T. KOMATSU
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EXTENSION OF BAKER'S ANALOGUE OF LITTLEWOOD'S DIOPHANTINE APPROXIMATION PROBLEM

BY TAKAO KOMATSU

1. Introduction.

The famous but still unsolved problem of Littlewood can be stated as follows: for each pair of real numbers θ and ϕ and each $\varepsilon>0$, does there exist a positive integer n such that

$$n\|n\theta\| \ \|n\phi\|<\varepsilon \ ?$$

Here $\|\alpha\|$ denotes the difference between α and the nearest integer. In 1963 Davenport and Lewis [1] obtained a negative answer for an analogous question concerning formal power series. The following year Baker [2] gave examples where the construction of Davenport and Lewis holds. And as a generalization of these results, he indicated the following result:

THEOREM (Baker (1964)). *If $\lambda_1, \cdots, \lambda_r$ are distinct real numbers, none of them 0, and $u(t), u_1(t), \cdots, u_r(t)$ are real polynomials with $u(t)\neq 0$, then*

$$|u(t)|_K \prod_{j=1}^{r} |u_j(t)-e^{\lambda_j/t}u(t)|_K \geqq e^{-R},$$

where $R=(1/2)(r^3+r)$.

The valuation of a formal power series relative to the real number field K is defined by

$$|a_m t^m + a_{m-1}t^{m-1}+ \cdots |_K=e^m \quad (a_m\neq 0,\ m \text{ is integer}).$$

The purpose of this paper is an extension of Baker's result, proving the following theorem:

THEOREM. *Let n, r be positive integers. If $\lambda_1, \cdots, \lambda_r$ are distinct real numbers, none of them 0, and $u(t), u_1(t), \cdots, u_r(t)$ are real polynomials with $u(t)\neq 0$, then*

$$(1) \qquad |u(t)|_K \prod_{j=1}^{r} |u_j(t)-e^{\lambda_j/t^n}u(t)|_K \geqq e^{-R(n,r)},$$

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where $R(n, r)=(1/2)n(r^3+r)$.

2. The construction of polynomials

Let m_j $(j=1, 2, \cdots, r)$ be positive integers. Then clearly there exist real polynomials $P_0(x)$, $P_0^{(1)}(x), \cdots, P_0^{(r)}(x)$ of degree at most $h=\sum_{j=1}^{r} m_j-r$, not all identically zero, such that for $j=1, 2, \cdots, r$

$$P_0^{(j)}(x)-e^{\lambda_j x}P_0(x)=b_{m_j+h}^{(j)}x^{m_j+h}+b_{m_j+h+1}^{(j)}x^{m_j+h+1}+\cdots, \tag{2}$$

where (j) in $P_0^{(j)}(x)$ denotes the suffix. $P_0(x)$, $P_0^{(j)}(x)$ $(j=1, 2, \cdots, r)$ cannot vanish identically.

We define further polynomials $P_i(x)$, $P_i^{(j)}(x)$ $(j=1, 2, \cdots, r)$, for $i=1, 2, \cdots, r$, by

$$P_{i+1}(x)=P_i'(x), \qquad P_{i+1}^{(j)}(x)=-\lambda_j P_i^{(j)}(x)+\{P_i^{(j)}(x)\}'. \tag{3}$$

where the accent denotes the derivative with respect to x. Next we define

$$\xi_i^{(j)}(x)=P_i^{(j)}(x)-e^{\lambda_j x}P_i(x) \qquad (i=0, 1, \cdots, r,\ j=1, 2, \cdots, r). \tag{4}$$

Then it follows that, for $i=0, 1, \cdots, r-1$, $j=1, 2, \cdots, r$

$$\xi_{i+1}^{(j)}(x)=-\lambda_j\xi_i^{(j)}(x)+\{\xi_i^{(j)}(x)\}'.$$

From (2) it follows that for $i=0, 1, 2, \cdots, r$ the lowest possible powers of x in $\xi_i^{(j)}(x)$ are x^{m_j+h-i}. Therefore, for any positive integer n

$$|\xi_i^{(j)}(t^{-n})|_K\leqq e^{-n(m_j+h-i)}. \tag{5}$$

Lastly, we define the determinant $\varDelta(x)$ by

$$\varDelta(x)=\begin{vmatrix} P_0(x) & P_0^{(1)}(x)\cdots P_0^{(r)}(x) \\ P_1(x) & P_1^{(1)}(x)\cdots P_1^{(r)}(x) \\ & \cdots\cdots\cdots \\ P_r(x) & P_r^{(1)}(x)\cdots P_r^{(r)}(x) \end{vmatrix} \tag{6}$$

From (3) the highest coefficient of the polynomial $\varDelta(x)$,

$$(-1)^{r(r+1)/2}\lambda_1\cdots\lambda_r\prod_{1\leq i<j\leq r}(-\lambda_i+\lambda_j)\cdot pp_1\cdots p_r$$

is nonzero, where p, p_j $(j=1, 2, \cdots, r)$ are the highest nonzero coefficients in $P_0(x)$, $P_0^{(j)}(x)$ $(j=1, 2, \cdots, r)$, respectively. Thus $\varDelta(x)$ is not identically zero.

3. Proof of the Theorem

Now let $u(t)$ be a polynomial with real coefficients, of degree $k \geqq 0$. And let $u_j(t)$ $(j=1, 2, \cdots, r)$ be any polynomials with real coefficients. Let

$$|u_j(t)-e^{\lambda_j/t^n}u(t)|_K=e^{-a_j} \qquad (j=1, 2, \cdots, r).$$

By the definition of the valuation, we can consider that all a_j are positive integers. And also the proof of (1) can be replaced by the proof of following inequality:

$$k-\sum_{j=1}^{r} a_j \geqq -R(n, r). \tag{7}$$

There are three cases in the proof.

(I) Suppose that for all integers j with $1 \leqq j \leqq r$

$$a_j \geqq L=L(n, r),$$

where $L(n, r)$, equation (12) later, is a positive constant depending on only n and r.

We use the construction of Section 2 with

$$m_j=[(a_j-L)/n]+1$$

for $j=1, 2, \cdots, r$, that is

$$nm_j=a_j-L+n-\tau_j \tag{8}$$

if $a_j-L\equiv\tau_j \bmod n$ $(0\leqq\tau_j\leqq n-1)$.

$E(t)$ is defined by

$$E(t)=\begin{vmatrix} P_{i_1}(t^{-n}) & P_{i_1}^{(1)}(t^{-n}) \cdots P_{i_1}^{(r)}(t^{-n}) \\ & \cdots\cdots\cdots\cdots \\ P_{i_r}(t^{-n}) & P_{i_r}^{(1)}(t^{-n}) \cdots P_{i_r}^{(r)}(t^{-n}) \\ u(t) & u_1(t) \quad \cdots \quad u_r(t) \end{vmatrix},$$

where $i_1, \cdots, i_r$ are some r distinct numbers chosen from $0, 1, \cdots, r$. Since the equality $E(t)=0$ contradicts the fact $\Delta(x)\neq 0$, we may assume that $E(t)$ is not identically zero.

We will compare two estimates for $|E(t)|_K$. First, we give the lower estimate. Since for $i=1, 2, \cdots, r$ the degrees of $P_i(x)$, $P_i^{(j)}(x)$ $(j=1, 2, \cdots, r)$ are at most h,

$$|P_i(t^{-n})|_K,\ |P_i^{(j)}(t^{-n})|_K \geqq e^{-nh} \qquad (j=1, 2, \cdots, r).$$

Therefore, we get

$$|E(t)|_K \geqq e^{-nrh}. \tag{9}$$

Next for each integer j with $1 \leqq j \leqq r$, by subtracting the first column multiplied by e^{λ_j/t^n} from the $(j+1)$-th column, we have

$$E(t)=\begin{vmatrix} P_{i_1}(t^{-n}) & \xi_{i_1}^{(1)}(t^{-n}) & \cdots & \xi_{i_1}^{(r)}(t^{-n}) \\ & \cdots\cdots\cdots\cdots & & \\ P_{i_r}(t^{-n}) & \xi_{i_r}^{(1)}(t^{-n}) & \cdots & \xi_{i_r}^{(r)}(t^{-n}) \\ u(t) & u_1(t)-e^{\lambda_1/t^n}u(t) & \cdots & u_r(t)-e^{\lambda_r/t^n}u(t) \end{vmatrix}.$$

Inequalities (5) and $|P_i(t^{-n})|_K \leqq 1$ give

$$|E(t)|_K \leqq e^M, \tag{10}$$

where

$$M=\max\left\{k-\sum_{j=1}^{r} a_j+rL+\sum_{j=1}^{r}\tau_j+\frac{1}{2}nr(r-1)-nrh,\right. \tag{11}$$

$$\left.\max_{1\leqq j\leqq r}\left\{\frac{1}{2}nr^2-\frac{1}{2}nr-L-\tau_j-nrh\right\}\right\}.$$

Now we define

$$L=L(n, r)=\frac{1}{2}nr^2-\frac{1}{2}nr+1. \tag{12}$$

Then from (9) and (10), using that $\sum_{j=1}^{r}\tau_j \leqq (n-1)r$,

$$k-\sum_{j=1}^{r} a_j \geqq -rL-\sum_{j=1}^{r}\tau_j-\frac{1}{2}nr(r-1)$$

$$\geqq -\frac{1}{2}n(r^3+r).$$

(II) Assuming that for all integers j with $1\leqq j\leqq r$

$$a_j \leqq L-1,$$

clearly,

$$k-\sum_{j=1}^{r} a_j \geqq -r(L-1)$$

$$\geqq -\frac{1}{2}n(r^3+r).$$

(III) Suppose that

$$a_1, \cdots, a_\kappa \geqq L, \quad a_{\kappa+1}, \cdots, a_r \leqq L-1 \qquad (\kappa=1, 2, \cdots, r-1).$$

If we rearrange $a_1, a_2, \cdots, a_r$, it will be reduced to this case. By the definition (12)

$$a_1, \cdots, a_\kappa \geqq L(n, r) \geqq L(n, \kappa).$$

Let $\kappa \geqq 2$. Since

$$k - \sum_{j=1}^{\kappa} a_j \geqq -R(n, \kappa).$$

therefore,

$$k - \sum_{j=1}^{r} a_j \geqq -R(n, \kappa) - (r-\kappa)\{L(n, r) - 1\}$$
$$\geqq -R(n, r).$$

When $\kappa=1$, by the following Lemma, the same result will hold.

LEMMA. *Let n be a positive integer. If λ is nonzero real number, and $u(t)$, $v(t)$ are real polynomials with $u(t) \neq 0$, then*

$$(13) \qquad |u(t)|_K \, |v(t) - e^{\lambda/t^n} u(t)|_K \geqq e^{-n}.$$

4. Proof of the Lemma

Let $u(t)$ be a polynomial with real coefficients, of degree $k \geqq 0$. And let

$$|v(t) - e^{\lambda/t^n} u(t)|_K = e^{-a}.$$

In order to prove (13), we just need to show $k - a \geqq -n$. We use the construction of polynomials with

$$h = m-1, \qquad m = [(a-1)/n] + 1.$$

Simply, set $P_i^{(1)} = Q_i$, $u_1 = v$. Consider the estimation of

$$E_1(t) = \begin{vmatrix} P_i(t^{-n}) & Q_i(t^{-n}) \\ u(t) & v(t) \end{vmatrix},$$

where $i=0$ or 1. Similarly as the first part of the proof of the Theorem, we can prove (13). Hence

$$|E_1(t)|_K \geqq e^{-nh}$$

and

$$|E_1(t)|_K \leqq e^{M_1},$$

where $M_1 = \max\{-a, k - n(m+h-i)\}$.

From the two estimates.

$$k - n(m+h-i) \geqq -nh.$$

Therefore, we get the result of Lemma.

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REFERENCES

[1] H. DAVENPORT AND D. J. LEWIS, An analogue of a problem of Littlewood, Michigan Math. J. 10 (1963), 157-160.

[2] A. BAKER, On an analogue of Littlewood's diophantine approximation probrem, Michigan Math. J. 11 (1964), 247-250.

DEPARTMENT OF MATHEMATICS
GAKUSHUIN UNIVERSITY
1-5-1 MEJIRO, TOSHIMA-KU
TOKYO 171 JAPAN