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ON SLOWLY INCREASING UNBOUNDED HARMONIC FUNCTIONS

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Abstract

In this paper we prove several growth results for slowly increasing unbounded harmonic functions in the unit disc. They generalize some of the theorems in [4] for our new definition of maximum growth in a finite number of directions.

§ 1. Introduction.

Let $u(z)$ be a harmonic function in a plane domain Ω, consider the level curves of $u(z)$, $l(c)=\{z\in\Omega: u(z)=c\}$ for $-\infty<c<\infty$, and let $\Theta(c)=\int_{l(c)}|*du|$ for $-\infty<c<\infty$, where $*du$ is the differential of the harmonic conjugate function of $u(z)$, with the agreement that $\Theta(c)=0$ if $l(c)=\emptyset$. We have the following definition.

DEFINITION A. Let $u(z)$ be a harmonic function in a domain Ω, if there exists $a\in u(\Omega)$ such that, $\int_a^b \frac{dc}{\Theta(c)}<\infty$ for every $b>a$, and $\lim_{b\to\infty}\int_a^b \frac{dc}{\Theta(c)}=\infty$, then we say that the function $u(z)$ is a slowly increasing unbounded harmonic function.

Observe that by our convention for any bounded harmonic function the above integral will take the value infinity for a finite value of b. If the function grows to infinity fast as we approach the boundary of Ω the integral $\int_a^b \frac{dc}{\Theta(c)}$ increases more slowly, thus in a sense the functions defined above are the most slowly increasing unbounded harmonic functions.

Consider the family of curves $\Gamma(a, b)=\{l(c): a<c<b\}$ for $-\infty\leqq a\leqq b<\infty$. We denote its module by the symbol $\mu(a, b)$, then it is known that if $\Gamma(a, b)\neq\emptyset$ and $(a, b)\subset u(\Omega)$ then

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$$\mu(a, b)=\int_a^b \frac{dc}{\Theta(c)}. \tag{1.1}$$

This equality has proved itself of being of great importance, in particular it gives a beautiful and short proof of the Ahlfors distortion theorem.

For any open set $G\subset\Omega$ if we denote by $\mu_G(a, b)$ the module og $\Gamma_G(a, b)=\{l(c)\cap G : a<c<b\}$, $\mu_G(a, b)=\int_a^b \frac{dc}{\Theta_G(c)}$, where $\Theta_G(c)=\int_{l(c)\cap G}|*du|$.

In Jenkins and Oikawa [4] we can find an account of inequalities for these modules, some of which shall be used in this paper giving the appropriate reference.

Our interest will be focused in the growth of the quantities $\mu[a, u(z)]$, $\mu_{G_j}[a, u(z)]$ as z approaches the boundary of Ω, because in some cases if we can estimate the growth of μ and μ_{G_j} we obtain estimates for the growth of the function $u(z)$.

Hayman in [2] showed that if $f(z)$ is an areally mean p-valent function in the unit disc and $M(r, f)=\max\{|f(z)| : |z|=r\}$, then

$$\alpha=\lim_{r\to 1}(1-r)^{2p}M(r, f)$$

exists and $0\leqq\alpha<\infty$, also if $\alpha\neq 0$ then there exists a unique direction $e^{i\theta_0}$ such that

(i) $$\lim_{r\to 1}\frac{|f(re^{i\theta_0})|}{(1-r)^{2p}}=\alpha,$$

as $r\to 1$, and for any $\varepsilon>0$ there exist constants K_1 and K_2 such that

(ii) $$\lim_{r\to 1}\frac{|f(re^{i\theta})|}{K_1(1-r)^{2p\varepsilon}|\theta-\theta_0|^{2p(1-\varepsilon)}}$$

if $|\theta-\theta_0|>K_2(1-r)$.

Jenkins and Oikawa observed that if the function $f(z)$ is zero free, areally mean p-valent and unbounded, then $u(z)=\log|f(z)|$ is a slowly increasing unbounded harmonic function satisfying the following inequality

$$[u(z)-a]\leqq 2\pi p\mu[a, u(z)]+\tau,$$

where τ is a universal constant.

Using the above inequality and (1.1) they obtained results on the growth of $\mu[a, u(z)]$ and $\mu_G[a, u(z)]$ from which they were able to recover Hayman's results.

We are going to define the reduced module,

DEFINITION B. If $(-\infty, b)\subset u(\Omega)$ and $\Theta(a)$ is constant for a small enough, then

$$\tilde{u}(b)=\frac{a}{\Theta(a)}+\mu(a, b)$$

is called the reduced module of the family $\Gamma(-\infty, b)$, $\tilde{\mu}(b)$ is independent of our choice of a.

Jenkins and Oikawa studied slowly increasing unbounded harmonic function $u(z)$ of maximum growth, i.e. $u(z)$ has maximum growth in the direction $e^{i\theta_0}$ if there exist sequences $\{b_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$ such that:

(i) $$a<b_n\leqq u(r_n e^{i\theta_0}),$$

for $n=1, 2, \cdots$

(ii) $$\lim_{n\to\infty} r_n=1.$$

(iii) $$\limsup_{n\to\infty}\left[\mu(a, b_n)-\frac{1}{\pi}\log\left[\frac{1}{1-r_n}\right]\right]>-\infty.$$

We generalize Jenkins and Oikawa definition to the case in which we have a finite number of directions of maximum growth and different growths in each direction.

For this new definition we generalize some of the results by Jenkins and Oikawa [4] and by Hamilton [1]. Our principal method is the method of the extremal metric making use of the identity (1.1), the basic tool which enables us to obtain these results is the generalized arithmetic-harmonic inequality which says that for $\lambda_1, \cdots, \lambda_n\geqq 0$ and $a_1, \cdots, a_n\geqq 0$ we have

$$\frac{\lambda_1+\cdots+\lambda_n}{\lambda_1/a_n+\cdots+\lambda_n/a_n}\leqq\frac{\lambda_1 a_1+\cdots+\lambda_n a_n}{\lambda_1+\cdots+\lambda_n}.$$

In section 2 we shall state the definitions and results of this paper and in section 3 we shall prove all our results. Finally in section 4 we shall show that Jenkins and Oikawa's condition for maximum growth in one direction can not be replaced by a weaker condition that we will state there, we shall construct a function satisfying this condition but it will not have maximum growth in just one direction.

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§ 2. Statements.

DEFINITION 1. Let $u(z)$ be a harmonic function in $\Omega=\{z: r_0<|z|<1\}$ with $[a, \infty)\subset u(\Omega)$ for some value a. We say that $u(z)$ attains maximum $(\alpha_1, \cdots, \alpha_n)$-

growth in the directions $e^{i\phi_1}, \cdots, e^{i\phi_n}(\sum_{j=1}^{k}\alpha_j=1)$, if there exist sequences $\{b_n\}_{n=1}^{\infty}$, $\{r_n^{[j]}\}_{n=1}^{\infty}$, $j=1, \cdots, k$ with the properties,

(i) $$a<b_n\leqq u[r_n^{[j]}e^{i\phi_j}]$$

for $n=1, 2, \cdots$ $j=1, \cdots, k$,

(ii) $$\lim_{n\to\infty} r_n^{[j]}=1$$

for each $j=1, \cdots, k$,

(iii) $$\limsup_{n\to\infty}\left[\mu(a, b_n)-\sum_{n=1}^{k}\frac{\alpha_j^2}{\pi}\log\left[\frac{1}{1-r_n^{[j]}}\right]\right]>-\infty.$$

In the following definition we are going to translate our situation to the case in which the domain of the function $u(z)$ is a horizontal strip.

DEFINITION 2. We say that a harmonic function $u(z)$ defined on an open horizontal strip $\bar{S}$ of width π attains $(\alpha_1, \cdots, \alpha_k)$-growth $(\sum_{j=1}^{k}\alpha_j=1)$, if the strip is cut along horizontal slits with a finite left end and an infinite right end, such that if we prolong the left ends to infinity the strip $\bar{S}$ is subdivided into k substrips $\bar{S}_j=\{\zeta: l_j<Im\zeta<l_j+\pi\alpha_j\}$ of width $\pi\alpha_j$, $j=1, \cdots, k$, $(-\infty, \infty)\subset u(\bar{S}_j)$, $j=1, \cdots, k$. If we define $\bar{S}_j(\delta)=\{\zeta: l_j+\delta<Im\zeta<l_j+\pi\alpha_j-\delta\}$ for $j=1, \cdots, k$, then

(i) $$\mu_{\bar{S}_j}(a, b)<\infty$$

for $-\infty<a<b<\infty$.

(ii) $$\mu_{\bar{S}_j}(a, b)\longrightarrow\infty$$

as $b\to\infty$.

(iii) $$\limsup_{(\xi_j+i\eta_j)\to\infty} u(\xi_j+i\eta_j)=\infty$$

where $(\xi_j+i\eta_j)\to\infty\in\bar{S}_j(\delta)$, $j=1, \cdots, k$.

Let us remark here that Jenkins and Oikawa's definition of maximum simultaneous growth in the directions $e^{i\phi_1}, \cdots, e^{i\phi_k}$ [4, p. 60] is just Definition 1 for the case $\alpha_1=\cdots=\alpha_k=1/k$.

The next two theorems extend results of Jenkins and Oikawa in [4].

THEOREM 1. *Let $u(z)$ be harmonic in $\Omega=\{z: r_0<|z|<1\}$. If $u(z)$ is as in Definition 1 and if we let $S_j=S_j(\phi_j, \delta_j)=\{z\in\Omega: |arg\, z-\phi_j|<\delta_j\}$, $j=1, \cdots, k$ be mutually disjoint sectors with $a\in u(S_j)$, $j=1, \cdots, k$, then*

(i) $$\lim_{z\to e^{i\phi_j}}\left[\mu_{S_j}[a, u(z)]-\frac{1}{\pi}\log\left[\frac{1}{|z-e^{i\phi_j}|}\right]\right]=\gamma_j$$

$j=1, \cdots, k$, $-\infty<\gamma_j<\infty$, uniformly in Stolz domain.

(ii) $$\mu[a, u(z)]=\frac{\alpha_j}{\pi}\log\left[\frac{1}{|z-e^{i\phi_j}|}\right]+o\left[\left[\log\frac{1}{1-|z|}\right]^{1/2}\right]$$

$j=1, \cdots, k$ uniformly as $z\to e^{i\phi_j}$ in a Stolz domain.

(iii) $$\mu[a, u(z)]=o\left[\left[\log\frac{1}{1-|z|}\right]^{1/2}\right]$$

uniformly as $|z|\to 1$ in every sector whose closure does not meet the rays $\arg z=\phi_j$, $j=1, \cdots, k$.

THEOREM 2. *Let $u(z)$ be a harmonic function in $\Omega=\{z: r_0<|z|<1\}$. If $u(z)$ is as in Definition 1, then for any fixed $\bar{c}>1$,*

(i) *there exist Stolz domains Δ_j with vertices at $e^{i\phi_j}$ and sectors $S_j=S(\phi_j, \delta_j)$, $j=1, \cdots, k$, the sectors being mutually disjoint, such that*

$$u(re^{i\phi_j})>u(se^{i\theta})$$

where $r^{\bar{c}}<s<r^{1/\bar{c}}$, and $se^{i\theta}\in(S_j\setminus\Delta_j)$ for $j=1, \cdots, k$, and r large enough.

(ii)

$$\lim_{r\to 1}\mu_{S_j}[u(re^{i\phi_j}), m_{S_j}(r)]=0, \qquad j=1, \cdots, k$$

where $m_{S_j}(r)=\max\{u(z): |z|=r, z\in S_j\}$.

The next three results improve results in Hamilton [1].

THEOREM 3. *Let $u(z)$ be a harmonic function on an open strip $\bar{S}$ of width π as in Definition 2. If $\zeta_j=\xi_j+i\eta_j\to\infty$ in a proper substrip of $\bar{S}$, and $b=u[\xi_j(b)+i\eta_j(b)]$, $j=1, \cdots k$ then we have that*

$$\lim_{b\to\infty}\left[\mu(a, b)-\frac{1}{\pi}\sum_{j=1}^{k}\alpha_j\xi_j(b)\right]=\alpha$$

with $-\infty\leqq\alpha<\infty$. Moreover, if $h(z)=1/2\log z$ and $u[h(z)]-\lambda/2\log|z|$ has a harmonic extension at the origin, then

$$\lim_{b\to\infty}\left[\tilde{\mu}(b)-\frac{1}{\pi}\sum_{j=1}^{k}\alpha_j\xi_j(b)\right]=\tilde{\alpha}$$

and $-\infty\leqq\tilde{\alpha}\leqq u_0/2\pi\lambda$ where $u_0=\lim_{z\to 0}[u[h(z)]-\lambda/2\log|z|]$, with equality $\tilde{\alpha}=u_0/2\pi\lambda$ only for the function $u(\zeta)=u_0+\lambda\xi$.

COROLLARY 1. *Let $f(z)$ be an areally mean p-valent function in the unit disc Δ with a Taylor series expansion at $z=0$ of the form $f(z)=z^p+\cdots$, such that $\lim_{z\to e^{i\phi_j}}|f(z)|=\infty$, $j=1, \cdots, k$ in some Stolz domains with vertices $e^{i\phi_1}, \cdots, e^{i\phi_k}$. Let us map the unit disc onto the slit strip $\bar{S}$ of width π by means of the conformal mapping*

$$\zeta=\xi+i\eta=\frac{1}{2\pi}\log\left[\frac{z}{(1-ze^{-i\phi_1})^{2\alpha_1}\cdots(1-ze^{-i\phi_k})^{2\alpha_k}}\right].$$

Assume that the function $\log|f[z(\zeta)]|=u(\zeta)$, *where* $z(\zeta)$ *is the inverse of the above function, has* $(\alpha_1, \cdots, \alpha_k)$*-growth as in Definition* 2.

Then under these hypotheses, if $z_j\to e^{i\phi_j}$ *in a Stolz domain and* $|f(z_j)|=R$, $j=1, \cdots, k$

$$\limsup_{R\to\infty} R\prod_{j=1}^{k}[1-|z_j|]^{2p\alpha_j^2}=\alpha\neq 0,$$

where

$$\alpha\leqq\prod_{j=1}^{k}\prod_{q\neq j}|e^{i\phi_j}-e^{i\phi_q}|^{-2p\alpha_j\alpha_q}$$

with equality only for

$$f(z)=\frac{z^p}{\prod_{j=1}^{k}(1-ze^{-i\phi_j})^{2p\alpha_j}}.$$

Before we state our next result, we give the following definition.

DEFINITION 4. We say that a domain Ω is a k-star domain if $\partial\Omega$ consists of k rays such that if we prolong them they pass through the origin, they are not necessarily of equal length, and these prolonged rays are separated by angles $2\pi\alpha_j$, $j=1, \cdots, k$ with $\sum_{j=1}^{k}\alpha_j=1$.

THEOREM 4. *Suppose that* $f(z)=z+\cdots$ *is a circumferentially mean* 1*-valent function defined on a* k*-star domain* Ω. *If* z_j *is a point in the* j*-th sector of* Ω *for which* $|f(z_j)|=R$ *and*

$$\lim_{z_j\to\infty}|f(z_j)|=R\longrightarrow\infty$$

for any $j=1, \cdots, k$, *then*

$$\limsup_{R\to\infty}\frac{R}{\prod_{j=1}^{k}|z_j|^{\alpha_j}}\leqq 1$$

with equality only for $f(z)=z$.

§ 3. Proofs of the results.

Proof of Theorem 1. Consider $\Theta_j(c)=\Theta_{S_j}(c)=\int_{l(c)\cap S_j}|*du|$, $j=1, \cdots, k$, by the generalized arithmetic-harmonic inequality we have that $\int_a^{b_n}\frac{dc}{\Theta(c)}\leqq\sum_{j=1}^{k}\alpha_j^2\int_a^{b_n}\frac{dc}{\Theta_j(c)}$, thus $\sum_{j=1}^{k}\alpha_j^2\mu_{S_j}(a, b_n)-\mu(a, b_n)\geqq 0$.

Since $u(z)$ satisfies Definition 1, we have that $\mu(a, b_n)-\sum_{j=1}^{k}\frac{\alpha_j^2}{\pi}\log\left[\frac{1}{1-r_n^{[j]}}\right]\geqq O(1)$. Thus

$$\sum_{j=1}^{k}\alpha_j^2\mu_{S_j}(a, b_n)-\mu(a, b_n)\leqq\sum_{j=1}^{k}\alpha_j^2\mu_{S_j}(a, b_n)-\sum_{j=1}^{k}\frac{\alpha_j^2}{\pi}\log\left[\frac{1}{1-r_n^{[j]}}\right]+O(1)\leqq O(1),$$

therefore

$$\sum_{j=1}^{k}\alpha_j^2\mu_{S_j}(a, b_n)-\mu(a, b_n)=O(1). \tag{3.1}$$

Proof of (i). Applying Theorem 2 in [4] to each $S_j=S(\phi_j, \delta_j)$, $j=1, \cdots, k$, we have that

$$\lim_{z\to e^{i\phi_j}}\left[\mu_{S_j}[a, u(z)]-\frac{1}{\pi}\log\left[\frac{1}{z-e^{i\phi_j}}\right]\right]=\gamma_j \tag{3.2}$$

in a Stolz domain, $-\infty\leqq\gamma_j<\infty$. Lemma 7 in [4], the arithmetic-harmonic inequality and Definition 1 give that for r large enough

$$-\infty<-M\leqq\sum_{j=1}^{k}\alpha_j^2\left[\left[\int_a^{b_n}\frac{dc}{\Theta_j(c)}-\frac{1}{\pi}\log\left[\frac{1}{1-r}\right]\right]\right]+C.$$

Letting $r\to 1$ and taking $z=re^{i\phi_j}$, $j=1, \cdots, k$ in (3.2) we have that $-\infty<-M\leqq\sum_{j=1}^{k}\alpha_j^2\gamma_j+C$, which implies that $\gamma_j>-\infty$, $j=1, \cdots, k$ and thus (i) is proved.

Proof of (iii). Using the notation and the definitions [4], (3.1) says that $\|\tilde{\rho}\|^2-\|\rho\|^2=O(1)$, where $\tilde{\rho}=\sum_{j=1}^{k}\alpha_j\rho_{S_j}$. By inequality (8) in [4], $\|\rho\|\leqq\|\rho_{S_1\cup\cdots\cup S_k}\|\leqq\|\tilde{\rho}\|$, then

$$\|\rho_{S_1\cup\cdots\cup S_k}\|^2_{\Omega(a,\infty)}-\|\rho\|^2_{\Omega(a,\infty)}=\|\rho_{S_1\cup\cdots\cup S_k}-\rho\|^2_{\Omega(a,\infty)}=O(1)$$

where $\Omega(a, \infty)=\{z\in\Omega: a<u(z)<\infty\}$. By Lemma 8 in [4] we obtain that

$$\mu[a, u(z)]=o\left[\left[\log\frac{1}{1-|z|}\right]^{1/2}\right]$$

uniformly as $|z|\to 1$ on any sector whose closure is disjoint from $(S_1\cup\cdots\cup S_k)$, this proves (iii).

Proof of (ii). As we saw in the proof of (iii) $\|\tilde{\rho}\|^2_{\Omega(a,\infty)}-\|\rho\|^2_{\Omega(a,\infty)}=O(1)$, then given $\varepsilon>0$ we can take b_0 large enough such that

$$\|\tilde{\rho}-\rho\|^2_{\Omega(b_0,\infty)}=\|\tilde{\rho}\|^2_{\Omega(b_0,\infty)}-\|\rho\|^2_{\Omega(b_0,\infty)}<[\min_{1\leqq j\leqq k}\{\alpha_j\}\varepsilon]^2.$$

It is not difficult to see that we have the following identities,

(i) $$\|\rho_{S_j}\|_{\Omega(b_0,b)}=\frac{1}{\alpha_j}\|\tilde{\rho}\|_{\Omega(b_0,b)\cap S_j}, \qquad j=1, \cdots, k,$$

(ii) $$\|\rho\|^2_{\Omega(b_0,b)}=(\rho, \rho_{S_j})_{\Omega(b_0,b)}=\frac{1}{\alpha_j}(\rho, \tilde{\rho})_{\Omega(b_0,b)\cap S_j}.$$

Using (i) and (ii) we find that,

$$\alpha_j[\mu(b_0, b)-\alpha_j\mu_j(b_0, b)]=\alpha_j[\|\rho\|^2_{\Omega(b_0,b)}-\alpha_j\|\rho_{S_j}\|_{\Omega(b_0,b)}]$$
$$=(\rho, \tilde{\rho})_{\Omega(b_0,b)\cap S_j}-\|\rho-\tilde{\rho}\|^2_{\Omega(b_0,b)\cap S_j}.$$

Hence,

$$|\mu(b_0, b)-\alpha_j\mu_j(b_0, b)|=\frac{1}{\alpha_j}|(\rho, \rho-\tilde{\rho})_{\Omega(b_0,b)\cap S_j}-\|\rho-\tilde{\rho}\|^2_{\Omega(b_0,b)\cap S_j}|.$$

Using the Cauchy-Schwarz inequality in the first term in the right hand side we have that

$$|\mu(b_0, b)-\alpha_j\mu_j(b_0, b)|\leqq\varepsilon[\|\rho\|_{\Omega(b_0,b)}+\varepsilon].$$

We are going to consider two possible cases, first if $u(z)>b_0$ we have that

$$|\mu[a, u(z)]-\alpha_j\mu_j[a, u(z)]|\leqq|\mu(a, b_0)-\alpha_j(a, b_0)|+|\mu[b_0, u(z)]-\alpha_j\mu_j[b_0, u(z)]|$$
$$\leqq\mu(a, b_0)+\alpha_j\mu_j(a, b_0)+\varepsilon[\|\rho\|_{\Omega[b_0,u(z)]}+\varepsilon]. \quad (3.3)$$

Since $\|\rho\|^2_\Omega=\mu[b_0, u(z)]$,

$$\|\rho\|_{\Omega[b_0,u(z)]}=[\mu[b_0, u(z)]]^{1/2}\leqq\left[\frac{1}{\pi}\log\left[\frac{1}{1-|z|}\right]+A\right]^{1/2},$$

thus

$$|\mu[a, u(z)]-\alpha_j\mu_j[a, u(z)]|<\varepsilon\left[\log\left[\frac{1}{1-|z|}\right]\right]^{1/2}$$

for $|z|$ close to 1.

If $a<u(z)\leqq b_0$ inequality (3.3) is obviously satisfied, and the same conclussion holds. Therefore the above inequality and (i) say that,

$$\mu[a, u(z)]=\frac{\alpha_j}{\pi}\log\left[\frac{1}{|z-e^{i\phi_j}|}\right]+o\left[\left[\log\frac{1}{1-|z|}\right]^{1/2}\right]$$

for $j=1, \cdots, k$ uniformly as $z\to e^{i\phi_j}$ in a Stolz domain, this ends the proof of (ii), and the Theorem.

Proof of Theorem 2. First we observe that (ii) follows from (i) using (i) in Theorem 1, thus we only need to prove (i).

Fix $j\in\{1, \cdots, k\}$ and a sector $S_j=S(\phi_j, \delta_j)$, and transform the variable z to ζ by $\zeta=\log(1/ze^{-i\phi_j})$, let $z(\zeta)$ be its inverse function. The image of S_j under the map $\zeta(z)$ is the rectangle $R_j=\{\zeta: 0<Re\,\zeta<\log 1/r_0, |Im\,\zeta|<\delta_j\}$, with $e^{i\phi_j}$ corresponding to $\zeta=0$. We consider the function $u(\zeta)=u[z(\zeta)]$, then $\limsup_{\xi\to 0}u(\xi)=\infty$. For $u(\zeta)$ part (i) of Theorem 1 says that

$$\limsup_{\zeta\to 0}\left[\mu_{R_j}[a, u(\zeta)]-\frac{1}{\pi}\log\left[\frac{1}{|\zeta|}\right]\right]=\gamma_j \quad (3.4)$$

$-\infty<\gamma_j<\infty$, uniformly as $\zeta\to 0$ in a Stolz domain in R_j.

We have to prove that, there exist a Stolz domain Δ_j in R_j with vertex at $\zeta=0$, a value $\xi^*_j(0<\xi^*_j<\log 1/r_0)$, and $\delta^*_j, 0<\delta^*_j<\delta_j$; such that for $0<\xi<\xi^*$,

$\xi/\tilde{c}\leqq\xi\leqq\tilde{c}\xi$, $|\eta|<\delta_j^*$ and $(\hat{\xi}+i\eta)\notin\Delta_j$, $u(\xi)>u(\tilde{\xi}+i\eta)$.

Call $\zeta=\hat{\xi}+i\eta$ it is enough to prove that $\int_{u(\zeta)}^{u(\xi)}\frac{dc}{\Theta_{R_j}(c)}>0$.

Consider $\Delta_j'=\{\zeta:\zeta=\xi+i\eta, |\eta|<2e^{\pi/2}\xi\}$, take $\tilde{\zeta}=\tilde{\xi}+i\eta\notin\Delta_j'$, and $|\eta|<\delta_0^{[j]}=\min[2\delta_j/3, 1/2\log 1/r_0]$, set $\xi'=(|\eta|/2e^{\pi/2})$, then

$$\int_{u(\zeta)}^{u(\xi)}\frac{dc}{\Theta_{R_j}(c)}=\int_{u(\zeta)}^{u(\xi')}\frac{dc}{\theta_{R_j}(c)}+\int_{u(\xi')}^{u(\xi)}\frac{dc}{\Theta_{R_j}(c)}.$$

We are going to estimate the two integrals in the right hand side. We start with the second integral,

$$\int_{u(\xi')}^{u(\xi)}\frac{dc}{\Theta_{R_j}(c)}=\mu_{R_j}[a, u(\xi)]-\mu_{R_j}[a, u(\xi')],$$

by (3.4) it is not difficult to show that

$$\int_{u(\xi')}^{u(\xi)}\frac{dc}{\Theta_{R_j}(c)}=\frac{1}{\pi}\log\frac{|\eta|}{\xi}+O(1) \tag{3.5}$$

as $\tilde{\zeta}\to 0$, $\tilde{\zeta}\notin\Delta_j'$.

We need to estimate now $\int_{u(\tilde{\zeta})}^{u(\xi')}\frac{dc}{\Theta_{R_j}(c)}$. For this we use the following estimate in [4, p. 55].

$$\int_{u(\xi')}^{u(\tilde{\zeta})}\frac{dc}{\Theta_{R_j}(c)}\leqq\left[\frac{1}{2}+o(1)\right]\left[\log\frac{|\eta|}{\tilde{\xi}}+O(1)\right]^{1/2} \tag{3.6}$$

as $\tilde{\zeta}\to 0$, $\tilde{\zeta}\notin\Delta_j'$.

Using (3.5) and (3.6) we have that

$$\int_{u(\tilde{\zeta})}^{u(\xi)}\frac{dc}{\Theta_{R_j}(c)}\geqq\frac{1}{\pi}\log\frac{|\eta|}{\xi}+O(1)-\left[\log\frac{|\eta|}{\tilde{\xi}}+O(1)\right]^{1/2}.$$

Since $\tilde{\zeta}=\tilde{\xi}+i\eta\notin\Delta_j'$ and $\xi/\tilde{c}\leqq\xi\leqq\tilde{c}\xi$, the right hand side is bounded below by

$$\frac{1}{\pi}\log\frac{|\eta|}{\tilde{\xi}}-\frac{1}{\pi}\log\tilde{c}+O(1)-\left[\log\frac{|\eta|}{\tilde{\xi}}+O(1)\right]^{1/2}$$

as $\tilde{\zeta}\to 0$, thus we can choose a wide Stolz domain Δ_j and small ξ_j^*, δ_j^* such that for $0<\xi<\xi^*$, $\xi/\tilde{c}\leqq\xi\leqq\tilde{c}\xi$, $|\eta|<\delta_j^*$ and $\tilde{\zeta}=(\tilde{\xi}+i\eta)\notin\Delta_j$, $\int_{u(\tilde{\zeta})}^{u(\xi)}\frac{dc}{\Theta_{R_j}(c)}>0$, and therefore $u(\xi)>u(\tilde{\zeta})$, since the integrand is positive, and this finishes the proof of the theorem.

Proof of Theorem 3. Applying Theorem 2 in [4] to each substrip $\bar{S}_j$, $j=1,\cdots,k$ we have that

$$\lim_{b\to\infty}\left[\mu_{\bar{s}_j}(a, b)-\frac{1}{\pi\alpha_j}Re\{\zeta_j(b)\}\right]=\gamma_j$$

$j=1, \cdots, k$, with $-\infty \leqq \gamma_j < \infty$, $\zeta_j(b)$ in a proper substrip of $\bar{S}_j$ and $u[\zeta_j(b)]=b$, $j=1, \cdots, k$. Multiplying each of these limits by α_j^2 and adding them we get that

$$\lim_{b\to\infty}\left[\sum_{j=1}^{k}\alpha_j^2\mu_{\bar{S}_j}(a, b)-\frac{\alpha_j}{\pi}Re\{\zeta_j(b)\}\right]=\gamma$$

with $-\infty \leqq \gamma < \infty$. By the generalized arithmetic-harmonic means inequality we obtain that there exists

$$\limsup_{b\to\infty}\left[\mu(a, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}Re\{\zeta_j(b)\}\right]=\tilde{\gamma}$$

with $-\infty \leqq \tilde{\gamma} < \infty$. We want to show that this lim inf is in fact a limit. To show the existence of the limit we shall need the following definitions on each substrip $\bar{S}_j$, $j=1, \cdots, k$.

Fix a point ξ^0 in the real axis and let $\zeta_j^0=\xi^0+i\eta_j^0 \in \bar{S}_j$, $j=1, \cdots, k$. For each $c>u(\zeta_j^0)$ let $D_j(c)$ be the component of $\{\zeta \in \bar{S}_j : u(\zeta)<c\}$ containing ζ_j^0. It is simply connected and each component of $[\bar{S}_j \cap [\partial D_j(c)]]$ is a piecewise analytic arc contained in $l(c)$. If $D_j(c)$ is bounded to the right, then the unbounded component $B_j(c)$ of $[\bar{S}_j \setminus cl\ D_j(c)]$ is uniquely determined. For each $j=1, \cdots, k$ fix a value $a_0^{[j]}$ and for each $c>a_0^{[j]}$ take $\gamma_j(c)=\bar{S}_j \cap cl\ D_j(c) \cap cl\ B_j(c)$, this is a piecewise open analytic arc joining the upper and lower edges of $\bar{S}_j$. Set $\xi_j'(c)=\inf\{Re\,\zeta : \zeta \in \gamma_j(c)\}$ and $\xi_j''(c)=\sup\{Re\,\zeta : \zeta \in \gamma_j(c)\}$, by Lemma 2 in [4], $\lim_{c\to\infty}\xi_j'(c)=\infty$. Let

$$\liminf_{b\to\infty}\left[\mu_{\bar{S}_j}(a_0^{[j]}, b)-\frac{1}{\pi}\xi_j'(b)\right]=\alpha_0^{[j]}$$

$j=1, \cdots, k$, and denote by $\bar{\mu}_j(a, b)$ the module of the family $\{\gamma_j(c) : a<c<b\}$, $a_0^{[j]}<a<b$. After these preliminaries we proceed to prove the existence of the limit

$$\lim_{b\to\infty}\left[\mu(a, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}Re\{\zeta_j(b)\}\right].$$

Let $a_0=\max_{1\leqq j\leqq k}\{a_0^{[j]}\}$, we are going to prove that there exists the limit

$$\lim_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}Re\{\zeta_j(b)\}\right].$$

For this we prove that

$$\lim_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j''(b)\right]=\lim_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j'(b)\right]$$

and then, since $\xi'(b) \leqq Re\{\zeta_j(b)\} \leqq \zeta_j''(b)$, $j=1, \cdots, k$ the result follows.

Let $a_0<b<b'$, then $\mu(a_0, b')-\mu(a_0, b)=\mu(b, b')$, clearly $\{\gamma_j(c) : j=1, \cdots, k, b<c<b'\} \subset \Gamma(b, b')$, thus we have that

$$\mu(b, b') \leqq \sum_{j=1}^{k}\alpha_j^2\tilde{\mu}_{\bar{S}_j}(b, b'),$$

and so

$$\mu(a_0, b')-\mu(a_0, b)\leqq\sum_{j=1}^{k}\frac{\alpha_j}{\pi}[\xi_j''(b')-\xi_j'(b)].$$

We can rewrite the above inequality as follows

$$\mu(a_0, b')-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j''(b')\leqq\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j'(b), \tag{3.7}$$

taking the lim sup in the left hand side of the above inequality as $b\to\infty$, and then taking the lim inf as $b\to\infty$ in the right hand side, we obtain

$$\limsup_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j''(b)\right]\leqq\liminf_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j'(b)\right]. \tag{3.8}$$

If $\xi_j''(b)<\xi_j'(b')$ for each $j=1, \cdots, k$ by a result in [3, p. 186]

$$\mu(b, b')\leqq\sum_{j=1}^{k}\frac{\alpha_j}{\pi}[\xi_j'(b')-\xi_j''(b)]+2\sum_{j=1}^{k}\alpha_j^2.$$

Using the above inequality in (3.7) and following the same steps than before we get

$$\limsup_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j'(b)-2\sum_{j=1}^{k}\alpha_j^2\right]\leqq\liminf_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j''(b)\right].$$

Let $\bar{\alpha}_0=\liminf_{b\to\infty}[\mu(a_0, b)-\sum_{j=1}^{k}(\alpha_j/\pi)\xi_j'(b)]$, $-\infty\leqq\bar{\alpha}_0<\infty$. We consider two different cases.

Case I. If $\bar{\alpha}_0=-\infty$, then clearly the limit is equal to $-\infty$.

Case II. If $\bar{\alpha}_0>-\infty$, by Lemma 4 in [4] we have the two identities

$$\limsup_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j''(b)\right]=\liminf_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j'(b)\right], \tag{3.9}$$

and

$$\liminf_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j''(b)\right]=\liminf_{b\to\infty}\left[\mu(a_0, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j'(b)\right]. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10) we obtain that $\lim_{b\to\infty}[\mu(a_0, b)-\sum_{j=1}^{k}(\alpha_j/\pi)$ $Re\{\zeta_j(b)\}]=\alpha_0$ with $-\infty\leqq\alpha_0<\infty$. Therefore

$$\lim_{b\to\infty}\left[\mu(a, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j(b)\right]=\alpha,$$

where $\alpha=\alpha_0+\mu(a, a_0)$, this proves the first part of Theorem 3.

By hypothesis $(u\circ h)(z)=\lambda/2\log|z|+v(z)$ in a neighborhood of the origin, with $v(z)$ harmonic. Therefore $e^{f(z)}=z^{\lambda/2}e^{g(z)}$ where $Re\{f(z)\}=(u\circ h)(z)$ and $Re\{g(z)\}=v(z)$, $g(z)$ is analytic in a neighborhood of the origin. For a small $\Theta(a)=\int_{l(a)}|*du|=\int_{h^{-1}[l(a)]}|*d(u\circ h)|=\pi\lambda$. Thus $\bar{\mu}(b)=a/\Theta(a)+\mu(a, b)=a/\pi\lambda+$ $\mu(a, b)$. Therefore the existence of

$$\tilde{\alpha}=\lim_{b\to\infty}\left[\tilde{\mu}(b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j(b)\right]$$

follows from the first part of the Theorem.

We proceed to estimate $\tilde{\alpha}$. For small a, $l(a)$ is a simple arc joining the upper and lower edges of the strip. Applying for each $j=1, \cdots, k$ the result in [3, p. 186] we obtain the inequality

$$\mu(a, b)\leqq\sum_{j=1}^{k}\frac{\alpha_j}{\pi}[\xi_j''(b)-\xi_j'(a)]-\sum_{j=1}^{k}\alpha_j^2 f[\xi_j''(c)-\xi_j'(c)].$$

Adding $a/\pi\lambda$ to both sides of the above inequality we have that

$$\frac{a}{\pi\lambda}+\mu(a, b)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j''(b)\leqq\frac{a}{\pi\lambda}-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j'(a)-\sum_{j=1}^{k}\alpha_j^2 f[\xi_j''(c)-\xi_j'(c)],$$

the left hand side goes to $\tilde{\alpha}$ for fixed a as $b\to\infty$, therefore by the definition of $\xi_j'(a)$ we have that

$$\tilde{\alpha}\leqq\frac{u_0}{\pi\lambda}+o(1)-\sum_{j=1}^{k}\alpha_j^2 f[\xi_j''(c)-\xi_j'(c)] \tag{3.11}$$

as $a\to\infty$, since $f(x)$ is a strictly increasing function, $\tilde{\alpha}\leqq u_0/\pi\lambda$.

Equality in (3.11) is possible if and only if $\xi_j''(c)=\xi_j'(c)$, $j=1, \cdots, k$ and c large, then $u(\zeta)$ must be a linear transformation of the form $(a+b\xi)$, since $[(u\circ h)(z)-\lambda/2\log|z|]$ is harmonic at the origin, $u(\zeta)=u_0+\lambda\xi$. This finishes the proof of the Theorem.

Proof of Corollary 1. Let

$$\zeta(z)=\xi(z)+i\eta(z)=\frac{1}{2}\log\left[\frac{z}{(1-ze^{-i\phi_1})^{2\alpha_1}\cdots(1-ze^{-i\phi_k})^{2\alpha_k}}\right],$$

then $\lim_{z\to e^{i\phi_j}}[\xi(z)+\alpha_j\log[1-|z|]+\log\beta_j]=0$ for each $j=1, \cdots, k$, where $\beta_j=\prod_{q\neq j}|e^{i\phi_q}-e^{i\phi_j}|^{\alpha_q}$.

We proceed to estimate the reduced module $\tilde{\mu}(\log R)$ from below. The function $[u(z)-p\log|z|]$ is harmonic at the origin, thus for $\bar{R}$ small enough we have that $\Theta(\log\bar{R})=2p\pi$, and since the function $f(z)$ is an areally mean p-valent function, we have that

$$\begin{aligned}\tilde{\mu}(\log R)&=\frac{\log\bar{R}}{2p\pi}+\frac{1}{2\pi}\int_{\bar{R}}^{R}\frac{d\tilde{R}}{\tilde{R}p(\tilde{R})}\\&\geqq\frac{\log\bar{R}}{2p\pi}+\frac{1}{2p\pi}\log\frac{R}{\bar{R}}=\frac{1}{2p\pi}\log R.\end{aligned}$$

Substracting $\sum_{j=1}^{k}(\alpha_j/\pi)\xi_j(\log R)$ from both sides of this inequality we have that

$$\frac{1}{2p\pi}\log R-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j(\log R)\leqq\tilde{\mu}(\log R)-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j(\log R).$$

The limit of the right hand side as $R\to\infty$ is less than or equal to zero by Theorem 3 ($u_0=0$), thus

$$\lim_{R\to\infty}\left[\frac{1}{2p\pi}\log R-\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\xi_j(\log R)\right]$$

$$=\lim_{R\to\infty}\left[\frac{\log R}{2p\pi}+\sum_{j=1}^{k}\frac{\alpha_j^2}{\pi}\log(1-|z_j|)+\sum_{j=1}^{k}\frac{\alpha_j}{\pi}\log\beta_j+o(1)\right]\leqq 0,$$

or equivalently,

$$\log R+\sum_{j=1}^{k}2p\alpha_j^2\log(1-|z_j|)\leqq-\sum_{j=1}^{k}2p\alpha_j\log\beta_j+o(1)$$

as $R\to\infty$. Exponentiating this inequality, and taking the lim sup in both sides as $R\to\infty$ we obtain that

$$\limsup_{R\to\infty}R\prod_{j=1}^{k}(1-|z_j|)^{2p\alpha_j^2}=\alpha\neq 0$$

where

$$\alpha\leqq\prod_{j=1}^{k}\beta_j^{-2p\alpha_j}=\prod_{j=1}^{k}\prod_{q\neq j}|e^{\iota\phi_q}-e^{\iota\phi_j}|^{-2p\alpha_j\alpha_q}.$$

By the equality statement in Theorem 3, we have equality only if $u(\zeta)=2p\xi$, which means that $\log|f[z(\zeta)]|=2p\xi$, thus $f(z)=\dfrac{z^p}{\prod_{j=1}^{k}(1-ze^{-\iota\phi_j})^{2p\alpha_j}}$. This ends the proof of Corollary 1.

Proof of Theorem 4. We can not apply Theorem 3 in this case, since we do not know where the points z_j are located, thus we shall use another technique in the proof of this result. To do that we have to restrict ourselves to functions $f(z)$ which are circumferentially mean 1-valent.

We map Ω onto a strip of width 2π with horizontal slits with their left ends finite and their right ends infinite by the map $\zeta=\log z$, if we prolong these slits to infinity we have subdivided the strip $\bar{S}$ into substrips $\bar{S}_j$ of width $2\pi\alpha_j$, $j=1,\cdots,k$.

Let the harmonic function $u(\zeta)=\log|f[z(\zeta)]|$, be defined on $\bar{S}$, where $z(\zeta)$ is the inverse function of the principal branch of $\zeta=\log z$. We apply to each $\bar{S}_j$, $j=1,\cdots,k$ Lemma 4 in [4] for $n=1$, to obtain

$$\mu_j(\log\rho,\log R)\leqq\frac{\xi_j''(\log R)-\xi_j'(\log\rho)}{2\pi\alpha_j}-f[\xi_j''(c)-\xi_j'(c)]$$

for $\log\rho<c<\log R$, thus

$$\sum_{j=1}^{k}\alpha_j^2\int_{\log\rho}^{\log R}\frac{dx}{\Theta_j(x)}=\sum_{j=1}^{k}\alpha_j^2\mu_j(\log\rho,\log R)$$

$$\leqq\sum_{j=1}^{k}\frac{\alpha_j}{2\pi}[\xi_j''(\log R)-\xi_j'(\log\rho)]-\sum_{j=1}^{k}\alpha_j^2 f[\xi_j''(c)-\xi_j'(c)]$$

where $\Theta_j(x)$ is the length of the image under the function $\log f(z)$ of $l_j(x)=l(x)\cap\bar{S}_j$. Using the generalized arithmetic-harmonic inequality in the left hand side of the above, and since $\sum_{j=1}^{k}\Theta_j(x)\leqq 2\pi$ for any value of x for a circumferentially mean 1-valent function, we have that

$$\frac{1}{2\pi}[\log R-\log\rho]\leqq\sum_{j=1}^{k}\frac{\alpha_j}{2\pi}[\xi_j''(\log R)-\xi_j'(\log\rho)]-\sum_{j=1}^{k}\alpha_j^2 f[\xi_j''(c)-\xi_j'(c)].$$

Now we proceed to estimate $\xi_j''(\log R)$ and $\xi_j'(\log\rho)$, $j=1,\cdots,k$ for R large and ρ close to zero. For $\xi_j''(\log R)$, we distinguish two cases:

Case (i). If $\alpha_0^{[j]}=\lim_{b\to\infty}[\mu_j(a_0^{[j]},b)-(1/2\pi\alpha_j)\xi_j'(b)]=-\infty$, then $\mu_j(a_0^{[j]},b)\leqq(1/2\pi\alpha_j)\xi_j'(b)$ for large b and choosing $b=\log R$ we obtain that

$$\mu_j(\log\rho,\log R)\leqq\frac{1}{2\pi\alpha_j}[\xi_j'(\log R)-(\log\rho)]\leqq\frac{1}{2\pi\alpha_j}[\log|z_j|-\log\rho].$$

Case (ii). If $\alpha_0^{[j]}>-\infty$ then by Lemma 4 in [4] we have that $\lim_{c\to\infty}[\xi_j''(c)-\xi_j'(c)]=0$, thus $\xi_j''(\log R)=\log|z_j|+o(1)$ as $r\to\infty$. By definition $\xi_j'(\log\rho)=\inf\{Re\,\zeta_j:\zeta_j\in\bar{S}_j,\ u(\zeta_j)=\log\rho\}$, since $f(z)=z+\cdots$ and $z(\zeta)=e^{\zeta}$, then $\log|f[z(\zeta)]|=Re\{\zeta_j\}+o(1)$ as $\zeta_j\to-\infty$ and so $u(\zeta_j)=\log|f[z(\zeta)]|=Re\{\zeta_j\}+o(1)=\log\rho$. Hence, $\log\rho+o(1)=Re\{\zeta_j\}$ as $\rho\to 0$, thus $\xi_j'(\log\rho)=\log\rho+o(1)$ as $\rho\to 0$.

Using the above estimates we obtain that

$$\frac{1}{2\pi}[\log R-\log\rho]\leqq\sum_{j=1}^{k}\frac{\alpha_j}{2\pi}[\log|z_j|+o(1)-\log\rho+o(1)]-\sum_{j=1}^{k}\alpha_j^2 f[\xi_j''(c)-\xi_j'(c)].$$

Multiplying the inequality by 2π and observing that the terms in both sides in $\log\rho$ cancel we get

$$\log R\leqq\sum_{j=1}^{k}[\log|z_j|^{\alpha_j}+o(1)]-2\pi\sum_{j=1}^{k}\alpha_j^2 f[\xi_j''(c)-\xi_j'(c)].$$

Since the second term in the right hand side is non-negative we have that

$$\log R\leqq\sum_{j=1}^{k}[\log|z_j|^{\alpha_j}+o(1)]$$

as $R\to\infty$, exponentiating this inequality we get that

$$\frac{R}{\prod_{j=1}^{k}|z_j|^{\alpha_j}}\leqq e^{o(1)}$$

as $R\to\infty$, thus taking the lim sup as $R\to\infty$ we obtain

$$\limsup_{R\to\infty}\frac{R}{\prod_{j=1}^{k}|z_j|^{\alpha_j}}\leqq 1.$$

Equality holds only if $f[\xi_j''(c)-\xi_j'(c)]=0$ for $j=1,\cdots,k$ and c large, since $f(z)$ is positive and strictly increasing this is possible only if $\xi_j''(c)=\xi_j'(c)$ for any $j=1,\cdots,k$ and c large, which means that $f(z)$ is a linear mapping of the form $(az+b)$, and since by hypothesis $f(z)=z+\cdots$ this implies that $f(z)=z$. This

ends the proof of Theorem 4.

§ 3. A counterexample.

Let $u(z)$ be a slowly increasing unbounded harmonic function in $\Omega=\{z: r_0<|z|<1\}$ with $[a, \infty)\subset u(\Omega)$, then $u(z)$ attains maximum growth in one direction $e^{i\varphi_0}$ if

$$\infty>\alpha=\lim_{r\to 1}\left[\mu[a, u(re^{i\varphi_0})]-\frac{1}{\pi}\log\left[\frac{1}{1-r}\right]\right]>-\infty,$$

or equivalently if $u(z)$ is defined in the strip $S=\{\zeta: |Im\,\zeta|<\pi/2\}$, if

$$\infty>\alpha=\lim_{\xi\to\infty}\left[\mu[a, u(\xi)]-\frac{\xi}{\pi}\right]>-\infty.$$

A natural question to ask is whether a weaker condition on $\mu[a, u(\xi)]$ still implies that we have maximum growth in one direction, namely, does $\lim_{\xi\to\infty}\left[\frac{\mu[a, u(\xi)]}{\xi}\right]=\gamma<\infty$ imply maximum growth in one direction?

In this section we are going to answer this question negatively by constructing a slowly increasing unbounded harmonic function $u(\zeta)$ in the strip S for which $\lim_{\xi\to\infty}\left[\frac{\mu[a, u(\xi)]}{\xi}\right]=\beta$ with β finite and such that $\lim_{\xi\to\infty}[\mu[a, u(\xi)]-\xi/\pi]=-\infty$.

We define the following strip-like domain H in the W-plane, $H=\{W=U+iV: \Phi_-(U)<V<\Phi_+(U), -\infty<U<\infty\}$ where,

$$\Phi_+(U)=\begin{cases}\pi, & \text{if } U\leqq 0;\\ \pi+\sqrt{U}, & \text{otherwise,}\end{cases}$$

and

$$\Phi_-(U)=\begin{cases}0, & \text{if } U\leqq 0;\\ \sqrt{U}, & \text{otherwise.}\end{cases}$$

Map H conformally onto S by the function $Z(W)=X(W)+iY(W)$, where $\lim_{U\to\infty}X(W)=\infty$. The domain H is a L-strip according to the definition in [5, p. 280], with boundary inclination $\gamma=0$ at $U=\infty$. Let $W(Z)=U(Z)+iV(Z)$ be the inverse function of $Z(W)$. We want to show that $\lim_{X\to\infty}\left[\frac{\mu[U_0, U(X)]}{X}\right]$ exists and is finite, and also that $\lim_{X\to\infty}[\mu[U_0, U(X)]-X/\pi]=-\infty$.

Using the notation in [5], $\theta(U)=\Phi_+(U)-\Phi_-(U)=\pi$ and,

$$\Psi(U)=\frac{1}{2}[\Phi_+(U)-\Phi_-(U)]=\begin{cases}\pi/2, & \text{if } U\leqq 0;\\ \pi/2+\sqrt{U}, & \text{otherwise.}\end{cases}$$

If we choose $U_0>0$, then $\int_{U_0}^{\infty}\frac{\theta'^2(U)}{\theta(U)}dU=0$ converges trivially. Then by [5, p. 281] we have the following asymptotic representation for $Z(W)$

$$Z(W)=\lambda+\pi\int_{U_0}^{U}\frac{1+\Psi'^2(t)}{\theta(t)}dt+i\pi\frac{V-\Psi(U)}{\theta(U)}+o(1)$$

as $U\to\infty$ uniformly with respect to V, and λ is a real constant. We can choose W in H so that $Z(W)=X$, $W=U(X)+iV(X)$, thus

$$X=\lambda+\pi\int_{U_0}^{U(X)}\frac{1+\Psi'^2(t)}{\theta(t)}dt+i\pi\frac{V(X)-\Psi(U(X))}{\theta(U(X))}+o(1),$$

as $X\to\infty$. Taking real parts in both sides of the above inequality we obtain that

$$X=\lambda+\pi\int_{U_0}^{U(X)}\frac{1+\Psi'^2(t)}{\theta(t)}dt+o(1)$$

as$X\to\infty$, for our choice of Φ_+ and Φ_-,

$$X=\lambda+\pi\int_{U_0}^{U(X)}\frac{dt}{\pi}+\int_{U_0}^{U(X)}\frac{dt}{4t}+o(1)$$

as $X\to\infty$. We know that $\int_{U_0}^{U(X)}\frac{dt}{\pi}=\mu[U_0, U(X)]$, hence

$$-\frac{1}{4\pi}\log\frac{U(X)}{U_0}-\frac{\lambda}{\pi}=\mu[U_0, U(X)]-\frac{X}{\pi}+o(1)$$

as $X\to\infty$. Taking limits in both sides as $X\to\infty$, we have that

$$\lim_{X\to\infty}\left[\mu[U_0, U(X)]-\frac{X}{\pi}\right]=\lim_{X\to\infty}\left[-\frac{1}{4\pi}\log\frac{U(X)}{U_0}-\frac{\lambda}{\pi}\right],$$

The limit in the right hand side is $-\infty$, thus $\lim_{X\to\infty}[\mu[U_0, U(X)]-X/\pi]=-\infty$. It remains to show that $\lim_{X\to\infty}\left[\frac{\mu[U_0, U(X)]}{X}\right]$ exists and is finite. We know that

$$X=\lambda+\pi\int_{U_0}^{U(X)}\frac{dt}{\pi}+\frac{1}{4}\log\frac{U(X)}{U_0}+o(1)$$

$$=\lambda+[U(X)-U_0]+\frac{1}{4}\log\frac{U(X)}{U_0}+o(1)$$

as $X\to\infty$, we divide both sides of the equality by πX to get that

$$-\frac{1}{4\pi}\frac{1}{X}\log\frac{U(X)}{U_0}-\frac{\lambda}{\pi X}=\frac{\mu[U_0, U(X)]}{X}-\frac{1}{\pi}+o(1)$$

as $X\to\infty$. It is not difficult to show that $\lim_{X\to\infty}1/X\log(U(X)/U_0)=0$, thus taking the limit as $X\to\infty$ in both sides we obtain that

$$\lim_{x\to\infty}\left[\frac{\mu[U_0, U(X)]}{X}\right]=\frac{1}{\pi},$$

as we wanted to show.

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