C. Z. HUANG
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SOME RESULTS ON THE COMPLEX OSCILLATION THEORY OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

BY CUN-ZHI HUANG

1. Introduction

We consider the second order linear differential equation

$$f''+Af=0, \tag{1}$$

where A is an entire function. For an entire function f, let $\rho(f)$ be its order, $\mu(f)$ its lower order and $\lambda(f)$ the exponent of convergence of its zeros. In addition, we assume that the reader is familiar with the standard notations of Nevanlinna theory (see [3]).

When A is a polynomial of degree $n\geqq 1$, S. Bank and I. Laine obtained the following ([1]).

THEOREM A. *Let A be a polynomial of degree $n\geqq 1$. If $f\not\equiv 0$ is a solution of* (1), *then*

$$\rho(f)=(n+2)/2, \tag{2}$$

and if f_1, f_2 are two linearly independent solutions of (1), *then*

$$\max(\lambda(f_1), \lambda(f_2))=(n+2)/2. \tag{3}$$

If A is transcendental, we apply the lemma on the logarithmic derivative in Nevanlinna theory to (1) and can deduce that any solution $f\not\equiv 0$ of (1) satisfies

$$\rho(f)=\infty. \tag{4}$$

We may hope that

$$\max(\lambda(f_1), \lambda(f_2))=\infty, \tag{5}$$

where f_1 and f_2 are linearly independent solutions of (1). However, examples in [1] show that this is not the case. Specifically, for $\rho(A)$ a positive integer or infinity, there exist A and independent solutions f_1, f_2 of (1) such that

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$$\max(\lambda(f_1), \lambda(f_2)) < \infty .$$

When the growth of A is suitably restricted, the following were obtained.

THEOREM B ([6]). *Let A be a transcendental entire function of finite order and of lower order $\mu \leqq 1/2$. If f_1 and f_2 are two linearly independent solutions of* (1), *then*

$$\max(\lambda(f_1), \lambda(f_2)) = \infty .$$

THEOREM C ([5]). *Let A be a transcendental entire function of order $\rho(A) < 1$. If f_1 and f_2 are two linearly independent solutions of* (1), *then*

$$\lambda(f_1 f_2) = \infty$$

or

$$\rho(A)^{-1} + \lambda(f_1 f_2)^{-1} \leqq 2 .$$

In this paper, we prove

THEOREM 1. *Let A be a transcendental entire function of lower order $\mu(A) < 1$. If f_1 and f_2 are two linearly independent solutions of* (1), *then*

$$\lambda(f_1 f_2) = \infty$$

or

$$\mu(A)^{-1} + \lambda(f_1 f_2)^{-1} \leqq 2 .$$

Remark. Theorem 1 generalizes Theorem B and Theorem C. Furthermore, we note that the condition $\rho(A)$ is finite in Theorem B is not necessary.

Before stating Theorem 2, we introduce some definitions. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a transcendental entire function. We denote by $\Lambda = \{\lambda_k\}$, $M = \{\mu_k\}$ $(k = 1, 2, \cdots)$ the sequences of exponents n for which $a_n \neq 0$ and $a_n = 0$ respectively, arranged in increasing order. We say that $f(z)$ has Fabry gap if

$$\frac{\lambda n}{n} \longrightarrow \infty \qquad (n \to \infty).$$

Now we have

THEOREM 2. *Let A be a transcendental entire function of lower order $\mu(A) < \infty$ and have Fabry gap. If f_1 and f_2 are two linearly independent solutions of* (1), *then*

$$\max(\lambda(f_1), \lambda(f_2)) = \infty .$$

2. Preliminaries

LEMMA 1. *Let E be an entire function of finite order, then there exist a positive integer q and a set $\Delta\subset[1, \infty)$ of finite linear measure, such that for $z\bar{\in}\Delta^*$, we have*

$$2|E''/E(z)|+|E'/E(z)|^2<|z|^q,$$

where $\Delta^*=\{z: |z|\in\Delta\}$.

This lemma can be deduced from [2].

From [4, Theorem 4], we have

LEMMA 2. *Suppose that $A(z)$ is an entire function and has Fabry gap such that for some arbitrarily large R we have*

$$\log M(R, A)<R^\lambda, \tag{6}$$

where λ is a positive constant. Let η_1, η_2 be constants between 0 and 1, then there exists a subset $\boldsymbol{E}$ of the real axis, such that the logarithmic measure of $\boldsymbol{E}\cap[1, R]$ is at least $(1-\eta_1)\log R+O(1)$, as $R\to\infty$ through values satisfying (6) and such that we have, for r in $\boldsymbol{E}$,

$$\log L(r, A)>(1-\eta_2)\log M(r, A),$$

where $L(r, A)=\min_{|z|=r}|A(z)|$ and $M(r, A)=\max_{|z|=r}|A(z)|$.

3. Proof of Theorem 1

Let f_1 and f_2 be two linearly independent solutions of (1). Set $E=f_1f_2$, then we note as in [1] that

$$-4A=(c/E)^2-(E'/E)^2+2(E''/E), \tag{7}$$

where c is the constant Wronskian of f_1 and f_2. Applying Nevanlinna theory to (7), we have

$$T(r, E)=N(r, 1/E)+\frac{1}{2}T(r, A)+O(\log(rT(r, E))) \tag{8}$$

as $r\to\infty$ outside a set of finite measure.

We assume that $\rho(E)<\infty$. Since $\mu(A)<1$, $A(z)$ must have infinitely many zeros. Let $a_1, a_2, \cdots, a_{q+1}$ be $q+1$ zeros of $A(z)$ with q as in Lemma 1. Define

$$H(z)=A(z)\Big/\prod_{i=1}^{q+1}(z-a_i),$$

then H is entire and of lower order $\mu(H)=\mu(A)<1$.

Set

$$D(H)=\{z: |H(z)|>1\},$$
$$D(E)=\{z: |E(z)|>1\}.$$

Since $H(z)$ and $E(z)$ are transcendental, there exist $z_i \in \boldsymbol{C}$ $(i=1, 2)$ such that

$$|H(z_1)|>e,$$
$$|E(z_2)|>e.$$

Let Ω_1 (Ω_2) be the connected component of $D(H)$ $(D(E))$ containing the point $z_1(z_2)$. By the maximum modulus principle, we conclude that $\Omega_i (i=1, 2)$ are unbounded.

Set

$$r_0=\max\{1, |z_1|, |z_2|\}.$$

Let $\theta_{it} (i=1, 2; r_0 \leqq t<\infty)$ be the part of the circle $|z|=t$ in Ω_i and $t\theta_i(t)$ the linear measure of θ_{it}.

By Lemma 1 and (7), we deduce that

$$4|A(z)| \leqq |c|^2+|z|^q, \qquad z \in D(E)-\Delta^*. \tag{9}$$

But for $z \in D(H)-\Delta^*$ and $|z|$ sufficiently large, we have

$$|A(z)| \geqq \frac{1}{2}|z|^{q+1}. \tag{10}$$

From (9) and (10), we have for r_0' large enough

$$(D(E)-\Delta^*) \cap (D(H)-\Delta^*) \cap \{z: |z|>r_0'\}=\phi. \tag{11}$$

(11) implies

$$\theta_1(t)+\theta_2(t) \leqq 2\pi, \qquad t \in \Delta. \tag{12}$$

By a theorem of Tsuji [7], we have

$$\log|H(z_1)| \leqq 9\sqrt{2}\exp\left(-\pi\int_{2r_0}^{(1/2)r}\frac{dt}{t\theta_1(t)}\right)\log M(r, H). \tag{13}$$

(13) gives

$$\pi\int_{2r_0}^{(1/2)r}\frac{dt}{t\theta_1(t)} \leqq \log\log M(r, H)+\log(9\sqrt{2}). \tag{14}$$

Set

$$G(r)=\left[2r_0, \frac{1}{2}r\right]-\Delta$$

and

$$\alpha=\varliminf_{r\to\infty}(\log r)^{-1}\pi\int_{G(r)}\frac{dt}{t\theta_1(t)},$$

then, from (14), we have

$$\frac{1}{2}\leqq\alpha\leqq\mu(H). \tag{15}$$

Similarly

$$\pi\int_{G(r)}\frac{dt}{t\theta_2(t)}\leqq\log\log M(r,\ E)+\log(9\sqrt{2}). \tag{16}$$

By Cauchy-Schwarz inequality

$$\int_{G(r)}\frac{\theta_i(t)}{t}dt\int_{G(r)}\frac{dt}{t\theta_i(t)}\geqq\left(\int_{G(r)}\frac{dt}{t}\right)^2, \qquad i=1,\ 2. \tag{17}$$

From (12) and (17) with $i=1$, we obtain

$$\begin{aligned}\int_{G(r)}\frac{\theta_2(t)}{t}dt&\leqq\int_{G(r)}\frac{2\pi-\theta_1(t)}{t}dt\\&=2\pi\int_{G(r)}\frac{dt}{t}-\int_{G(r)}\frac{\theta_1(t)}{t}dt\\&\leqq 2\pi\int_{G(r)}\frac{dt}{t}-\frac{\left(\int_{G(r)}\frac{dt}{t}\right)^2}{\int_{G(r)}\frac{dt}{t\theta_1(t)}}.\end{aligned} \tag{18}$$

(18) and (17) with $i=2$ give

$$\begin{aligned}\int_{G(r)}\frac{dt}{t\theta_2(t)}&\geqq\frac{\left(\int_{G(r)}\frac{dt}{t}\right)^2}{2\pi\int_{G(r)}\frac{dt}{t}-\frac{\left(\int_{G(r)}\frac{dt}{t}\right)^2}{\int_{G(r)}\frac{dt}{t\theta_1(t)}}}\\&=\frac{\int_{G(r)}\frac{dt}{t}}{2\pi-\frac{\int_{G(r)}\frac{dt}{t}}{\int_{G(r)}\frac{dt}{t\theta_1(t)}}}.\end{aligned} \tag{19}$$

Since Δ is of finite linear measure, from (16) and (19), we obtain

$$\begin{aligned}\rho(E)&=\overline{\lim_{r\to\infty}}\frac{\log\log M(r,\ E)}{\log r}\\&\geqq\overline{\lim_{r\to\infty}}(\log r)^{-1}\pi\int_{G(r)}\frac{dt}{t\theta_2(t)}\\&\geqq\frac{1}{2-\frac{1}{\alpha}}\\&=\frac{\alpha}{2\alpha-1}.\end{aligned} \tag{20}$$

Inequalities (15) and (20) give

$$\rho(E) \geqq \frac{\mu(H)}{2\mu(H)-1},$$

which implies

$$\rho(E)^{-1}+\mu(A)^{-1} \leqq 2. \tag{21}$$

We assert that $\rho(E)=\infty$ implies $\lambda(E)=\infty$. If $\lambda(E)<\infty$, we would arrive at a contradiction.

Let $E=Pe^Q$ where P is a canonical product formed by the zeros of E and Q is an entire function, then $\rho(P)<\infty$. Since $\rho(E)=\infty$, Q must be transcendental. From here, we conclude that $\mu(E)=\infty$.

For any $\alpha>1$, we have by (8)

$$\frac{1}{2}T(r, E) \leqq N(\alpha r, 1/E)+\frac{1}{2}T(\alpha r, A), \qquad (r \text{ large enough}). \tag{22}$$

(22), $\mu(A)<1$ and $\lambda(E)<\infty$ imply

$$\mu(E)<\infty.$$

This gives a contradiction.

Similarly, we can prove that if $\rho(E)<\infty$, then $\lambda(E)=\rho(E)$. From (21), we have

$$\lambda(E)^{-1}+\mu(A)^{-1} \leqq 2.$$

We have completed the proof of Theorem 1.

4. Proof of Theorem 2

Applying the Wiman-Valiron theory to (7), we conclude that there exists a set D in $[1, \infty)$ of finite logarithmic measure such that if $r \bar{\in} D$ and z is a point on $|z|=r$ at which $|E(z)|=M(r, E)$, then

$$2|A(z)| \leqq \left(\frac{\nu(r)}{r}\right)^2, \tag{23}$$

where $\nu(r)$ denotes the central index of E. It follows from Lemma 2 that there exists a sequence $r_n \to \infty$ such that $r_n \in \boldsymbol{E}-D$

$$L(r_n, A) \leqq M(r_n, A)^{1/2}. \tag{24}$$

From (23) and (24), we have for any positive integer N

$$r_n{}^N \leqq \frac{\nu(r_n)}{r_n}, \tag{25}$$

which implies

$$\rho(E)=\infty.$$

The proof of Theorem 2 can be completed in the same way as that of Theorem 1.

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DEPARTMENT OF MATHEMATICS
EAST CHINA NORMAL UNIVERSITY
SHANGHAI 200062
P.R. CHINA