1}(\Delta_n{}^{-1})=\{z;-\pi<\arg z<-\pi/2\}$ and it holds that $\gamma_n\circ r_n=r_n\circ\gamma_n$. Hence the conformal homeomorphism $f_n\circ r_n\circ f_n{}^{-1}$ maps $\Delta_n{}^+$ onto $\Delta_n{}^-$ and $f_n\circ r_n\circ f_n{}^{-1}\circ g_n=g_n\circ f_n\circ r_n\circ f_n{}^{-1}$. Therefore $\Delta_n{}^+/\langle g_n\rangle$ is conformal to $\Delta_n{}^-/\langle g_n\rangle$ and $\lambda_{1,n}=\lambda_{4,n}$. This contradiction yields us to conclude that our assertion is true. □

3.5. Let u_n be a point of the closure of $\omega_n-\bigcup_{s=1}^{t}D_{s,n}$ with $\rho_L(u_n)^{-2}|[f_n(u_n)]|=\sup_{z\in L}\rho_L(z)^{-2}|[f_n(z)]|$. The existence of such a point is immediate from Lemma 1. Without loss of generality we may assume that $d_L(u_n,A_n)\leqq d_L(u_n,\gamma(A_n))$ for all $\gamma\in\Gamma_n$ and that $0\leqq\operatorname{Re}f_n(u_n)<b_n$. As is stated in No. 3.2, the point $z_n\in\omega_n$ lies on the axis of γ_n.

Now two cases can occur: (i) $\{d_L(u_n,z_n)\}_{n=1}^{\infty}$ is bounded. (ii) Otherwise.

We shall prove that (ii) never happens. Assume that (ii) does. Then since $\{d_{\Delta_n}(f_n(u_n),f_n(z_n))\}_{n=1}^{\infty}$ is unbounded, a subsequence, again denoted by $\{f_n(u_n)\}_{n=1}^{\infty}$, of $\{f_n(u_n)\}_{n=1}^{\infty}$ converges to a point ζ_0, which is either $\pi i/2$ or $-\pi i/2$. Let, say, ζ_0 by $\pi i/2$. Then each $f_n(u_n)$ is contained in $\Delta_n{}^+$. Set $\eta_n(\zeta)$ $(\zeta-\operatorname{Re}f_n(u_n)-\pi i/2)/|\operatorname{Im}f_n(u_n)-\pi/2|$. Then η_n takes the point $f_n(u_n)$ and the line $\operatorname{Im}\zeta=\pi/2$ into $-i$ and the real axis, respectively, and $\eta_n(\Delta_n)\subset L$. Note that $\eta_n(\Delta_n)$ includes the domain surrounded by $\cup\eta_n(h(f_n(A_n)))$, where the union is taken over all $h\in H_n{}^+$. The parabolic transformation $\eta_n\circ g_n\circ\eta_n{}^{-1}$ is of the form $\zeta\mapsto\zeta+e_n$, $e_n>0$. Note that

$$d_L(u_n,\gamma_n(u_n))=d_{\eta_n\circ f_n(L)}(\eta_n\circ f_n(u_n),\eta_n\circ f_n(\gamma_n(u_n)))$$
$$\geqq d_L(\eta_n\circ f_n(u_n),\eta_n\circ g_n(f_n(u_n)))=d_L(\eta_n\circ f_n(u_n),\eta_n\circ f_n(u_n)+e_n).$$

Since $\{d_L(u_n,\gamma_n(u_n))\}_{n=1}^{\infty}$ is less than a constant e_0 by Lemma 2, so is $\{e_n\}_{n=1}^{\infty}$. This together with Shimizu's lemma shows that each element of $\eta_nJ_n{}^+\eta_n{}^{-1}$ has the isometric circle whose radius is less than or equel to e_0. Since $K_n=\inf_{\zeta\in\eta_n(f_n(A_n))}|\operatorname{Im}\zeta|\to\infty$ by Lemma 4 and since for each $h\in J_n{}^+$ the arc $\eta_n(h(f_n(A_n)))\subset\eta_n(\Delta_n)$ is included in $\{\zeta\in L;\operatorname{Im}\zeta>-e_0{}^{-2}/K_n\}$, the kernel of $\{\eta_n(\Delta_n)\}_{n=1}^{\infty}$ is L. Let ξ_n be the element of $PSL(2,\boldsymbol{R})$ such that $\xi_n(-1)=u_n$ and $(\eta_n\circ f_n\circ\xi_n)'(-i)>0$. Then by Carathéodory kernel theorem $\eta_n\circ f_n\circ\xi_n$ converges locally uniformly to a conformal homeomorphism F which maps L onto the kernel L of $\{\eta_n(\Delta_n)\}_{n=1}^{\infty}$. Obviously F is a Möbius transformation and $[F](z)=0$. Using a theorem of Weierstrass, we have

$$\|[f_n]\| = \|[\eta_n \circ f_n \circ \xi_n]\|$$
$$= \sup_{z\in L} |(2|\mathrm{Im}\, z|)^2 [\eta_n \circ f_n \circ \xi_n](z)|$$
$$= (2(-1))^2 |[\eta_n \circ f_n \circ \xi_n](-i)| \longrightarrow 4|[F](-i)| = 0\,.$$

This contradicts the fact $\|[f_n]\| \geqq 2$ due to Ahlfors-Weill [3], and the case (ii) never happens.

3.6. Now we shall complete the proof of Theorem 3 under the condition (i). Since $d_{\Delta_n}(f_n(u_n), f_n(A_n)) = d_L(u_n, A_n) \leqq d_L(u_n, z_n)$ is less than a constant for each n, Lemmas 3 and 4 show the existence of a subsequence, again denoted by $\{f_n(u_n)\}_{n=1}^{\infty}$, of $\{f_n(u_n)\}_{n=1}^{\infty}$ which converges to a point ζ_0 with $\mathrm{Re}\,\zeta_0 = 0$ and $|\mathrm{Im}\,\zeta_0| < \pi/2$. Let μ_n be the element of $PSL(2, \boldsymbol{R})$ such that $\mu_n(-i) = z_n$ and $(f_n \circ \mu_n)'(-i) > 0$. Carathéodory kernel theorem together with Lemma 3 shows that $\{f_n \circ \mu_n(z) - f_n \circ \mu_n(-i)\}_{n=1}^{\infty}$ converges locally uniformly to $F(z) = 3\pi i/2 + \log z$ which maps L onto the kernel $\{\zeta;\ |\mathrm{Im}\,\zeta| < \pi/2\}$ of $\{f_n \circ \mu_n(L)\}_{n=1}^{\infty}$, where we take the branch of $\log z$ satisfying $F(-i) = 0$. Let E be a compact subset of L containing all $\mu_n^{-1}(u_n)$. Then we see that

$$\|[f_n]\| = \rho_L(u_n)^{-2} |[f_n](u_n)|$$
$$= \rho_L \circ \mu_n(\mu_n^{-1}(u_n))^{-2} |[f_n \circ \mu_n](\mu_n^{-1}(u_n))|$$
$$= \sup_{z\in E} \rho_L(z)^{-2} |[f_n](z)|$$
$$\longrightarrow \sup_{z\in E} \rho_L(z)^{-2} |[3\pi i/2 + \log z]| = 2\,.$$

Recall that $f_n \Gamma_n f_n^{-1}$ is a Koebe group. Then $T(\Gamma_n)$ does not contain the point $[f_n]$ and neither does $T(\Gamma)$. Therefore $2 \leqq i(\Gamma) \leqq \|[f_n]\| \to 2$. Now we obtain $i(\Gamma) = 2$ and complete the proof of Theorem 3.

Added in proof. After this note was completed, Professor T. Nakanishi informed the author that T. Nakanishi and J. A. Velling know a proof of the following Theorem A which is a generalization of Theorems 1, 2 and 3.

THEOREM A. *Let Γ be a Fuchsian group keeping L invariant. Then $i(\Gamma)$ is equal to 2 if Γ satisfies one of the following:*

(I_1) *For any positive number d, there exists a hyperbolc disc of radius d which is precisely invariant under the trivial subgroup of Γ.*

(I_2) *For any positive number d, there exists the collar of width d about the axis of a hyperbolic element of Γ.*

He also informed the author that their proof of Theorem A is different from the proof of Theorem 3 and depends on properties of a family of functions constructed in Kalme [11].

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