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LIE CONTACT STRUCTURES AND CONFORMAL STRUCTURES

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§ 0. Introduction.

In [SY] and [M1], the notion of Lie contact structures on a $(2n-1)$-dimensional contact manifold is established as a geometry on a manifold corresponding to the classical Lie sphere geometry [CC]. Following the connection theory by N. Tanaka [T], we construct a normal Cartan connection ω (called *Tanaka connection*, for brevity) corresponding to the structure in [M1], which is the main tool to solve the equivalence problem (see [SY]).

A typical and important example of the structure exists on the unit tangent bundle T_1M of an n-dimensional Riemannian manifold M. In this paper, we calculate the curvature K of Tanaka connection of this structure on T_1M. We call *K the Lie curvature* of T_1M. In particular, when $K\equiv 0$, T_1M is called *Lie flat*, and is locally Lie equivalent to the model space$=T_1S^n$, the unit tangent bundle of the standard n-sphere [SY]. This is apparently the case when M is conformally flat (§ 1). The inverse problem is presented by Sato [S]: Is M conformally flat when T_1M is Lie flat?

The purpose of this paper is to answer this problem affirmatively. The description of Tanaka connection and its curvature for this structure is given in Theorem in § 5, where the Lie curvature is expressed in terms of all coefficients of Weyl's conformal curvature. As a result, we know that the structure depends only on the conformal structure of M, and moreover we obtain

COROLLARY 1. *Let M be a Riemannian manifold of* $\dim\geqq 3$. *Then M is conformally flat if and only if T_1M is Lie flat.*

COROLLARY 2. *Let M and M' be two Riemannian manifolds of* $\dim\geqq 3$. *Let $\tilde{f}: T_1M\to T_1M'$ be a bundle map which preserves the Lie curvature. Then the induced map $f: M\to M'$ preserves the conformal curvature.*

A resume of [M1] and the present paper is given in [M2].

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§ 1. Preliminaries.

In this paper, we follow the argument in [M1] and use the notations in it.

Let $\boldsymbol{R}_2^{n+3}=\{x=(x^0, \cdots, x^{n+2}), x^i\in\boldsymbol{R}\}$ be an $(n+3)$-dimensional real vector space endowed with a scalar product $\langle , \rangle$ with signature $(+, \cdots, +, -, -)$ and let $\boldsymbol{R}_1^{n+2}=\{x\in\boldsymbol{R}_2^{n+3}, x^{n+2}=0\}$. Denote by P^{n+2} and P^{n+1} the associated projective spaces. Furthermore, let $\boldsymbol{R}^{n+1}=\{x\in\boldsymbol{R}_1^{n+2}, x^{n+1}=0\}$ be the $(n+1)$-dimentional space-like subspace of $\boldsymbol{R}_1^{n+2}$. By $\langle , \rangle$, we denote the induced scalar product on $\boldsymbol{R}_1^{n+2}$ or on $\boldsymbol{R}^{n+1}$. Now, the unit sphere $S^n=\{x\in\boldsymbol{R}^{n+1}|\langle x, x\rangle=1\}$ is naturally embedded in P^{n+1} as a Möbius space Q^n,

$$S^n\cong Q^n=\{[y]\in P^{n+1}|\langle y, y\rangle=0\},$$

by $x\to(x, 1)\in\boldsymbol{R}_1^{n+2}$. On the other hand, let Σ be the set of all oriented hyperspheres in S^n; $\Sigma=\{(m, \theta)\in S^n\times[0, \pi) \mid$ an oriented hypersphere with center m and radius $\theta\}$. Then Σ is naturally embedded in P^{n+2} as a quadratic Q^{n+1},

$$\Sigma\cong Q^{n+1}=\{[k]\in P^{n+2}|\langle k, k\rangle=0\},$$

by $(m, \theta)\to(m, \cos\theta, \sin\theta)\in\boldsymbol{R}_2^{n+3}$.

The Möbius group L is, by definition, a group consists of projective transformations of P^{n+1} preserving Q^n, and we have $L=PO(n+1, 1)$. The Lie transformation group G is, by definition, a group consists of projective transformations of P^{n+2} preserving Q^{n+1}, and we get $G=PO(n+1, 2)$. Clearly we have $L\subset G$. Now, let $\Lambda^{2n-1}=\{$lines in Q^{n+1} generated by $([k_1], [k_2])\in Q^{n+1}\times Q^{n+1}$, $\langle k_1, k_2\rangle=0\}$. Then we have

$$T_1S^n=\{(u, v)\in S^n\times S^n|\langle u, v\rangle=0\}\cong\Lambda^{2n-1}$$

under a mapping $(u, v)\to([k_1], [k_2])$, where $k_1=(u, 1, 0)$, $k_2=(v, 0, 1)$. Since G preserves $\langle , \rangle$, it induces an action on Λ^{2n-1}. This action resticted to L is translated as follows: A Möbius transformation $\sigma: S^n\to S^n$ is lifted to Lie transformations $\sigma_\pm: T_1S^n\to T_1S^n$, by

$$(*)\qquad \sigma_\pm(X)=\pm\sigma_*X/\|\sigma_*X\|.$$

We denote the subgroup $\sigma_+(L)$ of G by G_M. It is easy to see that G_M, and so G acts on Λ^{2n-1} transitively. Let G'_M and G' be isotropy subgroups:

FACT 1.1. $\Lambda^{2n-1}=G/G'=G_M/G'_M$.

As is shown in [M1], the Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g}=\sum_{p=-2}^{2}\mathfrak{g}_p, \quad [\mathfrak{g}_i, \mathfrak{g}_j]=\mathfrak{g}_{i+j},$$

$$\mathfrak{g}_{-2}={}^t\mathfrak{g}_2=\left\{\begin{pmatrix}0 & 0 & c_p\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, c_p=\begin{pmatrix}0 & p\\ -p & 0\end{pmatrix}\right\},$$

$$\mathfrak{g}_{-1}={}^t\mathfrak{g}_1=\left\{\begin{pmatrix}0 & b & 0\\ 0 & 0 & {}^tb\\ 0 & 0 & 0\end{pmatrix},\ {}^tb\in \boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}\right\},$$

$$\mathfrak{g}_0=\left\{\begin{pmatrix}a & 0 & 0\\ 0 & e & 0\\ 0 & 0 & -{}^ta\end{pmatrix},\ a\in\mathfrak{gl}(2,\boldsymbol{R}),\ e\in\mathfrak{o}(n-1)\right\}.$$

Here, note that a base of $\boldsymbol{R}_2^{n+3}$ is chosen so that

$$\langle u, v\rangle=-2u^0v^{n+1}-2u^1v^{n+2}+\sum_{i=2}^{n}u^iv^i,$$

for $u=(u^0, u^1, \cdots, u^{n+2})$ and $v=(v^0, v^1, \cdots, v^{n+2})\in\boldsymbol{R}_2^{n+3}$. Thus we have $\boldsymbol{R}_1^{n+2}=\{u\in\boldsymbol{R}_2^{n+3},\ u^{n+2}=-(1/2)u^1\}$, and $\boldsymbol{R}^{n+1}=\{u\in\boldsymbol{R}_1^{n+2},\ u^{n+1}=-(1/2)u^0\}$. We may assume that $G_M=\{h\in G\,|\,h$ preserves $\boldsymbol{R}_1^{n+2}\}$. Then the Lie algebra $\mathfrak{g}_M$ of G_M is given by

$$\mathfrak{g}_M=\left\{\begin{pmatrix}a & b & c_p\\ d & e & {}^tb\\ {}^tc_q & {}^td & -{}^ta\end{pmatrix}\in\mathfrak{g},\ a=\begin{pmatrix}* & -\frac{1}{2}p\\ -2q & 0\end{pmatrix},\ d=(d_1, d_2),\ {}^tb=({}^tb_1, -2{}^td_2)\right\}.$$

Now, we have $\mathfrak{g}'=\mathfrak{g}_0+\mathfrak{g}_1+\mathfrak{g}_2$, $\mathfrak{g}'_M=\mathfrak{g}'\cap\mathfrak{g}_M$, where $\mathfrak{g}'$ and $\mathfrak{g}'_M$ are Lie algebras of G' and G'_M, respectively. Note that $\mathfrak{o}(n-1)\subset\mathfrak{co}(n-1)\subset\mathfrak{g}_0\cap\mathfrak{g}_M\subset\mathfrak{g}'_M\subset\mathfrak{g}'$. From these facts, we get

FACT 1.2. [see Lemma 1.2, M1].

$$G'=\left\{h=\begin{pmatrix}A & 0 & 0\\ gd & g & 0\\ {}^tA^{-1}\left\{\frac{1}{2}{}^tdd+f\right\} & {}^tA^{-1}\,{}^td & {}^tA^{-1}\end{pmatrix},\ \begin{matrix}g\in O(n-1) \quad A\in GL(2,\boldsymbol{R})\\ d\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1},\ f=\begin{pmatrix}0 & -q\\ q & 0\end{pmatrix}\end{matrix}\right\},$$

$$G'_M=\left\{h\in G'\,|\,A=\begin{pmatrix}\alpha & 0\\ \gamma & 1\end{pmatrix},\ gd=(*, 0),\ \alpha\neq 0\right\},$$

and $O(n-1)\subset CO(n-1)\subset G'_M\subset G'$.

Put, $\mathfrak{m}=T_0(G/G')$, $\tilde{G}=\rho(G')$ and $\tilde{G}_M=\rho(G'_M)$, where $\rho: G', G'_M\to GL(\mathfrak{m})=GL(2n-1)$ is the linear isotropy representation. Since $\operatorname{Ker}\rho=\exp\mathfrak{g}_2$, denoting $\rho(O(n-1))=O(n-1)$ and $\rho(CO(n-1))=CO(n-1)$, we get

FACT 1.3. [see Proposition 1.3, M1].

$$\tilde{G}=\left\{\begin{pmatrix}\det A & 0 & 0\\ * & g\otimes A & \\ * & & \end{pmatrix}\right\},\quad \tilde{G}_M=\left\{\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha g & 0\\ * & \gamma g & g\end{pmatrix}\right\},$$

where A, g, α, γ are given in Fact 1.2, and

$$O(n-1)\subset CO(n-1)\subset\tilde{G}_M\subset\tilde{G}\,. \tag{1.1}$$

Now, let N be a $(2n-1)$-dimensional contact manifold. It is well-known that the linear frame bundle $L(N)$ has a reduction $L^\#(N)$ with structure group $G_0(\mathfrak{m})^\#=\left\{\begin{pmatrix}a & 0\\ \zeta & CSp(n-1, \boldsymbol{R})\end{pmatrix},\ a\neq 0\right\}$. Noting that $\tilde{G}\subset G_0(\mathfrak{m})^\#$, [M1], we define:

DEFINITION. *A $\tilde{G}$-reduction of $L^\#(N)$ is called a Lie contact structure on N.*

Now, recall the way of construction of Lie contact structure on the unit tangent bundle T_1M of an n-dimensional riemannian manifold (M, g). Let Q_g be the principal fibre bundle over M with structure group $O(n)$. According to [KN, p. 57], $P_g=(Q_g/O(n-1), O(n-1))$ is a principal fibre bundle over T_1M with structure group $O(n-1)$. It is shown in [M1] that the extended bundle

$$\tilde{P}_g=Q_g\times_{O(n-1)}\tilde{G}\,,$$

gives a Lie contact structure on T_1M.

It is obvious that T_1M is Lie flat if M is conformally flat, since a conformal transformation is lifted to a Lie transformation by (*). But it is a non-trivial matter to see whether M is conformally flat when T_1M is Lie flat, since the structure group is enlarged. The purpose of this paper is to solve this question.

For later use, recall the geometry of the unit tangent bundle T_1M of an n-dimensional riemannian manifold M. Let $z_1\in T_1M$ and let $(z_1, \cdots, z_n)$ be an orthonormal frame of M at $p=\pi_1(z_1)\in M$, where $\pi_1\colon T_1M\to M$ is the projection. By using the horizontal lift z_i^h and the vertical lift z_i^v of $z_i\in T_pM$ to $T_{z_1}TM$, we make a frame $u(z)=(u_1, \cdots, u_{2n-1})$ of T_1M at z_1, where $u_i=z_i^h$, $1\leqq i\leqq n$, and $u_{n+i-1}=z_i^v$, $2\leqq i\leqq n$. Note that z_1^v is a normal vector of T_1M in TM. It is well-known that $u(z)$ is an orthonormal frame of $T_{z_1}T_1M$ with respect to the metric on T_1M induced from the Sasaki metric on TM. Now, let $h\in O(n-1)$ and put $\tilde{h}=\begin{pmatrix}1 & 0\\ 0 & h\end{pmatrix}\in O(n)$. We make h act on $u(z)$ by

$$u(z)h=u(z\tilde{h})\,. \tag{1.2}$$

Then we obtain an $O(n-1)$-bundle $\pi\colon P_g\to T_1M$, where $P_g=\{u(z)|z=(z_i)$ is an orthonormal frame of M at $\pi_1\circ\pi(z)=\pi_1(z_1)\}$. We have shown in the end of the proof of [M1, Proposition 2.3] that $u(z)$ is a frame adapted to the Lie contact structure.

§ 2. Construction of a normal Cartan connection (Q, χ).

In this section, following the argument in [M1], we construct a normal Cartan connection of type H/H_0 on an H_0-reduction (Q, ζ) of the Lie contact structure $\tilde{P}$ over T_1M, when M is a Rimannian manifold. Let $\pi: \tilde{P} \to T_1M$ be the projection. Here, we start with the $O(n-1)$-reduction P_g of $\tilde{P}$.

Let $\mathfrak{k} = \mathfrak{m} + \mathfrak{o}(n-1)$, where $\mathfrak{o}(n-1)$ is the Lie algebra of $O(n-1)$, and let K be a Lie subgroup of G of which Lie algebra is $\mathfrak{k}$. As is mentioned in § 1, an element $u(z) \in P_g$ is an orthonormal base of $T_{z_1}T_1M$, at $z_1 = \pi(u(z))$ with respect to the metric s_g induced from the Sasaki metric s_g on TM. Therefore, as a basic form on P_g, we should take

$$\zeta^i(X) = s_g(\pi_* X, u_i), \quad 1 \leqq i \leqq n,$$

$$\zeta^{\bar{i}}(X) = s_g(\pi_* X, u_{\bar{i}}), \quad 2 \leqq i \leqq n,$$

where $X \in T_{u(z)}P_g$ and we put $u_{\bar{i}} = u_{n+i-1}$, $2 \leqq i \leqq n$. We will express them in a local coordinate of P_g. Around $u_0 = u(z_0) \in P_g$, where $z_0 = (z_1, \cdots, z_n)$, $\pi(u_0) = z_1$, and $\pi_1(z_1) = p \in M$, we choose a local coordinate (x^i, z^i_j), $1 \leqq i, j \leqq n$, as follows: let $(x^1, \cdots, x^n)$ be the geodesic normal coordinate of M around p such that $z_i(p) = \partial/\partial x^i$, and let $(z^i_j) \in GL(n, \boldsymbol{R})$ be such that

$$g_{ij} z^i_k z^j_m = \delta_{km}, \tag{2.1}$$

where g_{ij} is the component of the Riemannian metric g on M with respect to $(x^1, \cdots, x^n)$. Here and hereafter, we use Einstein convension for $1 \leqq i, j, k, m, r, s, t, u, v \leqq n$, unless otherwise stated. Note that we have

$$\begin{cases} z^i_j(p) = \delta^i_j, \\ g_{ij}(p) = \delta_{ij}, \\ \dfrac{\partial}{\partial x^k} g_{ij}(p) = 0, \\ \left\{ {i \atop jk} \right\}(p) = 0, \end{cases} \tag{2.2}$$

where $\left\{ {i \atop jk} \right\}$ is the Christoffel's symbol. Let (x^i, v^i) be a local coordinate of TM expressing $v^i(\partial/\partial x^i) \in T_{(x^i)}M$, and $(x^i, v^i, \xi^i, \eta^i)$ be a coordinate of TTM expressing $\xi^i(\partial/\partial x^i) + \eta^i(\partial/\partial v^i) \in T_{(x^i, v^i)}TM$. The coeffecients of the Sasaki metric ${}_g s$ on TM are then given by [SS]

$$G_{ij} = g_{ij} + g_{ru} \left\{ {r \atop si} \right\} \left\{ {u \atop tj} \right\} v^s v^t,$$

$$G_{i\, n+j} = g_{rj} \left\{ {r \atop ui} \right\} v^u,$$

$$G_{n+i\,n+j}=g_{ij}.$$

Since we can express $u\in P_g$ in a neighbourhood of u_0 by

$$(2.3)\qquad \begin{cases} u_i=\left(z_i^j,\ -\left\{{j \atop st}\right\}z_1^s z_i^t\right), & 1\leq i\leq n \\ u_{\bar{i}}=(0,\ z_i^j), & 2\leq i\leq n \end{cases}$$

we can take (x^i, z_j^i), $1\leq i,\ j\leq n$, satisfying (2.1) as a local coordinate of P_g. Moreover, in this local coordinate, we can show easily that the basic forms are expressed by

$$\zeta^i=g_{jk}z_i^k dx^j,\quad 1\leq i\leq n,$$

$$\zeta^{\bar{i}}=g_{jk}z_i^k\left(dz_1^j+\left\{{j \atop st}\right\}z_1^s dx^t\right),\quad 2\leq i\leq n.$$

Now, define

$$\chi'^i_r=g_{jk}z_i^k\left(dz_r^j+\left\{{j \atop st}\right\}z_r^s dx^t\right),\quad 2\leq i,\ r\leq n,$$

and put $\chi'^i=\zeta^i$, $1\leq i\leq n$, $\chi'^{\bar{i}}=\zeta^{\bar{i}}$, $2\leq i\leq n$.

LEMMA 2.1. *χ' is a Cartan connection of type $K/O(n-1)$ on P_g.*

Proof. Obviously, χ' is an $\mathfrak{k}$-valued 1-form on P_g. Then χ' is a Cartan connection of type $K/O(n-1)$ iff

C1) For $X\in TP_g$, $\chi'(X)=0$ implies $X=0$.
C2) $\chi'(A^*)=A$, $A\in\mathfrak{o}(n-1)$ and A^* is the fundamental vector field.
C3) $R_a^*\chi'=Ad(a^{-1})\chi'$, $a\in O(n-1)$.

For $X=(dx^i, dz_j^i)\in T_{u_0}P_g$, we have $\chi'^i(X)=dx^i$, $\chi'^{\bar{i}}(X)=dz_1^i$, $\chi'^i_r(X)=dz_r^i$, and so C1) is obvious. For $A=(A_r^i)\in\mathfrak{o}(n-1)$, put $a_t=\exp tA=(a_j^i(t))\in O(n-1)$. We use the Einstein convension over $2\leq i,\ j\leq n$ as far as $a_j^i(t)$ is concerned. Since $u_0a_t=u(z_0\tilde{a}_t)$ by (1.2), the local coordinate expression of $A_{u_0}^*$ is given by $dx^i=0$, $dz_1^i=0$, and $dz_r^i=(da_r^i)(0)=A_r^i$, and we get $\chi'(A_{u_0}^*)=A$. Now, for $a=(a_j^i)\in O(n-1)$ and $X=(dx^i, dz_j^i)\in T_{u_0}P_g$, from $R_a^*\chi'(X)=\chi'(R_{a*}X)$ and $R_{a*}X=(dx^i,\ dz_1^i,\ d(z_k^i a_r^k))$, $1\leq i\leq n$, $2\leq j$, $r\leq n$ at $u_0a\in P_g$, it follows

$$\zeta^1(R_{a*}X)=dx^1,$$

$$\zeta^i(R_{a*}X)=\sum_{k=2}^n a_i^k dx^k,\quad 2\leq i\leq n,$$

$$\zeta^{\bar{i}}(R_{a*}X)=\sum_k a_i^k dz_1^k,\quad 2\leq i\leq n,$$

$$\chi'^{i}_{r}(R_{a*}X)=\sum_{j,k}a^{j}_{i}(dz^{j}_{k})a^{k}_{r}\,,\quad 2\leqq i,\ r\leqq n\,.$$

On the other hand $Ad(a^{-1})\chi'(X)=a^{-1}\chi'(X)a={}^{t}a\chi'(X)a$ is given by

$$\begin{pmatrix} I_2 & 0 & 0 \\ 0 & {}^{t}a & 0 \\ 0 & 0 & I_2 \end{pmatrix}\begin{pmatrix} 0 & \hat{\zeta} & \hat{\zeta}^1 \\ 0 & \hat{\chi}' & {}^{t}\hat{\zeta} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} I_2 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & I_2 \end{pmatrix}=\begin{pmatrix} 0 & \hat{\zeta}a & \hat{\zeta}^1 \\ 0 & {}^{t}a\hat{\chi}'a & {}^{t}a{}^{t}\hat{\zeta} \\ 0 & 0 & 0 \end{pmatrix},$$

where $\hat{\zeta}=\begin{pmatrix} \zeta^2 \cdots \zeta^n \\ \zeta^{\bar{2}} \cdots \zeta^{\bar{n}} \end{pmatrix}$, $\hat{\zeta}^1=\begin{pmatrix} 0 & \zeta^1 \\ -\zeta^1 & 0 \end{pmatrix}$, and $\hat{\chi}'=(\chi'^{i}_{r})$, $2\leqq i,\ r\leqq n$. Thus by an easy calculation, we get C3). q.e.d.

Now, enlarging the structure group to $H_0=\{a\in G_0 \mid \det a=\pm 1\}$, we get a principal fibre bundle $Q=P_g\times_{O(n-1)}H_0$ over T_1M. A local coordinate of Q is given by (x^i, z^i_j, h^a_b), where $(h^a_b)\in \pm SL(2, \boldsymbol{R})$, since $H_0/O(n-1)\cong \pm SL(2, \boldsymbol{R})$. Denote by $\tilde{\chi}'$ the Cartan connection on Q naturally extended from χ' on P_g, that is, at $u=(x^i, z^i_j, h^a_b)=u_0h$, where $h=\rho\left(\begin{pmatrix} h^a_b & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & {}^{t}(h^a_b)^{-1} \end{pmatrix}\right)=\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & h^0_0 I_{n-1} & h^0_1 I_{n-1} \\ 0 & h^1_0 I_{n-1} & h^1_1 I_{n-1} \end{pmatrix}$, using C3), we have

$$\tilde{\zeta}^1=\zeta^1\,,$$

$$\tilde{\zeta}^i=h^1_1\zeta^i-h^0_1\zeta^{\bar{i}}\,,$$

$$\tilde{\zeta}^{\bar{i}}=-h^1_0\zeta^i+h^0_0\zeta^{\bar{i}}\,,$$

$$(\tilde{\chi}'^{i}_{r})=(\chi'^{i}_{r})\,,$$

$$(\tilde{\chi}'^{a}_{b})=\begin{pmatrix} (sv+tu)\chi'^0_0+uv\chi'^0_1-st\chi'^1_0 & 2tv\chi'^0_0+v^2\chi'^0_1-t^2\chi'^1_0 \\ -2su\chi'^0_0-u^2\chi'^0_1+s^2\chi'^1_0 & -(tu+sv)\chi'^0_0-uv\chi'^0_1+st\chi'^1_0 \end{pmatrix},$$

for $(a, b)=(0, 0), (0, 1)$ and $(1, 0)$, where

$$\chi'^{a}_{b}=dh^a_b\,,$$

putting $\begin{pmatrix} s & t \\ u & v \end{pmatrix}=(h^a_b)$, and ζ^r and χ' are evaluated at u_0. In the following, we use the notation χ' instead of $\tilde{\chi}'$ for simplicity. Let Ψ' be the curvature form of χ' and put $\Psi'=(1/2)T'\zeta\wedge\zeta$. With respect to the base of $\mathfrak{g}$ given in [M1, §3], the $\mathfrak{g}_p$-component T'_p is given by $T'_{-2}=T'^{1}_{\beta\gamma}e_1$, $T'_{-1}=T'^{i}_{\beta\gamma}e_i+T'^{\bar{i}}_{\beta\gamma}e_{\bar{i}}$, $2\leqq i\leqq n$, and $T'_0=T'^{i}_{r\beta\gamma}e^r_i+T'^{0}_{0\beta\gamma}e^0_0+T'^{0}_{1\beta\gamma}e^1_0+T'^{1}_{0\beta\gamma}e^0_1$, $2\leqq i,\ r\leqq n$, and $\beta, \gamma\in\{1, \cdots, n, \bar{2}, \cdots, \bar{n}\}$.

PROPOSITION 2.2. *The curvature T' of χ' at $u_0\in P_g$ is given by*

$$T'_{-2}=0\,,$$

$$T'^{i}_{1\bar{j}}=\delta^i_j\,,\quad T'^{i}_{\beta\gamma}=0 \quad \textit{otherwise},$$

$$T'^{\bar{i}}_{1j}=R^{i}_{11j},\quad T'^{\bar{i}}_{jk}=R^{i}_{1jk},\quad T'^{\bar{i}}_{j\bar{r}}=0,$$

$$T'^{i}_{jst}=R^{i}_{jst},\quad T'^{i}_{j\bar{k}\bar{m}}=\delta^{i}_{k}\delta_{jm}-\delta^{i}_{m}\delta_{jk},\quad T'^{i}_{j\beta\gamma}=0\quad \textit{otherwise},$$

$$T'^{0}_{0\beta\gamma}=T'^{0}_{1\beta\gamma}=T'^{1}_{0\beta\gamma}=0,$$

where $i, j, k\in\{2, \cdots, n\}$, $s, t\in\{1, \cdots, n\}$, $\beta, \gamma\in\{1, \cdots, n, \bar{2}, \cdots, \bar{n}\}$, *and* R^{i}_{jkm} *denotes the coefficients of the Riemannian curvature of the base manifold* M *with respect to* (x^{i}) *at* $p=\pi_{1}\circ\pi(u_{0})$.

To prove this, we prepare:

LEMMA 2.3. *We have the following formulas:*

(2.5) $$dx^{i}=z^{i}_{j}\zeta^{j},\quad \text{and at } u_{0},\quad dx^{i}=\zeta^{i},\quad 1\leqq i\leqq n,$$

(2.6) $$dz^{i}_{1}=z^{i}_{j}\zeta^{\bar{j}}-\left\{{i \atop st}\right\}z^{s}_{1}dx^{t},\quad \text{and at } u_{0},\quad dz^{i}_{1}=\zeta^{\bar{i}},\quad 2\leqq i\leqq n,$$

(2.7) $$dz^{i}_{r}=z^{i}_{j}\chi'^{j}_{r}-\left\{{i \atop st}\right\}z^{s}_{1}z^{t}_{r},\quad \text{and at } u_{0},\quad dz^{i}_{r}=\chi'^{i}_{r},\quad 2\leqq i, r\leqq n,$$

(2.8) $$0=dz^{i}_{j}+dz^{j}_{i}\quad \text{at } u_{0},\quad 1\leqq i\leqq n,$$

(2.9) $$0=\partial_{k}z^{i}_{j}\quad \text{for } 1\leqq j<i\leqq n \text{ and at } u_{0} \text{ for } 1\leqq i, j, k\leqq n,$$

(2.10) $$\partial_{\bar{r}}z^{i}_{1}=\delta^{i}_{r},\quad 2\leqq i, r\leqq n,$$

(2.11) $$\partial_{\bar{r}}(z^{i}_{j})=\delta^{i}_{r}\delta_{j1}-\delta_{rj}\delta^{i}_{1}\quad \text{at } u_{0} \text{ for } 1\leqq i, j\leqq n,\quad 2\leqq r\leqq n,$$

where we use $\partial_{i}=\partial/\partial x^{i}$ *and* $\partial_{\bar{r}}=\partial/\partial z^{r}_{1}$.

Proof. Since $g_{jk}z^{k}_{i}z^{j}_{m}=\delta_{im}$, (y^{i}_{j}) given by $y^{i}_{j}=g_{jk}z^{k}_{i}$ is the inverse matrix of (z^{i}_{j}). The first three are direct consequence of this fact and (2.2). From $0=d(g_{ij}z^{i}_{k}z^{j}_{m})=\delta_{ij}\{(dz^{i}_{k})\delta^{j}_{m}+\delta^{i}_{k}dz^{j}_{m}\}$ at u_{0} follows (2.8). Since we may consider z^{i}_{j}, $1\leqq j<i$, as free variables, we have $\partial_{k}(z^{i}_{j})\equiv0$, for $1\leqq k\leqq n$, $1\leqq j<i\leqq n$. Especially at u_{0}, by virtue of (2.8), we get (2.9). In the same way, since $\partial_{\bar{r}}=\partial/\partial z^{r}_{1}$, $2\leqq r\leqq n$, we get (2.10) for $2\leqq i$, $r\leqq n$, and $\partial_{\bar{r}}z^{i}_{j}\equiv0$ for $2\leqq i<j\leqq n$. The last formula follows from (2.8). q. e. d.

Proof of Proposition 2.2. Since $\chi'^{a}_{b}=dh^{a}_{b}$, $(a, b)=(0, 0)$, $(0, 1)$, $(1, 0)$, and since $\Psi'=(1/2)T'\zeta\wedge\zeta$, we may ignore the terms χ'^{a}_{b} in the structure equation (see (3.1) in § 3), when we calculate the curvature. It is ovbious that $T'^{a}_{b\beta\gamma}=0$. Now, we obtain

$$\Psi'^{1}=d\zeta^{1}+\sum_{i=2}^{n}\zeta^{i}\wedge\zeta^{\bar{i}}$$

$$=d(g_{jk}z^{k}_{1}dx^{j})+\sum_{i=2}^{n}dx^{i}\wedge dz^{i}_{1}$$

$$=\sum_{j=1}^{n} dz_1^j\wedge dx^j+\sum_{i=2}^{n} dx^i\wedge dz_1^i$$

$$=0,$$

i.e. $T'_{-2}=0$. Next, we get

$$\Psi'^i=d\zeta^i+\sum_{r=2}^{n}\chi'^i_r\wedge\zeta^r$$

$$=d(g_{jk}z_i^k dx^j)+\sum_{r=2}^{n} g_{jk}z_i^k\left(dz_r^j+\left\{{j \atop st}\right\}z_1^s dx^t\right)\wedge\zeta^r$$

$$=\sum_{j=1}^{n} dz_i^j\wedge dx^j+\sum_{r=2}^{n} dz_r^i\wedge dx^r$$

$$=dz_i^1\wedge dx^1$$

$$=-\zeta^{\bar{i}}\wedge\zeta^1,\quad i\geqq 2,$$

i.e. $T'^{i}_{1\bar{i}}=1$ and $T'^{i}_{\beta\gamma}=0$ otherwise. From

$$\Psi'^{\bar{i}}=d\zeta^{\bar{i}}+\sum_{r=2}^{n}\chi'^i_r\wedge\zeta^{\bar{r}}$$

$$=d\left(g_{jk}z_i^k\left(dz_1^j+\left\{{j \atop st}\right\}z_1^s dx^t\right)\right)+\sum_{r=2}^{n} dz_r^i\wedge dz_1^r$$

$$=\sum_{j=1}^{n} dz_i^j\wedge dz_1^j+d\left\{{i \atop 1t}\right\}\wedge dx^t+\sum_{r=2}^{n} dz_r^i\wedge dz_1^r$$

$$=\left(\partial_s\left\{{i \atop 1t}\right\}\right)dx^s\wedge dx^t,\quad i\geqq 2,$$

follow $T'^{\bar{i}}_{st}=R^i_{1st}$ and $T'^{\bar{i}}_{j\gamma}=0$. Now, we have

$$\Psi'^i_r=d\chi'^i_r+\sum_{k=2}^{n}\chi'^i_k\wedge\chi'^k_r$$

$$=d\left(g_{jk}z_i^k\left(dz_r^j+\left\{{j \atop st}\right\}z_r^s dx^t\right)\right)+\sum_{k=2}^{n} dz_k^i\wedge dz_r^k$$

$$=\sum_{j=1}^{n} dz_i^j\wedge dz_r^j+d\left\{{i \atop rt}\right\}\wedge dx^t+\sum_{k=2}^{n} dz_k^i\wedge dz_r^k$$

$$=dz_i^1\wedge dz_r^1+\partial_s\left\{{i \atop rt}\right\}dx^s\wedge dx^t$$

$$=\zeta^{\bar{i}}\wedge\zeta^{\bar{r}}+\partial_s\left\{{i \atop rt}\right\}dx^s\wedge dx^t,$$

i.e. $T'^{i}_{rst}=R^{i}_{rst}$, $T'^{i}_{rj\bar{k}}=\delta^{i}_{j}\delta_{rk}-\delta^{i}_{k}\delta_{rj}$ and $T'^{i}_{r\beta\gamma}=0$ otherwise. q.e.d.

Since (Q, χ') is an H_0-reduction desired in [M1, Proposition 5.1], we can apply [M1, Proposition 5.2] to it. Namely, as in the existence proof of the connection (Q, χ) there, put

$$A^{1}_{0j}=-\frac{1}{n-2}\sum_{i=2}^{n}T'^{\bar{i}}_{ji}=\frac{1}{n-2}\sum_{i}R^{i}_{1ij}=\frac{1}{n-2}R_{1j},$$

$$A^{0}_{0j}=A^{0}_{1j}=A^{1}_{0\bar{j}}=A^{0}_{0\bar{j}}=A^{0}_{1\bar{j}}=A^{i}_{ki}=A^{k}_{i\bar{i}}=A^{k}_{ij}=A^{k}_{i\bar{j}}=0,$$

for $2\leqq i, j, k\leqq n$, where we use the Ricci curvature tensor $R_{jk}=\sum_{i=1}^{n}R^{i}_{jik}$ of M at p. The scalar curvature tensor of M is denoted by $R=\sum_{j=1}^{n}R_{jj}$. In the following, we also use the notation $\tilde{R}_{jk}=\sum_{i=2}^{n}R^{i}_{jik}$, $1\leqq j, k\leqq n$, and $\tilde{R}=\sum_{i=2}^{n}\tilde{R}_{jj}$. Immediately, we have

$$
\begin{aligned}
&R_{1j}=\tilde{R}_{1j}, \quad 1\leqq j\leqq n,\\
(2.12)\qquad &R_{jk}=\tilde{R}_{jk}+R^{1}_{j1k}, \quad 2\leqq j, k\leqq n,\\
&R=\sum_{i=2}^{n}R_{ii}+R_{11}=\tilde{R}+2R_{11}.
\end{aligned}
$$

Let $\Psi''=(1/2)T''\zeta\wedge\zeta$ be the curvature of the connection χ'' defined by

$$\chi''^{1}_{0}=\chi'^{1}_{0}+\sum_{j=2}^{n}A'_{0j}\zeta^{j}$$

and $\chi''=\chi'$ for other indices. Then we have $\Psi''=\Psi'$ except for

$$\Psi''^{1}_{0}-\Psi'^{1}_{0}=d\Big(\sum_{j=2}^{n}A^{1}_{0j}\zeta^{j}\Big)-2\chi'^{0}_{0}\wedge A^{1}_{0j}\zeta^{j}.$$

Note that $T''^{1}_{0i\bar{i}}=-\partial_{\bar{i}}A^{1}_{0i}$. Now, putting

$$\chi^{\alpha}_{\beta}=\chi''^{\alpha}_{\beta}+A^{\alpha}_{\beta 1}\zeta^{1},$$

where

$$A^{0}_{01}=-\frac{1}{2(n-1)}\sum(T''^{i}_{1i}-T''^{0}_{0i\bar{i}})=0,$$

$$A^{0}_{11}=-\frac{1}{2(n-1)}\Big(\sum_{i=2}^{n}T''^{i}_{1i}-T''^{0}_{0i\bar{i}}\Big)=-\frac{1}{2},$$

$$A^{1}_{01}=-\frac{1}{2(n-1)}\sum_{i=2}^{n}(T''^{\bar{i}}_{1i}-T''^{1}_{0i\bar{i}})=\frac{1}{2(n-1)}(R_{11}-\sum_{i}\partial_{\bar{i}}A^{1}_{0i}),$$

$$A^{i}_{j1}=-\frac{1}{n+3}(T''^{i}_{1j}-T''^{j}_{1i}+T''^{\bar{i}}_{1\bar{j}}-T''^{\bar{j}}_{1\bar{i}}-\sum_{k}T''^{i}_{jk\bar{k}})=0,$$

we obtain the desired (Q, χ). Here, note that the curvatures given in Proposition 2.2 and so these coefficients are given pointwise. Thus, to get A^1_{01} explicitely, we must compute

$$\partial_j A^1_{0i} = -\frac{1}{n-2}\sum_{m=2}^{n}(\partial_j T'^{\bar{m}}_{im}).$$

From

$$\Psi'^{\bar{m}} = d\left(g_{rk}z^k_m\left(dz^r_1 + \begin{Bmatrix} r \\ st \end{Bmatrix} z^s_1 dx^t\right)\right) + \sum_{r=2}^{n} \chi'^m_r \wedge \zeta^{\bar{r}}$$

$$\equiv \partial_u\left(g_{rk}z^k_m \begin{Bmatrix} r \\ st \end{Bmatrix} z^s_1\right) z^u_i z^t_v \zeta^i \wedge \zeta^v \quad (\text{mod } \zeta^{\bar{r}} \wedge \zeta^r),$$

we have for $j, m \geqq 2$,

$$\partial_j \Psi'^{\bar{m}} \equiv \left\{\partial_j \partial_u\left(g_{rk}\begin{Bmatrix} r \\ st \end{Bmatrix} z^k_m z^s_1\right) z^u_i z^t_v + \partial_u\left(g_{rk}\begin{Bmatrix} r \\ st \end{Bmatrix} z^k_m z^s_1\right)\partial_j(z^u_i z^t_v)\right\}\zeta^i \wedge \zeta^v$$

$$= \left\{\partial_u\left(g_{rk}\begin{Bmatrix} r \\ st \end{Bmatrix}\partial_j(z^k_m z^s_1)\right)\delta^u_i \delta^t_v\right.$$

$$\left. + \left(\partial_u \begin{Bmatrix} m \\ 1t \end{Bmatrix}\right)\{(\delta^u_j \delta_{i1} - \delta_{ji}\delta^u_1)\delta^t_v + \delta^u_i(\delta^t_j \delta_{v1} - \delta_{jv}\delta^t_1)\}\right\}\zeta^i \wedge \zeta^v$$

$$= \left\{\partial_u \begin{Bmatrix} k \\ st \end{Bmatrix}(-\delta_{jm}\delta^k_1 \delta^s_1 + \delta^k_m \delta^s_j)\delta^u_i \delta^t_v + \partial_j \begin{Bmatrix} m \\ 1v \end{Bmatrix}\delta_{i1} - \partial_1 \begin{Bmatrix} m \\ 1v \end{Bmatrix}\delta_{ji}\right.$$

$$\left. + \partial_i \begin{Bmatrix} m \\ 1j \end{Bmatrix}\delta_{v1} - \partial_i \begin{Bmatrix} m \\ 11 \end{Bmatrix}\delta_{jv}\right\}\zeta^i \wedge \zeta^v$$

$$= \left\{-\partial_i \begin{Bmatrix} 1 \\ 1v \end{Bmatrix}\delta_{jm} + \partial_i \begin{Bmatrix} m \\ jv \end{Bmatrix} + \partial_j \begin{Bmatrix} m \\ 1v \end{Bmatrix}\delta_{i1} - \partial_1 \begin{Bmatrix} m \\ 1v \end{Bmatrix}\delta_{ji} + \partial_i \begin{Bmatrix} m \\ 1j \end{Bmatrix}\delta_{v1}\right.$$

$$\left. - \partial_i \begin{Bmatrix} m \\ 11 \end{Bmatrix}\delta_{jv}\right\}\zeta^i \wedge \zeta^v.$$

Thus we obtain

$$\partial_j T'^{\bar{m}}_{iv} = R^m_{jiv} + R^m_{1jv}\delta_{i1} - R^m_{11v}\delta_{ji} + R^m_{1ij}\delta_{v1} - R^m_{1i1}\delta_{jv}, \tag{2.13}$$

for $1 \leqq i,\ v \leqq n,\ 2 \leqq j,\ m \leqq n$, and it follows for $2 \leqq i,\ j \leqq n$,

$$\partial_j A^1_{0i} = -\frac{1}{n-2}(-\tilde{R}_{ji} + R_{11}\delta_{ji} - R^j_{1i1})$$
$$= \frac{1}{n-2}(R_{ij} - R_{11}\delta^i_j), \tag{2.14}$$

using (2.12). Finally we have

$$A^1_{01}=\frac{1}{2(n-1)}\Big(R_{11}-\sum_{i=2}^{n}\partial_{\bar i}A^1_{0i}\Big)$$
$$=\frac{1}{2(n-1)}\Big\{R_{11}-\frac{1}{n-2}\Big(\sum_{i=2}^{n}R_{ii}-(n-1)R_{11}\Big)\Big\}$$
$$=\frac{1}{n-2}R_{11}-\frac{1}{2(n-1)(n-2)}R\,.$$

Thus, the Cartan connection (Q, χ) of type H/H_0 defined by

$$\chi_-=\chi'_-\,,\quad \chi^i_r=\chi'^i_r\,,\quad \chi^0_0=\chi'^0_0\,,$$
$$\chi^0_1=\chi'^0_1+A^0_{11}\zeta^1\,,\quad \chi^1_0=\chi'^1_0+\sum_{j=2}^{n}A^1_{0j}\zeta^j+A^1_{01}\zeta^1\,,$$

is normal (i.e. $T^{-1}=\partial^*T^0=(\partial^*T^1)(e_1)=0$, where $\Psi=(1/2)T\zeta\wedge\zeta$ is its curvature) [M1, Proposition 5.2].

PROPOSITION 2.4. *The curvature* $\Psi=(1/2)T\zeta\wedge\zeta$ *of* (Q, χ) *is given by*

$$T_{-2}=0\,,$$
$$T^i_{1j}=\frac{1}{2}\delta^i_j\,,\quad T^i_{\beta\gamma}=0\quad \textit{otherwise},$$
$$T^{\bar i}_{1j}=R^i_{11j}+\delta^i_j\Big\{\frac{1}{n-2}R_{11}-\frac{R}{2(n-1)(n-2)}\Big\}\,,$$
$$T^{\bar i}_{1\bar j}=0\,,$$
$$T^{\bar i}_{jk}=R^i_{1jk}-\frac{1}{n-2}(R_{1k}\delta^i_j-R_{1j}\delta^i_k)\,,$$
$$T^{\bar i}_{j\gamma}=0\,,$$
$$T^i_{jkm}=R^i_{jkm}\,,\quad T^i_{j\bar k\bar m}=\delta^i_k\delta_{jm}-\delta^i_m\delta_{jk}\,,\quad T^i_{j\beta\gamma}=0\quad \textit{otherwise},$$
$$T^0_{01i}=-\frac{1}{2(n-2)}R_{1i}\,,\quad T^0_{0\beta\gamma}=0\quad \textit{otherwise},$$
$$T^0_{1i\bar i}=\frac{1}{2}\,,\quad T^0_{1\beta\gamma}=0\quad \textit{otherwise},$$
$$T^1_{01j}=\frac{1}{n-2}\partial_1R_{1j}-\frac{1}{n-2}\partial_jR_{11}+\frac{1}{2(n-1)(n-2)}\partial_jR\,,$$
$$T^1_{01\bar j}=-\partial_{\bar j}A^1_{01}+\frac{1}{n-2}R_{1j}\,,$$
$$T^1_{0ij}=\frac{1}{n-2}(\partial_iR_{1j}-\partial_jR_{1i})\,,$$

$$T^1_{0\bar{i}j}=\frac{1}{n-2}R_{ij}-\frac{R}{2(n-1)(n-2)}\delta^i_j$$

$$T^1_{0\bar{i}\bar{j}}=0\,,$$

where $2\leqq i,\ j,\ k\leqq n$ *and* $\beta,\gamma\in\{1,\cdots,n,\bar{2},\cdots,\bar{n}\}$.

Proof. We use the relation between T and T' obtained in [M1, § 5]. The first one is obvious. Next, we have

$$T^i_{1j}=T'^i_{1j}+A^0_{11}\delta^i_j=\frac{1}{2}\delta^i_j\,,$$

$$T^{\bar{i}}_{1j}=T'^{\bar{i}}_{1j}+A^1_{01}\delta^i_j$$

$$=R^i_{11j}+\delta^i_j\left(\frac{1}{n-2}R_{11}-\frac{R}{2(n-1)(n-2)}\right)$$

$$T^{\bar{i}}_{jk}=T'^{\bar{i}}_{jk}-A^1_{0k}\delta^i_j+A^1_{0j}\delta^i_k$$

$$=R^i_{1jk}-\frac{1}{n-2}(R_{1k}\delta^i_j-R_{1j}\delta^i_k)\,,\quad 2\leqq j,\ k\leqq n\,,$$

$$T^{\bar{i}}_{\bar{j}\gamma}=T'^{\bar{i}}_{\bar{j}\gamma}=0\,,$$

$$T^i_{r\beta\gamma}=T'^i_{r\beta\gamma}=R^i_{rst}\delta^s_\beta\delta^t_\gamma+\delta^{\bar{i}}_\beta\delta_{\bar{r}\gamma}-\delta^{\bar{i}}_\gamma\delta_{\bar{r}\beta}\,,\quad 2\leqq i,\ r,\ s,\ t\leqq n\,.$$

Now, from

$$\Psi^0_0-\Psi'^0_0=\chi^0_1\wedge\chi^1_0-\chi'^0_1\wedge\chi'^1_0$$

$$=(\chi'^0_1+A^0_{11}\zeta^1)\wedge(\chi'^1_0+A^1_{0j}\zeta^j+A^1_{01}\zeta^1)-\chi'^0_1\wedge\chi'^1_0$$

$$A^0_{11}\zeta^1\wedge\left(\sum_{j=2}^n A^1_{0j}\zeta^j+A^1_{01}\zeta^1\right),$$

we get

$$T^0_{01i}=T'^0_{01i}+A^0_{11}A^1_{0i}=-\frac{1}{2(n-2)}R_{1i}\,,\quad T^0_{0\beta\gamma}=T'^0_{0\beta\gamma}=0\,,\quad \text{otherwise.}$$

Similarly, from

$$\Psi^0_1-\Psi'^0_1=d(A^0_{11}\zeta^1)=-\frac{1}{2}d\zeta^1=\frac{1}{2}\sum_{i=2}^n\zeta^i\wedge\zeta^{\bar{i}}\,,$$

we obtain

$$T^0_{1i\bar{i}}=\frac{1}{2}\,,\quad T^0_{1\beta\gamma}=0\,,\quad \text{otherwise.}$$

Then, from

$$\Psi^1_0-\Psi'^1_0=d\left(\sum_{j=2}^n A^1_{0j}\zeta^j+A^1_{01}\zeta^1\right)=\sum_{j=2}^n dA^1_{0j}\wedge\zeta^j+dA^1_{01}\wedge\zeta^1+\sum_{j=2}^n A^1_{0j}d\zeta^j+A^1_{01}d\zeta^1\,,$$

to obtain $T^1_{01j}=T'^1_{01j}+\partial_1 A^1_{0j}-\partial_j A^1_{01}$, we compute

$$\partial_i A^1_{0j} = -\frac{1}{n-2}\partial_i(\sum_m T'^{\bar{m}}_{jm}), \quad 2 \leqq i, j \leqq n.$$

Here we use (2.9) and get

$$\partial_i \Psi'^{\bar{m}} \equiv \partial_i\Big(\partial_u\Big(g_{rk} z^k_m \begin{Bmatrix} r \\ st \end{Bmatrix} z^s_1\Big) z^u_j z^t_v\Big)\zeta^j \wedge \zeta^v \quad (\bmod \zeta^{\bar{r}} \wedge \zeta^r)$$

$$= \partial_i \partial_u\Big(g_{rk} z^k_m \begin{Bmatrix} r \\ st \end{Bmatrix}\Big)\delta^s_1 \delta^u_j \delta^t_v \zeta^j \wedge \zeta^v$$

$$= \partial_i \partial_j \begin{Bmatrix} m \\ 1v \end{Bmatrix} \zeta^j \wedge \zeta^v,$$

and so

$$\partial_i A^1_{0j} = -\frac{1}{n-2}\partial_i \sum_m R^m_{1jm} = \frac{1}{n-2}\partial_i R_{1j}, \quad 1 \leqq i \leqq n \quad \text{and} \quad 2 \leqq j \leqq n.$$

Similarly, we get

$$\partial_j A^1_{01} = \frac{1}{n-2}\partial_j R_{11} - \frac{1}{2(n-1)(n-2)}\partial_j R.$$

Therefore, we have

$$T^1_{01j} = T'^1_{01j} + \partial_1 A^1_{0j} - \partial_j A^1_{01}$$

$$= \frac{1}{n-2}\partial_1 R_{1j} - \frac{1}{n-2}\partial_j R_{11} + \frac{1}{2(n-1)(n-2)}\partial_j R,$$

$$T^1_{0ij} = T'^1_{0ij} + \partial_i A^1_{0j} - \partial_j A^1_{0i}$$

$$= \frac{1}{n-2}(\partial_i R_{1j} - \partial_j R_{1i}),$$

$$T^1_{0\bar{i}j} = T'^1_{0\bar{i}j} + \partial_{\bar{i}} A^1_{0j} + A^1_{01}\delta^i_j$$

$$= \frac{1}{n-2}(R_{ij} - R_{11}\delta^i_j) + \delta^i_j\Big\{\frac{1}{n-2}R_{11} - \frac{R}{2(n-1)(n-2)}\Big\}$$

$$= \frac{1}{n-2}R_{ij} - \frac{R}{2(n-1)(n-2)}\delta^i_j,$$

$$T^1_{0\bar{i}\bar{j}} = T'^1_{0\bar{i}\bar{j}} = 0.$$

Now, we have

$$T^1_{01\bar{j}} = T'^1_{01\bar{j}} - \partial_{\bar{j}} A^1_{01} + A^1_{0j} = -\partial_{\bar{j}} A^1_{01} + \frac{1}{n-2}R_{1j},$$

where $\partial_{\bar{j}} A^1_{01}$ is given later (Lemma 4.3). q.e.d.

§ 3. A Cartan connection.

Using the method of the proof of [M1, Proposition 5.3], we construct Tanaka connection on $P=Q\times_{H_0}G'$ from the normal Cartan connection (Q, χ) constructed in § 2.

First of all, for later use, recall the structure equation [M1, (3.1)], of a Cartan connection (P, ω) of the G/G', where $\theta=\omega_{-2}+\omega_{-1}$ is the basic form, and Ω is the curvature form:

$$d\theta^1=-(\omega_0^0+\omega_1^1)\wedge\theta^1-\sum_{i=2}^{n}\theta^i\wedge\theta^{\bar{i}}+\Omega^1,$$

$$d\theta^i=\omega_{\bar{i}}\wedge\theta^1-\sum_{j=2}^{n}\omega_j^i\wedge\theta^j+\theta^i\wedge\omega_0^0+\theta^{\bar{i}}\wedge\omega_1^0+\Omega^i,$$

$$d\theta^{\bar{i}}=-\omega_i\wedge\theta^1-\sum_{j=2}^{n}\omega_j^i\wedge\theta^{\bar{j}}+\theta^i\wedge\omega_0^1+\theta^{\bar{i}}\wedge\omega_1^1+\Omega^{\bar{i}},$$

$$d\omega_0^0=-\omega_1^0\wedge\omega_0^1-\sum_{i=2}^{n}\theta^i\wedge\omega_i-\theta^1\wedge\omega_1+\Omega_0^0,$$

$$d\omega_1^0=-(\omega_0^0-\omega_1^1)\wedge\omega_1^0-\sum_{i=2}^{n}\theta^i\wedge\omega_{\bar{i}}+\Omega_1^0,$$

$$d\omega_0^1=(\omega_0^0-\omega_1^1)\wedge\omega_0^1-\sum_{i=2}^{n}\theta^{\bar{i}}\wedge\omega_i+\Omega_0^1, \tag{3.1}$$

$$d\omega_1^1=-\omega_0^1\wedge\omega_1^0-\sum_{i=2}^{n}\theta^{\bar{i}}\wedge\omega_{\bar{i}}-\theta^1\wedge\omega_1+\Omega_1^1,$$

$$d\omega_j^i=-\omega_i\wedge\theta^j-\omega_{\bar{i}}\wedge\theta^{\bar{j}}-\sum_{k=2}^{n}\omega_k^i\wedge\omega_j^k-\theta^i\wedge\omega_j-\theta^{\bar{i}}\wedge\omega_{\bar{j}}+\Omega_j^i,$$

$$d\omega_i=-\omega_i\wedge\omega_0^0-\omega_{\bar{i}}\wedge\omega_0^1-\theta^{\bar{i}}\wedge\omega_1+\sum_{j=2}^{n}\omega_i^j\wedge\omega_j+\Omega_i,$$

$$d\omega_{\bar{i}}=-\omega_i\wedge\omega_1^0-\omega_{\bar{i}}\wedge\omega_1^1+\theta^i\wedge\omega_1+\sum_{j=2}^{n}\omega_i^j\wedge\omega_{\bar{j}}+\Omega_{\bar{i}},$$

$$d\omega_1=(\omega_0^0+\omega_1^1)\wedge\omega_1+\sum_{i=2}^{n}\omega_i\wedge\omega_{\bar{i}}+\Omega_1.$$

Now, let ω' be the Cartan connection on P naturally extended from χ. Namely, let $(x^i, z_j^i, s_b^a, s_i, s_{\bar{i}}, s_1)$ be a local coordinate on P where (x^i, z_j^i) is the local coordinate of P_g around a point $u_0\in P_g$ chosen in § 2,$(s_b^a)\in GL(2, \boldsymbol{R})$, $0\leqq a$, $b\leqq 1$, $s_i=s_i^{n+1}$, $s_{\bar{i}}=s_i^{n+2}$, $2\leqq i\leqq n$ and $s_1=s_0^{n+2}$. As in § 2, we define ω' by $\omega'_{-2}+\omega'_{-1}=\chi_{-2}+\chi_{-1}$, $\omega'^a_b=\chi_b^a$, $0\leqq a$, $b\leqq 1$, $\omega'_i=ds_i$, $\omega'_{\bar{i}}=ds_{\bar{i}}$ and $\omega'_1=ds_1$ at $u\in Q$ (note that $s_b^a|_Q=h_b^a$), and then extend it to P by $R_a^*\omega'=Ad(a^{-1})\omega'$ where $a\in G'$. Obviously, (P, ω') is a Cartan connection of type G/G' with basic form $\theta=\omega'_{-2}+\omega'_{-1}$.

PROPOSITION 3.1. *The curvature $\Omega'=(1/2)K'\theta\wedge\theta$ of ω' at u_0 satisfies*

$$\iota^*K'=T$$

where $\iota: Q\to P$ is the inclusion map and T is given in Proposition 2.4,

$$K'^1_{11j}=\frac{1}{2(n-2)}R_{1j},\quad K'^1_{1\beta\gamma}=0,\quad otherwise,$$

and

$$K'_{i\beta\gamma}=K'^{\bar{i}}_{i\beta\gamma}=K'^{1}_{1\beta\gamma}=0.$$

Proof. The non-trivial case is Ω'^1_1. From the structure equation (3.1), we get

$$\Omega'^1_1=d\omega'^1_1+\Big(\chi'^1_0+\sum_{j=2}^n A^1_{0j}\theta^j+A^1_{01}\theta^1\Big)\wedge(\chi'^0_1+A^0_{11}\theta^1)+\sum_{i=2}^n\theta^{\bar{i}}\wedge\omega_{\bar{i}}+\theta^1\wedge\omega_1$$

$$=\sum_{j=2}^n A^1_{0j}A^0_{11}\theta^j\wedge\theta^1,$$

and so

$$K'^1_{1j1}=-\frac{1}{2(n-2)}R_{1j},\quad K'^1_{1\beta\gamma}=0\quad\text{otherwise.}\qquad\qquad \text{q.e.d.}$$

Now, we construct a Cartan connection (P,ω'') as in [M1]. To obtain A_{ij} for $i\neq j$ in [M1], by Proposition 3.1, 2.4 and (2.12), we get

$$-K'^{\bar{i}}_{1j}+K'^0_{0ij}+K'^1_{0\bar{i}j}-\sum_{k=2}^n K'^i_{kkj}=R^i_{1j1}+\frac{1}{n-2}R_{ij}-\sum_{k=2}^n R^i_{kkj}$$

$$=\frac{n-1}{n-2}R_{ij},$$

so that noting $A_{ij}=A_{ji}$, we have

$$A_{ij}=-\frac{1}{n-2}R_{ij}.$$

For $i=j$, we get

$$-K'^{\bar{i}}_{1i}+K'^0_{0ii}+K'^1_{0\bar{i}i}-\sum_{k=2}^n K'^i_{kki}=R^i_{1i1}-\frac{1}{n-2}R_{11}+\frac{R}{2(n-1)(n-2)}$$

$$+\frac{1}{n-2}R_{ii}-\frac{R}{2(n-1)(n-2)}+\tilde{R}_{ii}$$

$$=\frac{n-1}{n-2}R_{ii}-\frac{1}{n-2}R_{11},$$

and the summation over $2\leqq i\leqq n$ gives

$$\sum_{i=2}^n\Big(-K'^{\bar{i}}_{1i}+K'^0_{0ii}-\sum_{k=2}^n K'^i_{kki}+K'^1_{0\bar{i}i}\Big)=\frac{n-1}{n-2}(R-R_{11})-\frac{n-1}{n-2}R_{11}$$

$$=\frac{n-1}{n-2}(R-2R_{11}).$$

Therefore, we get

$$A_{ii}=-\frac{1}{n-2}R_{ii}+\frac{1}{(n-1)(n-2)}R_{11}+\frac{1}{2(n-1)(n-2)}(R-2R_{11})$$

$$=-\frac{1}{n-2}R_{ii}+\frac{R}{2(n-1)(n-2)},$$

or,

$$A_{ij}=-\frac{1}{n-2}R_{ij}+\frac{R}{2(n-1)(n-2)}\delta^i_j. \tag{3.2}$$

The Left hand sides of (7)′ and (8)′ vanish so that we get

$$A_{\bar{i}j}=A_{i\bar{j}}=0. \tag{3.3}$$

As for (9)′, since we have

$$K'^i_{1j}+K'^0_{1ij}+K'^1_{1\bar{i}j}-\sum_{k=2}^n K'^i_{k\bar{k}j}=\frac{1}{2}\delta^i_j+\frac{1}{2}\delta^i_j-\sum_{k=2}^n(\delta^i_k\delta_{kj}-\delta^i_j)$$

$$=\delta^i_j-\delta^i_j+(n-1)\delta^i_j$$

$$=(n-1)\delta^i_j,$$

we get

$$A_{\bar{i}\bar{j}}=-\frac{1}{2}\delta^i_j. \tag{3.4}$$

Therefore, ω'' is given by

$$\omega''_p=\omega'_p,\quad p\leqq 0,\quad \omega''_i=\omega'_i+\sum_{j=2}^n A_{ij}\theta^j,\quad \omega''_{\bar{i}}=\omega'_{\bar{i}}-\frac{1}{2}\theta^{\bar{i}},\quad \omega''_1=\omega'_1.$$

PROPOSITION 3.2. *The curvature* $\Omega''=\frac{1}{2}K''\theta\wedge\theta$ *of* ω'' *is given at* $u_0\in P_g$ *by,*

$$K''_{-2}=0,$$

$$K''^i_{\beta\gamma}=0,$$

$$K''^{\bar{i}}_{1j}=R^i_{11j}+\frac{1}{n-2}R_{ij}+\frac{1}{n-2}R_{11}\delta^i_j-\frac{R}{(n-1)(n-2)}\delta^i_j,$$

$$K''^{\bar{i}}_{\bar{1}j}=K'^{\bar{i}}_{\bar{1}j}=0,$$

$$K''^{\bar{i}}_{jk}=R^i_{1jk}-\frac{1}{n-2}(R_{1k}\delta^i_j-R_{1j}\delta^j_k),\quad K''^{\bar{i}}_{\beta\gamma}=0\quad \textit{otherwise},$$

$$K''^i_{jk1}=R^i_{jk1},$$

$$K''^{i}_{jkm}=R^{i}_{jkm}-\frac{1}{n-2}(R_{ik}\delta^{j}_{m}-R_{im}\delta^{j}_{k}+R_{jm}\delta^{i}_{k}-R_{jk}\delta^{i}_{m})$$

$$+\frac{R}{(n-1)(n-2)}(\delta^{i}_{k}\delta_{jm}-\delta^{i}_{m}\delta_{jk})$$

$$K''^{i}_{j\bar{k}\gamma}=K'^{i}_{j\bar{k}\gamma}=0\,,$$

$$K''^{0}_{01j}=-\frac{1}{2(n-2)}R_{1j}\,,$$

$$K''^{0}_{0ij}=0\,,$$

$$K''^{0}_{0\bar{i}\gamma}=K'^{0}_{0\bar{i}\gamma}=0\,,$$

$$K''^{0}_{1\beta\gamma}=K'^{0}_{1\beta\gamma}=0\,,$$

$$K''^{1}_{01j}=\frac{1}{n-2}\partial_{1}R_{1j}-\frac{1}{n-2}\partial_{j}R_{11}+\frac{1}{2(n-1)(n-2)}\partial_{j}R\,,$$

$$K''^{1}_{01\bar{j}}=-\partial_{\bar{j}}A^{1}_{01}+\frac{1}{n-2}R_{1j}\,,$$

$$K''^{1}_{0ij}=\frac{1}{n-2}(\partial_{i}R_{1j}-\partial_{j}R_{1i})\,,\quad K''^{1}_{0\beta\gamma}=0\quad \textit{otherwise}\,,$$

$$K''^{1}_{11j}=\frac{1}{2(n-2)}R_{1j}\,,\quad K''^{1}_{1\beta\gamma}=K'^{1}_{1\beta\gamma}=0\,,\quad \textit{otherwise}\,,$$

$$K''_{i1j}=\partial_{1}A_{ij}\,,$$

$$K''_{i1\bar{j}}=A_{ij}+\frac{1}{2}A^{1}_{01}\delta^{i}_{j}\,,$$

$$K''_{ijk}=-\partial_{k}A_{ij}+\partial_{j}A_{ik}\,,$$

$$K''_{ij\bar{k}}=-\partial_{\bar{k}}A_{ij}+\frac{1}{2(n-2)}R_{1j}\delta^{i}_{k}\,,$$

$$K''_{\bar{i}1j}=-\frac{1}{2}R^{i}_{11j}+\frac{1}{2}A_{ij}\,,$$

$$K''_{\bar{i}jk}=-\frac{1}{2}R^{i}_{1jk}\,,\quad K''_{\bar{i}\beta\gamma}=0\quad \textit{otherwise}\,,$$

$$K''_{1\bar{i}j}=-\frac{1}{2}A_{ij}\,,$$

$$K''_{1\beta\gamma}=0\quad \textit{otherwise}\,.$$

Proof. We can compute $\Omega''-\Omega'$ by using (3.1) as follows:

$$\Omega''^{1}-\Omega'^{1}=0 \longleftrightarrow K''_{-2}=0\,,$$

$$\Omega''^{i}-\Omega'^{i}=\frac{1}{2}\theta^{\bar{i}}\wedge\theta^{1}$$

$$\longleftrightarrow K''^{i}_{1j}=K'^{i}_{1j}-\frac{1}{2}\delta^{i}_{j}=0$$

$$K''^{i}_{\beta\gamma}=K'^{i}_{\beta\gamma}=0 \quad \text{otherwise},$$

$$\Omega''^{\bar{i}}-\Omega'^{\bar{i}}=A_{ij}\theta^{j}\wedge\theta^{1}$$

$$\longleftrightarrow K''^{\bar{i}}_{1j}=K'^{\bar{i}}_{1j}-A_{ij}=R^{i}_{11j}+\frac{1}{n-2}R_{ij}, \qquad i\neq j,$$

$$K''^{\bar{i}}_{1i}=R^{i}_{11i}+\frac{1}{n-2}R_{11}-\frac{R}{2(n-1)(n-2)}+\frac{1}{n-2}R_{ii}-\frac{R}{2(n-1)(n-2)}$$

$$=R^{i}_{11i}+\frac{1}{n-2}R_{ii}+\frac{1}{n-2}R_{11}-\frac{R}{(n-1)(n-2)},$$

$$K''^{\bar{i}}_{jk}=K'^{\bar{i}}_{jk}=R^{i}_{1jk}-\frac{1}{n-2}(R_{1k}\delta^{i}_{j}-R_{1j}\delta^{i}_{k})$$

$$K''^{\bar{i}}_{j\gamma}=K'^{\bar{i}}_{j\gamma}=0,$$

$$\Omega''^{i}_{j}-\Omega'^{i}_{j}=\sum_{k=2}^{n}A_{ik}\theta^{k}\wedge\theta^{j}-\frac{1}{2}\theta^{\bar{i}}\wedge\theta^{\bar{j}}+\sum_{k=2}^{n}\theta^{i}\wedge A_{jk}\theta^{k}-\frac{1}{2}\theta^{\bar{i}}\wedge\theta^{\bar{j}}$$

$$\longleftrightarrow K''^{i}_{jk1}=K'^{i}_{jk1}=R^{i}_{jk1},$$

$$K''^{i}_{jkm}=K'^{i}_{jkm}+A_{ik}\delta^{j}_{m}-A_{im}\delta^{j}_{k}+A_{jm}\delta^{i}_{k}-A_{jk}\delta^{i}_{m},$$

$$=R^{i}_{jkm}-\frac{1}{n-2}(R_{ik}\delta^{j}_{m}-R_{im}\delta^{j}_{k}+R_{jm}\delta^{i}_{k}-R_{jk}\delta^{i}_{m})$$

$$+\frac{R}{(n-1)(n-2)}(\delta^{i}_{k}\delta_{jm}-\delta^{i}_{m}\delta_{jk})$$

$$K''^{i}_{j\bar{i}\bar{j}}=K'^{i}_{j\bar{i}\bar{j}}-1=0,$$

$$K''^{i}_{j\beta\gamma}=K'^{i}_{j\beta\gamma}=0 \quad \text{otherwise},$$

$$\Omega''^{0}_{0}-\Omega'^{0}_{0}=\sum_{i,j=2}^{n}\theta^{i}\wedge A_{ij}\theta^{j}$$

$$\longleftrightarrow K''^{0}_{01j}=K'^{0}_{01j}=-\frac{1}{2(n-2)}R_{1j},$$

$$K''^{0}_{0ij}=K'^{0}_{0ij}+A_{ij}-A_{ji}=0,$$

$$K''^{0}_{0\beta\gamma}=K'^{0}_{0\beta\gamma}=0 \quad \text{otherwise},$$

$$\Omega''^{0}_{1}-\Omega'^{0}_{1}=-\frac{1}{2}\sum_{i=2}^{n}\theta^{i}\wedge\theta^{\bar{i}}$$

$$\longleftrightarrow K''^{0}_{1\bar{i}i}=K'^{0}_{1\bar{i}i}-\frac{1}{2}=0\,,$$

$$K''^{0}_{1\beta\gamma}=K'^{0}_{1\beta\gamma}=0 \quad \text{otherwise}\,,$$

$$\Omega''^{1}_{0}-\Omega'^{1}_{0}=\sum_{i,j=2}^{n}\theta^{\bar{i}}\wedge A_{ij}\theta^{j}$$

$$\longleftrightarrow K''^{1}_{0\bar{i}j}=K'^{1}_{0\bar{i}j}+A_{ij}=0\,,$$

$$K''^{1}_{01j}=K'^{1}_{01j}=\frac{1}{n-2}\partial_{1}R_{1j}-\frac{1}{n-2}\partial_{j}R_{11}+\frac{1}{2(n-1)(n-2)}\partial_{j}R\,,$$

$$K''^{1}_{01\bar{j}}=K'^{1}_{01\bar{j}}=-\partial_{\bar{j}}A^{1}_{01}+\frac{1}{n-2}R_{1j}\,,$$

$$K''^{1}_{0ij}=K'^{1}_{0ij}=\frac{1}{n-2}(\partial_{i}R_{1j}-\partial_{j}R_{1i})\,,$$

$$K''^{1}_{0\bar{i}\bar{j}}=K'^{1}_{0\bar{i}\bar{j}}=0\,,$$

$$\Omega''^{1}_{1}-\Omega'^{1}_{1}=-\frac{1}{2}\theta^{\bar{i}}\wedge\theta^{\bar{i}}=0$$

$$\longleftrightarrow K''^{1}_{11j}=\frac{1}{2(n-2)}R_{1j}\,,\quad K''^{1}_{1\beta\gamma}=0 \quad \text{otherwise}\,,$$

$$\Omega''_{i}-\Omega'_{i}=d\Big(\sum_{j=2}^{n}A_{ij}\theta^{j}\Big)+A_{\bar{i}\bar{i}}\theta^{\bar{i}}\wedge\Big(\chi'^{1}_{0}+\sum_{j=2}^{n}A^{1}_{0j}\theta^{j}+A^{1}_{01}\theta^{1}\Big)$$

$$=\sum_{k=1}^{n}\sum_{j=2}^{n}(\partial_{k}A_{ij})\theta^{k}\wedge\theta^{j}+\sum_{j,k=2}^{n}(\partial_{\bar{k}}A_{ij})\theta^{\bar{k}}\wedge\theta^{j}-\sum_{j=2}^{n}A_{ij}\theta^{\bar{j}}\wedge\theta^{1}$$

$$-\sum_{j=2}^{n}\frac{1}{2(n-2)}R_{1j}\theta^{\bar{i}}\wedge\theta^{j}-\frac{1}{2}A^{1}_{01}\theta^{\bar{i}}\wedge\theta^{1}$$

$$\longleftrightarrow K''_{i1j}=\partial_{1}A_{ij}\,,$$

$$K''_{ijk}=-\partial_{k}A_{ij}+\partial_{j}A_{ik}\,,$$

$$K''_{i1\bar{j}}=A_{ij}+\frac{1}{2}A^{1}_{01}\delta^{i}_{j}$$

$$K''_{ij\bar{k}}=-\partial_{\bar{k}}A_{ij}+\frac{1}{2(n-2)}R_{1j}\delta^{i}_{k}\,,$$

$$\Omega''_{\bar{i}}-\Omega'_{\bar{i}}=d\Big(-\frac{1}{2}\theta^{\bar{i}}\Big)+\sum_{j=2}^{n}A_{ij}\theta^{j}\wedge(\chi'^{0}_{1}+A^{0}_{11}\theta^{1})$$

$$=-\frac{1}{2}d\theta^{\bar{i}}+\sum_{j=2}^{n}A_{ij}A^{0}_{11}\theta^{j}\wedge\theta^{1}$$

$$\longleftrightarrow K''_{\bar{i}1j}=-\frac{1}{2}R^{i}_{11j}+\frac{1}{2}A_{ij}$$

$$K''_{\bar{i}1\bar{j}}=0\,,$$

$$K''_{\bar{i}jk}=-\frac{1}{2}R^{i}_{1jk}\,,$$

$$K''_{\bar{i}\bar{j}\gamma}=K'_{\bar{i}\bar{j}\gamma}=0\,,$$

$$\Omega''_1-\Omega'_1=-\sum_{i,j}^{n}A_{ij}\theta^j\wedge\left(-\frac{1}{2}\theta^{\bar{i}}\right)$$

$$\longleftrightarrow K''_{11j}=K''_{11\bar{j}}=K''_{1ij}=K''_{1\bar{i}\bar{j}}=0\,,$$

$$K''_{1\bar{i}j}=-\frac{1}{2}A_{ij}\,.$$

COROLLARY. *Let* C^{i}_{jkm} *and* $C_{ijk}=\Pi_{ij,k}-\Pi_{ik,j}$ *be the coefficients of Weyl's conformal curvature tensor at* $p=\pi_1\circ\pi(u_0)\in M$. *Then we have*

$$K''^{i}_{1j}=C^{i}_{11j}\,,\qquad K''^{\bar{i}}_{jk}=C^{\bar{i}}_{1jk}\,,$$

$$K''^{1}_{01j}=C_{11j}\,,\qquad K''^{1}_{0ij}=C_{1ij}\,,$$

$$K''^{i}_{jkm}=C^{i}_{jkm}\,,$$

where $2\leqq i,\ j,\ k,\ m\leqq n$

Proof. Since

$$C^{i}_{jkm}=R^{i}_{jkm}+\frac{1}{n-2}(R_{jk}\delta^{i}_{m}-R_{jm}\delta^{i}_{k}+g_{jk}R^{i}_{m}-g_{jm}R^{i}_{k})-\frac{R}{(n-1)(n-2)}(g_{jk}\delta^{i}_{m}-g_{jm}\delta^{i}_{k})\,,$$

noting $g_{ij}(p)=\delta_{ij}$, the last formula follows immediately. Then for $i,\ j\geqq 2$, we have

$$C^{i}_{11j}=R^{i}_{11j}+\frac{1}{n-2}(R_{11}\delta^{i}_{j}+R^{i}_{j})-\frac{R}{(n-1)(n-2)}\delta^{i}_{j}$$

$$=K''^{\bar{i}}_{1j}\,.$$

Similarly, we have

$$C^{i}_{1jk}=R^{i}_{1jk}+\frac{1}{n-2}(R_{1j}\delta^{i}_{k}-R_{1k}\delta^{i}_{j})$$

$$=K''^{\bar{i}}_{jk}\,.$$

Moreover, from

$$\Pi_{jk}=-\frac{R_{jk}}{n-2}+\frac{Rg_{jk}}{2(n-1)(n-2)}\,,$$

and $\partial_k R_{ij}=R_{ij,k}$, $\partial_k R=R_{,k}$ at p, we get

$$\begin{aligned} C_{11j}&=\Pi_{11,j}-\Pi_{1j,1} \\ &=-\frac{1}{n-2}\partial_j R_{11}+\frac{1}{2(n-1)(n-2)}\partial_j R+\frac{1}{n-2}\partial_1 R_{1j} \\ &=K''^{1}_{01j}, \end{aligned}$$

and similarly,

$$\begin{aligned} C_{1ij}&=-\frac{1}{n-2}\partial_j R_{1i}+\frac{1}{n-2}\partial_i R_{1j} \\ &=K''^{1}_{0ij}. \end{aligned}$$

q. e. d.

§ 4. The curvatures and the main result.

As in [M1, § 5], we can construct Tanaka connection (P, ω) using Proposition 3.2. For the explicite description of ω, we need some more calculations, but the essential information of the curvature K of ω is given by its corollaly. In fact, it is shown in [SY] that the harmonic part $H^{p,2}(K)$ of the curvature K of ω determines the structure essentially. Moreover, in the case of Lie contact structures, $H^{p,2}(K)$ vanishes except for $p=0$ if $n\geqq 4$, and $p=0, 1$ if $n=3$ [SY]. Therefore it is sufficient to compute K_{-1} for $n\geqq 4$ and K_{-1} and K_0 for $n=3$. It is easy to see that $K''_{-1}=K_{-1}$ [M1]. Immediately, we obtain from Proposition 3.2 and its corollary:

PROPOSITION 4.1. *Let C^{i}_{jkm} be the coefficients of Weyl's conformal curvature at $p=\pi_1\circ\pi(u_0)\in M$, $u_0\in P_g$. Then the curvature K_{-1} of Tanana connection ω on $\pi: P\to T_1M$ is given by*

$$K^{\bar{i}}_{1j}(u_0)=C^{i}_{11j}(p), \qquad K^{\bar{i}}_{jk}(u_0)=C^{i}_{1jk}(p).$$

and all other coefficients vanish.

In particular, when $n=3$, K_{-1} vanishes identically, which is already proved in [SY] from the view point of integrability of CR-structures and twistor geometry. Thus in this case, we should compute K_0. As is shown in [M1], we can see that $K^{1}_{01j}=K''^{1}_{01j}$ and $K^{1}_{0ij}=K''^{1}_{0ij}$. Now, we prove:

PROPOSITION 4.2. *When $n=3$, by using the coefficients C_{ijk} of Weyl's conformal curvature tensors, the curvature K_0 of Tanaka conection is given by*

$$K^{1}_{01j}(u_0)=C_{11j}(p), \qquad K^{1}_{0ij}(u_0)=C_{1ij}(p).$$

and all other coefficients vanish.

Proof. We may prove the last statement. For the present, we do not assume $n=3$. Using (10)′ of [M1, §5], we have

$$(4.1)\qquad -(2n-1)A_{i1}$$

$$=K''^{0}_{0i1}+K''^{1}_{0\bar{i}1}-\sum_{k=2}^{n}(K''^{i}_{kk1}+K''_{ik\bar{k}})$$

$$=\frac{1}{2(n-2)}R_{1i}+\partial_{\bar{i}}A^{1}_{01}-\frac{1}{n-2}R_{1i}+R_{1i}-\sum_{k}\left(-\partial_{\bar{k}}A_{ik}+\frac{1}{2(n-2)}R_{1k}\delta^{i}_{k}\right)$$

$$=\frac{n-3}{n-2}R_{1i}+\partial_{\bar{i}}A^{1}_{01}+\sum_{k=2}^{n}\partial_{\bar{k}}A_{ik}.$$

Now we prepare:

LEMMA 4.3 *We have at* $u_0\in P_g$,

$$\partial_{\bar{r}}R^{1}_{jkm}=-R^{1}_{jkm}\delta_{ri}-R^{1}_{1km}\delta_{rj}+R^{1}_{jrm}\delta_{k1}-R^{1}_{j1m}\delta_{rk}+R^{1}_{jkr}\delta_{m1}-R^{1}_{jk1}\delta_{rm},$$

$$\partial_{\bar{r}}R_{11}=2R_{1r},$$

$$\partial_{\bar{r}}A^{1}_{01}=\frac{2}{n-2}R_{1r},$$

$$\partial_{\bar{r}}A_{ij}=\frac{1}{n-2}(R_{1j}\delta_{ri}+R_{1i}\delta_{rj}),$$

where $2\leqq r\leqq n$, $1\leqq i, j, k, m\leqq n$.

Proof. Since we have $R^{i}_{jkm}=\Psi'^{i}_{jkm}$, $2\leqq i, j\leqq n$, $1\leqq k, m\leqq n$, from

$$\partial_{\bar{r}}\Psi'^{i}_{j}=\partial_{\bar{r}}d\left(g_{uv}z^{v}_{i}\left(dz^{u}_{j}+\left\{{u\atop st}\right\}z^{s}_{j}dx^{t}\right)\right)$$

$$\equiv\partial_{\bar{r}}\left(\partial_{h}\left(g_{uv}z^{v}_{i}\left\{{u\atop st}\right\}z^{s}_{j}\right)z^{h}_{k}z^{t}_{m}\right)\zeta^{k}\wedge\zeta^{m}\qquad(\mathrm{mod}\,\zeta^{\bar{k}}\wedge\zeta^{r})$$

$$=\left\{\left(\partial_{h}\left\{{v\atop st}\right\}\right)\partial_{\bar{r}}(z^{v}_{i}z^{s}_{j})\delta^{h}_{k}\delta^{t}_{m}+\partial_{h}\left\{{i\atop jt}\right\}\partial_{\bar{r}}(z^{h}_{k}z^{t}_{m})\right\}\zeta^{k}\wedge\zeta^{m}$$

$$=\Big[\partial_{k}\left\{{v\atop sm}\right\}\{(\delta^{v}_{r}\delta_{i1}-\delta_{ri}\delta^{v}_{1})\delta^{s}_{j}+\delta^{v}_{i}(\delta^{s}_{r}\delta_{j1}-\delta_{rj}\delta^{s}_{1})\}$$

$$+\partial_{h}\left\{{i\atop jt}\right\}(\delta^{h}_{r}\delta_{k1}-\delta_{rk}\delta^{h}_{1})\delta^{t}_{m}+\delta^{h}_{k}(\delta^{t}_{r}\delta_{m1}-\delta_{rm}\delta^{t}_{1})\Big]\zeta^{k}\wedge\zeta^{m}$$

$$=\Big[-\partial_{k}\left\{{1\atop jm}\right\}\delta_{ri}-\partial_{k}\left\{{i\atop 1m}\right\}\delta_{rj}+\partial_{r}\left\{{i\atop jm}\right\}\delta_{k1}$$

$$-\partial_1\left\{{i \atop jm}\right\}\delta_{rk}+\partial_k\left\{{i \atop jr}\right\}\delta_{m1}-\partial_k\left\{{i \atop j1}\right\}\delta_{rm}\Big]\zeta^k\wedge\zeta^m,$$

we get

$$\partial_{\bar r}R^i_{jkm}=-R^1_{jkm}\delta_{ri}-R^i_{1km}\delta_{rj}+R^i_{jrm}\delta_{k1}-R^i_{j1m}\delta_{rk}+R^i_{jkr}\delta_{m1}-R^i_{jk1}\delta_{rm}.$$

Therefore, we obtain

$$\partial_{\bar r}\tilde R_{jm}=\sum_{i=2}^n\partial_{\bar r}R^i_{jim}$$

$$=-R^1_{jrm}-R_{1m}\delta_{rj}-R^r_{j1m}-R_{j1}\delta_{rm}$$

for j, $m\geqq 2$, and

$$\partial_{\bar r}\tilde R=\sum_{j=2}^n\partial_{\bar r}\tilde R_{jj}$$

$$=-4R_{1r}.$$

Now, using (2.13), we have

$$\partial_{\bar r}R_{11}=-\sum_{m=2}^n\partial_{\bar r}T'^{\bar m}_{1m}=-(-R_{1r}-R_{1r})=2R_{1r}.$$

Since R is a function on M, $\partial_{\bar r}R=0$ is trivial, but also follows from

$$\partial_{\bar r}R=\partial_{\bar r}(\tilde R+2R_{11})=0,$$

and so we obtain

$$\partial_{\bar r}A^1_{01}=\frac{1}{n-2}\partial_{\bar r}R_{11}-\frac{1}{2(n-1)(n-2)}\partial_{\bar r}R$$

$$=\frac{2}{n-2}R_{1r}.$$

Moreover, using (2.13) and (3.2), we get

$$\partial_{\bar r}A_{ij}=-\frac{1}{n-2}\partial_{\bar r}(R^i_{1j1}+\tilde R_{ij})+\frac{1}{2(n-1)(n-2)}\partial_{\bar r}R\delta^i_j$$

$$=-\frac{1}{n-2}(R^i_{rj1}+R^i_{1jr}-R^1_{irj}-R_{1j}\delta_{ri}-R^r_{i1j}-R_{i1}\delta_{rj})$$

$$=\frac{1}{n-2}(R_{1j}\delta_{ri}+R_{1i}\delta_{rj}).$$ q.e.d.

Finally, we obtain from (4.1),

$$A_{i1}=-\frac{1}{n-2}R_{1i}. \tag{4.2}$$

Next, the left hand sides of (11)′ and (13)′ of [M1, § 5] are zero and we get

(4.3) $$A_{\bar{i}1}=A_{1\bar{j}}=0\,.$$

From (12)′, we have

$$(K''^{0}_{01j}+K''^{1}_{11\jmath})-\sum_{\imath=2}^{n}(K''_{\bar{i}i\jmath}-K''_{i\bar{i}\jmath})=-\frac{1}{2(n-2)}R_{1j}+\frac{1}{2(n-2)}R_{1\jmath}$$

$$+\frac{1}{2}R_{1j}+\frac{n+1}{2(n-2)}R_{1\jmath}$$

$$=\frac{2n-1}{2(n-2)}R_{1\jmath}\,,$$

so that

(4.4) $$A_{1\jmath}=-\frac{1}{2(n-2)}R_{1\jmath}\,.$$

Using (3.1), we obtain

$$K^{i}_{jk1}=K''^{i}_{jk1}-A_{i1}\delta^{\jmath}_{k}+A_{j1}\delta^{i}_{k}$$

$$=R^{\imath}_{jk1}+\frac{1}{n-2}(R_{1i}\delta^{\jmath}_{k}-R_{1j}\delta^{\imath}_{k})=C^{\imath}_{jk1}\,,$$

$$K^{\imath}_{jkm}=K''^{i}_{jkm}=C^{\imath}_{jkm}\,,$$

$$K^{i}_{j\beta\gamma}=K''^{i}_{j\beta\gamma}=0 \qquad \text{otherwise}\,,$$

$$K^{0}_{0i1}=K''^{0}_{0i1}+A_{i1}-A_{1\imath}$$

$$=\left\{\frac{1}{2(n-2)}-\frac{1}{n-2}+\frac{1}{2(n-2)}\right\}R_{1\imath}=0\,,$$

$$K^{0}_{0\beta\gamma}=K''^{0}_{0\beta\gamma}=0 \qquad \text{otherwise},$$

$$K^{0}_{1\beta\gamma}=K''^{0}_{1\beta\gamma}=0\,,$$

$$K^{1}_{01\jmath}=K''^{1}_{01j}=C_{11\jmath}$$

$$K^{1}_{0\imath j}=K''^{1}_{0\imath\jmath}=C_{1\imath\jmath}\,,$$

$$K^{1}_{01\bar{j}}=K''^{1}_{01\bar{j}}-A_{j1}=-\partial_{\bar{j}}A^{1}_{01}+\frac{1}{n-2}R_{1j}+\frac{1}{n-2}R_{1\jmath}=0\,,$$

$$K^{1}_{0\beta\gamma}=K''^{1}_{0\beta\gamma}=0 \qquad \text{otherwise}\,,$$

$$K^{1}_{1i1}=K''^{1}_{1i1}-A_{1\imath}=\left\{-\frac{1}{2(n-2)}+\frac{1}{2(n-2)}\right\}R_{1\imath}=0\,,$$

$$K^{1}_{1\beta\gamma}=K''^{1}_{1\beta\gamma}=0 \qquad \text{otherwise}\,.$$

Since when $n=3$ we have

$$C^{\imath}_{jkm}=0\,,$$

the proposition is proved. q.e.d.

From the property C3) of Cartan connections, or, since K is a tensorial 2-form of type $(Ad, \mathfrak{g})$ on P, the harmonic part of K is determined all over P by Proposition 4.1 and 4.2. Moreover, noting that the index 1 in Proposition 4.1 and 4.2 denotes the direction of the base point $\pi(u_0) \in T_1 M$, which is arbitrarily chosen, we obtain the main results, Corollary 1 and 2.

Since we have used here the fact described in the beginning of this section, we will give a direct proof of these results in the next section, to be self-contained.

§ 5. Tanaka connection and the Lie curvature in a local coordinate.

In order to give a complete description of ω, determine A_{11} by (14)′ of [M1]. We prepare first

LEMMA 5.1. *We have at* $u_0 \in P_g$,

$$\partial_{\bar{j}} A_{i1} = -\frac{1}{n-2}(R_{ij} - R_{11}\delta_{ij}),$$

$$\partial_{\bar{j}} A_{1i} = -\frac{1}{2(n-2)}(R_{ij} - R_{11}\delta_{ij}).$$

Proof. This follows easily from (2.13) and

$$\partial_{\bar{j}} R_{1i} = -\partial_{\bar{j}} \sum_{m=2}^{n} T'^{\bar{m}}_{i m} = \tilde{R}_{ij} - R_{11}\delta_{ij} + R^{j}_{1i1}$$

$$= R_{ij} - R_{11}\delta_{ij}.$$

q.e.d.

Now, we obtain

$$\sum_{i=2}^{n} (\partial_{\bar{i}} A_{i1} + \partial_{\bar{i}} A_{1i}) = -\left\{\frac{1}{n-2} + \frac{1}{2(n-2)}\right\} \sum_{i=2}^{n} \partial_{\bar{i}} R_{1i}$$

$$= -\frac{3}{2(n-2)}\left\{\sum_{i=2}^{n} R_{ii} - (n-1)R_{11}\right\}$$

$$= -\frac{3}{2(n-2)}(R - nR_{11}),$$

and

$$\sum_{i=2}^{n} (-K''_{\bar{i}i1} + K''_{i\bar{i}1} - K''_{1i\bar{i}}) = \frac{1}{2}R_{11} + \frac{1}{2}\sum_{i=2}^{n} A_{ii} - \sum_{i=2}^{n} A_{ii} - \frac{1}{2}(n-1)A^{1}_{01} - \frac{1}{2}\sum_{i=2}^{n} A_{ii}$$

$$= \frac{1}{2}R_{11} + \frac{1}{n-2}\sum_{i=2}^{n} R_{ii} - \frac{R}{2(n-2)} - \frac{n-1}{2(n-2)}R_{11} + \frac{R}{4(n-2)}$$

$$=-\frac{3}{2(n-2)}R_{11}+\frac{3R}{4(n-2)}.$$

Thus, we obtain

$$A_{11}=-\frac{1}{2(n-2)}R_{11}+\frac{R}{4(n-1)(n-2)}.$$

As for the curvature $\Omega=1/2K\theta\wedge\theta$ of ω, the $\mathfrak{g}_p$-components K_p is given in Proposition 3.2 for $p<0$, and in the proof of Proposition 4.2 for $p=0$. For $p>0$, we have from

$$\Omega_i-\Omega''_i=d(A_{i1}\theta^1)+\theta^{\bar{i}}\wedge\sum_{j=2}^{n}A_{1j}\theta^j+\theta^{\bar{i}}\wedge A_{11}\theta^1,$$

$$\longleftrightarrow K_{i1j}=K''_{i1j}-\partial_j A_{i1}$$

$$=\partial_1 A_{ij}-\partial_j A_{i1}=\Pi_{ij,1}-\Pi_{i1,j}=C_{ij1},$$

$$K_{i1\bar{j}}=K''_{i1\bar{j}}\partial_{\bar{j}}A_{i1}-A_{11}\delta^i_j$$

$$=-\frac{1}{n-2}R_{ij}+\frac{R}{2(n-1)(n-2)}\delta^i_j+\frac{1}{2(n-2)}\left\{R_{11}-\frac{R}{2(n-1)}\right\}\delta^i_j$$

$$+\frac{1}{n-2}(R_{ij}-R_{11}\delta^i_j)-\left\{-\frac{1}{2(n-2)}R_{11}+\frac{R}{4(n-1)(n-2)}\right\}\delta^i_j$$

$$=0,$$

$$K_{ijk}=K''_{ijk}=-C_{ijk},$$

$$K_{i\bar{j}k}=K''_{i\bar{j}k}+A_{i1}\delta^j_k+A_{1k}\delta^i_j$$

$$=\frac{1}{n-2}(R_{1i}\delta^j_k+R_{1k}\delta^i_j)-\frac{1}{2(n-2)}R_{1k}\delta^i_j-\frac{1}{n-2}R_{1i}\delta^j_k-\frac{1}{2(n-2)}R_{1k}\delta^i_j$$

$$=0,$$

$$K_{i\bar{j}\bar{k}}=K''_{i\bar{j}\bar{k}}=0.$$

Similarly, we obtain

$$\Omega_{\bar{i}}-\Omega''_{\bar{i}}=-\theta^i\wedge\left(\sum_{j=2}^{n}A_{1j}\theta^j+A_{11}\theta^1\right)$$

$$\longleftrightarrow K_{\bar{i}1j}=K''_{\bar{i}1j}+A_{11}\delta^i_j$$

$$=-\frac{1}{2}R^i_{11j}-\frac{1}{2(n-2)}R_{ij}+\frac{R}{4(n-1)(n-2)}\delta^i_j$$

$$-\frac{1}{2(n-2)}R_{11}+\frac{R}{4(n-1)(n-2)}\delta^i_j$$

$$=-\frac{1}{2}C^{i}_{11j},$$

$$K_{\bar{i}1\bar{j}}=K''^{\bar{i}}_{1\bar{j}}=0,$$

$$K_{\bar{i}jk}=K''^{\bar{i}}_{jk}-A_{1k}\delta^{i}_{j}+A_{1j}\delta^{i}_{k}$$

$$=-\frac{1}{2}R^{i}_{1jk}+\frac{1}{2(n-2)}R_{1k}\delta^{i}_{j}-\frac{1}{2(n-2)}R_{1j}\delta^{i}_{k}$$

$$=-\frac{1}{2}C^{i}_{1jk},$$

$$K_{\bar{i}\bar{j}k}=K''^{\bar{i}}_{\bar{j}k}=0,$$

$$\Omega_1-\Omega''_1=d\Big(\sum_{j=2}^{n}A_{1j}\theta^{j}+A_{11}\theta^{1}\Big)\sum_{i=2}^{n}A_{i1}\theta^{1}\wedge A_{\bar{i}\bar{i}}\theta^{\bar{i}}$$

$$\longleftrightarrow K_{11j}=\partial_1A_{1j}-\partial_jA_{11}=\frac{1}{2}C_{1j1},$$

$$K_{11\bar{j}}=A_{1j}-\partial_{\bar{j}}A_{11}-A_{j1}A_{\bar{j}\bar{j}}$$

$$=-\frac{1}{2(n-2)}R_{1j}+\frac{1}{2(n-2)}\partial_{\bar{j}}R_{11}-\frac{1}{2(n-2)}R_{1j}$$

$$=-\frac{1}{n-2}R_{1j}+\frac{1}{n-2}R_{1j}=0,$$

$$K_{1jk}=\partial_jA_{1k}-\partial_kA_{1j}=-\frac{1}{2}C_{1jk},$$

$$K_{1j\bar{k}}=K''^{1}_{j\bar{k}}-\partial_{\bar{k}}A_{1j}-A_{11}\delta^{j}_{k}$$

$$=\frac{1}{2}\Big\{-\frac{1}{n-2}R_{jk}+\frac{R}{2(n-1)(n-2)}\delta^{j}_{k}\Big\}+\frac{1}{2(n-2)}(R_{jk}-R_{11}\delta^{i}_{j})$$

$$-\Big\{-\frac{1}{2(n-2)}R_{11}+\frac{R}{4(n-1)(n-2)}\Big\}\delta^{j}_{k}$$

$$=0,$$

$$K_{1\bar{j}\bar{k}}=0,$$

Now, we summarize our results as a theorem.

THEOREM. *Let* $(x^{i}, z^{i}_{j}, s^{a}_{b}, s_{i}, s_{\bar{i}}, s_{1})$ *be the local cordinate around* $u_0\in P_g$ *chosen as in* § 3. *Then at* $u=u(z)=(x^{i}, z^{i}_{j}, \delta^{a}_{b}, \mathbf{0}, \mathbf{0}, 0)\in P_g$, *Tanaka connection* $\boldsymbol{\omega}$ *on* $\pi: P\to T_1M$ *is given by*

$$\theta^{i}=g_{jk}z^{k}_{i}dx^{j}, \qquad 1\leqq i\leqq n,$$

$$\theta^{\bar{i}}=g_{jk}z_i^k\left(dz_1^j+\left\{{j \atop st}\right\}z_1^s dx^t\right), \qquad 2\leqq i\leqq n,$$

$$\omega_j^i=g_{uv}z_i^v\left(dz_j^u+\left\{{u \atop st}\right\}z_j^s dx^t\right), \qquad 2\leqq i,\ j\leqq n,$$

$$\omega_0^0=ds_0^0,$$

$$\omega_1^0=ds_1^0+A_{11}^0\theta^1,$$

$$\omega_0^1=ds_0^1+\sum_{j=2}^n A_{0j}^1\theta^j+A_{01}^1\theta^1,$$

$$\omega_1^1=ds_1^1,$$

$$\omega_i=ds_i+\sum_{j=2}^n A_{ij}\theta^j+A_{i1}\theta^1, \qquad 2\leqq i\leqq n,$$

$$\omega_{\bar{i}}=ds_{\bar{i}}+A_{\bar{i}\bar{i}}\theta^{\bar{i}}, \qquad 2\leqq i\leqq n,$$

$$\omega_1=ds_1+\sum_{j=2}^n A_{1j}\theta^j+A_{11}\theta^1,$$

where

$$A_{11}^0=-\frac{1}{2},\quad A_{0j}^1=\frac{1}{n-2}R_{1j},\quad A_{01}^1=\frac{1}{n-2}R_{11}-\frac{R}{2(n-1)(n-2)},$$

$$A_{ij}=-\frac{1}{n-2}R_{ij}+\frac{R}{2(n-1)(n-2)}\delta_j^i,\quad A_{\bar{i}\bar{i}}=-\frac{1}{2},$$

$$A_{i1}=-\frac{1}{n-2}R_{1i},\quad A_{1i}=-\frac{1}{2(n-2)}R_{1i},$$

$$A_{11}=-\frac{1}{2(n-2)}R_{11}+\frac{R}{4(n-1)(n-2)},$$

using the Ricci curvature $R_{ij}=z_i=z_i^k\partial/\partial x_k$, *and the scalar curvature* R *of* M *at* (x^i). *The curvature* K *of* ω *at* u_0 *is given by*

$$K_{1j}^{\bar{i}}=C_{11j}^i,\quad K_{jk}^{\bar{i}}=C_{1jk}^i,$$

$$K_{01i}^1=C_{11i},\quad K_{0ij}^1=C_{1ij},\quad K_{jk1}^i=C_{jk1}^i,\quad K_{jkm}^i=C_{jkm}^i,$$

$$K_{i1j}=-C_{i1j},\quad K_{ijk}=-C_{ijk},$$

$$K_{\bar{i}1j}=-\frac{1}{2}C_{11j}^i,\quad K_{\bar{i}jk}=-\frac{1}{2}C_{1jk}^i,$$

$$K_{11j}=-\frac{1}{2}C_{11j},\quad K_{1jk}=-\frac{1}{2}C_{1jk},$$

for $2 \leqq i, j, k, m \leqq n$, *and all other components vanish, where* C^i_{jkm} *and* C_{ijk} *are the coefficients of Weyl's conformal curvature.*

Since $R^*_a\omega = Ad(a^{-1})\omega$, $a \in G'$, ω and K are determined all over P by this theorem.

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