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PERIODIC EXTENSIONS OF TWO-DIMENSIONAL BROWNIAN MOTION ON THE HALF PLANE, II

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This paper is a continuation of the one with the same title [2]. Notation follow the previous paper. Theorems, propositions and formula in [2] are cited by their numbers without special mention.

Main results of this paper are summarized as follows:

(1) For any $B=\{\sigma, \mu, k, p\}$ in $\mathcal{L}$, there exists a B-process $\boldsymbol{P}$ with $B_p=B$, which satisfies conditions [M] and [V]. (See theorem [19.16]. Uniqueness of B-process for given B has already been proved in theorem [7.7] in the previous paper [2].)

(2) A B-process has continuous path functions in $\bar{D}$ if and only if σ and μ are positive for any open set. (See theorem [14.9] and theorem [19.16].)

(3) A process P in $\mathcal{P}$ has continuous path functions and is of Feller type in $\bar{D}$ if and only if P is a B-process, such that σ and μ are positive for any open set and σ has no discrete mass. (See theorem [15.10] and theorem [19.16].)

IV. Characterization of the class $\mathcal{P}_c$.

§12. Certain recurrence relations.

Throughout this section, we shall fix a process P in $\mathcal{P}$, on which we shall assume no additional condition.

Let $\sigma_a(w)$ be the hitting time of ∂a, and for any positive a and b with $a\neq b$, we define $\rho_n=\rho_n(a, b, w)$ and $\tau_n=\tau_n(a, b, w)$ by

$$
\begin{aligned}
(12.1)\qquad & \rho_0=\sigma_a\,, \\
& \tau_n=\rho_n+\sigma_b(\theta_{\rho_n}w)\,, \\
& \rho_{n+1}=\tau_n+\sigma_a(\theta_{\tau_n}w)\,, \qquad n=0,1,2,\cdots.
\end{aligned}
$$

Since one-dimensional reflecting Brownian motion is recurrent, by [1.5] and [1.6] and continuity of the process in D^* we can easily see:

[12.1] ρ_n and τ_n $(n=0, 1, 2, \cdots)$ are finite except on a set of P_z-measure

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zero for any z in D, and $\lim_{n\to\infty}\rho_n=\lim_{n\to\infty}\tau_n=\infty$ holds.

Set, for $t\geqq 0$ and $h>0$,

$$L_a^h(t, w)=\frac{1}{2h}\int_0^t I_{[a-h, a+h]}(y(s, w))ds, \tag{12.2}$$

where $z(t, w)=(x(t, w), y(t, w))$ for $z(t, w)\in D$ and I_A is the indicator of a set A. Noting [1.6], the following results are well known in theory of Brownian local time [1].

[12.2] For any z in D,

(1) $L_a(t, w)=\lim_{h\to 0} L_a^h(t, w)$ exists a.s. P_z,

(2) $E_z(L_a(t, w))=\lim_{h\to\infty} E_z(L_a^h(t, w))$.

(3) $L_a(t, w)$ is continuous and increasing in t and satisfies

$$L_a(t+s, w)=L_a(t, w)+L_a(s, \theta_t w) \tag{12.3}$$

for any s and t a.s. P_z.

(4) $L_a(t, w)$ increases only on t with $z(t, w)\in\partial_a$, that is,

$$L_a(t, w)=\int_0^t I_{\partial_a}(z(s, w))dL_a(s, w) \quad \text{a.s. } P_z. \tag{5}$$

$$E_z(L_a^h(t, w)), \quad E_z(L_a(t, w))\leqq C_1\sqrt{t}, \tag{12.4}$$

$$E_z(L_a^h(t, w)^2), \quad E_z(L_a(t, w)^2)\leqq C_2 t, \tag{12.5}$$

where C_1 and C_2 are absolute constants.

[12.3] Let a and b are any positive numbers and z be a point in D.

(1) If $y\leqq a<b$ or $y\geqq a>b$,

$$E_z(L_a(\sigma_b))=2|b-a|.$$

(2) In general, it holds that

$$E_z(L_a(\sigma_b))\leqq 2|b-a|$$

and

$$E_z(L_a(\sigma_b)^2)\leqq 8(b-a)^2.$$

[12.4] Let ϕ be a bounded continuous function defined on $D^{[a-c, a+c]}$ with $0<c<a$, and λ be a positive number. Then

$$\lim_{h\to 0} E_z\left(\int_0^\infty e^{-\lambda t}\phi(z(t))dL_a^h\right)=E_z\left(\int_0^\infty e^{-\lambda t}\phi(z(t))dL_a\right).$$

Proof.

1° Let ε be any positive number. By (12.3) and (12.4) we can choose T such that

$$E_z\Big(\int_T^\infty e^{-\lambda t}|\phi(z(t))|\,dL_a^h\Big),\qquad E_z\Big(\int_T^\infty e^{-\lambda t}|\phi(z(t))|\,dL_a\Big)<\varepsilon.$$

2° Choose positive ε_1 such that $(\varepsilon_1 C_1+8\|\phi\|\sqrt{\varepsilon_1 C_2})\sqrt{T}<\varepsilon/2$, where C_1 and C_2 are constants appearing in (12.4) and (12.5). The function ϕ can be extended to a function $\tilde{\phi}$ which is continuous in D with $\|\phi\|=\|\tilde{\phi}\|$, and there exists a positive integer N such that, for

$$\mathfrak{U}=\mathfrak{U}(T,N,\varepsilon_1)=\{w:\sup_{s,t\leqq T,\,|s-t|\leqq 1/N}|\tilde{\phi}(z(s))-\tilde{\phi}(z(t))|<\varepsilon_1\},$$

$P_z(\mathfrak{U}^c)<\varepsilon_1$ and $(\lambda/N)\|\phi\|C_1\sqrt{T}<\varepsilon/2$ hold. Set

$$t_k=\frac{kT}{N}\qquad (k=0,1,2,\cdots,N)\qquad\text{and}$$

$$I_N^h=E_z\Big\{\sum_{k=0}^{N-1}e^{-\lambda t_k}\tilde{\phi}(z(t_k))(L_a^h(t_{k+1})-L_a^h(t_k))\Big\},$$

$$I_N=E_z\Big\{\sum_{k=0}^{N-1}e_-^{\lambda t_k}\tilde{\phi}(z(t_k)(L_a(t_{k+1})-L_a(t_k))\Big\}.$$

Then by (12.4) and (12.5)

$$\Big|E_z\Big(\int_0^T e^{-\lambda t}\phi(z(t)dL_a^h)-I_N^h\Big|$$

$$\leqq(1-e^{-\lambda T/N})\|\phi\|E_z(L_a^h(t))$$

$$+E_z\Big(\sum_{k=0}^{N-1}e^{-\lambda t_k}\int_{t_l}^{t_{k+1}}|\tilde{\phi}(z(t))-\tilde{\phi}(z(t_k))|\,dL_a^h\Big)$$

$$\leqq\frac{\lambda T}{N}\|\phi\|E_z(L_a^h(T))+\varepsilon_1E_z(I_{\mathfrak{U}}L_a^h(T))+2\|\tilde{\phi}\|E_z(I_{\mathfrak{U}^c}L_a^h(T))$$

$$\leqq\Big(\frac{\lambda T}{N}\|\phi\|+\varepsilon_1\Big)E_z(L_a^h(T))+2\|\phi\|P_z(\mathfrak{U}^c)^{1/2}E_z(L_a^h(T)^2)^{1/2}$$

$$\leqq\Big(\frac{\lambda T}{N}\|\phi\|+\varepsilon_1\Big)C_1\sqrt{T}+4\sqrt{2}\|\phi\|\sqrt{\varepsilon_1C_2T}<\varepsilon.$$

Similarly, by (12.4) and (12.5),

$$\Big|E_z\Big(\int_0^T e^{-\lambda t}\phi(z(t))dL_a\Big)-I_N\Big|<\varepsilon.$$

3° On the other hand, by (12.3) and Markov property of the process, for fixed N and T we have

$$\begin{aligned}\lim_{h\to 0} I_N^h &= \lim_{h\to 0} E_z\left\{\sum_{k=0}^{N-1} e^{-\lambda t_k}\tilde{\phi}(z(t_k))E_{z(t_k)}\left(L_a^h\left(\frac{T}{N}\right)\right)\right\}\\ &= E_z\left\{\sum_{k=0}^{N-1} e^{-\lambda t_k}\tilde{\phi}(z(t_k))E_{z(t_k)}\left(L_a\left(\frac{T}{N}\right)\right)\right\}\\ &= I_N .\end{aligned}$$

By 1°, 2° and 3° proof of [12.4] is completed.

[12.5] Let a and δ be any positive numbers, then

(1) $\lim_{b\to a} \sup_x P_{(x,a)}(\sigma_b \geqq \sigma)=0,$

(2) $\lim_{b\to a} \sup_x P_{(x,a)}(\sup_{s\leqq \sigma_b}|z(s)-z(0)|>\delta,\ \sigma_b<\sigma)=0,$

(3) $\lim_{b\to a} \sup_x P_{(x,a)}(\sigma_b \geqq \delta)=0.$

Proof. Noting $P_{(x,a)}(\sigma_b\geqq\delta)\leqq P_{(x,a)}(\sigma_b\geqq\sigma)+P_{(x,a)}(\sigma>\sigma_b\geqq\delta)$, [12.5] is obvious by (p. 4) in [1.1]

[12.6] For ϕ in $C_p(R)$ and $a>0$

$$\lambda E_{\tilde{m}}\left(\int_0^\infty e^{-\lambda t}\phi(x(t))dL_a\right)=\int_0^{2\pi}\phi(x)m_P(x, a)dx, \tag{12.6}$$

where $E_{\tilde{m}}(\cdot)=\int_{\tilde{D}} E_z(\cdot)m_P(z)dz$ and $\tilde{D}=\{z=(x, y)\in D: 0<x<2\pi\}$.

Proof. Set $\tilde{\phi}(z)=\phi(x)$ for $z=(x, y)$ in D, then by [8.20]

$$\begin{aligned}\lambda E_{\tilde{m}}\left(\int_0^\infty e^{-\lambda t}\tilde{\phi}(z(t))dL_a^h\right) &= \frac{\lambda}{2h}\int_{\tilde{D}} G_\lambda(I_{[a-h,a+h]}\tilde{\phi})(z)m_P(z)dz\\ &= \frac{1}{2h}\int_{a-h}^{a+h}dy\int_0^{2\pi}\phi(x)m_P(x, y)dx\\ &\longrightarrow \int_0^{2\pi}\phi(x)m_P(x, a)dx \qquad (h\to 0).\end{aligned}$$

On the other hand, since

$$\begin{aligned}\left|E_z\left(\int_0^\infty e^{-\lambda t}\tilde{\phi}(z(t))dL_a^h\right)\right| &\leqq \|\phi\| E_y^{R,1}\left(\int_0^\infty e^{-\lambda t}dL_a^h\right)\\ &= \|\phi\| e^{-\sqrt{2\lambda(y-c)}}E_c^{R,1}\left(\int_0^\infty e^{-\lambda t}dL_a^h\right)\end{aligned}$$

for $a+h<c$ and $y\geqq c$, we have by [12.4], (4) in [12.2] and the dominated convergence theorem we have

$$\lim_{h\to 0}\lambda E_{\tilde{m}}\left(\int_0^\infty e^{-\lambda t}\tilde{\phi}(z(t))dL_a^h\right)$$

$$=\lambda E_{\tilde{m}}\left(\int_0^\infty e^{-\lambda t}\tilde{\phi}(z(t))dL_a\right)$$

$$=\lambda E_{\tilde{m}}\left(\int_0^\infty e^{-\lambda t}\phi(x(t))dL_a\right).$$

[12.7] For any positive a and b with $0<|b-a|\leqq 1$, ρ_n and τ_n are defined as in (12.1). Then, for any positive λ, it holds that

$$E_z\left(\sum_{n=0}^\infty e^{-\lambda\rho_n}\right)\leqq\frac{E_z(e^{-\lambda\sigma_a})}{1-e^{-\sqrt{2\lambda}|b-a|}}. \tag{12.7}$$

Especially,

$$|b-a|E_z\left(\sum_{n=0}^\infty e^{-\lambda\rho_n}\right)\leqq K(\lambda)\operatorname{Min}\{e^{-\sqrt{2\lambda}(y-a)}, 1\}, \tag{12.8}$$

where $K(\lambda)$ is a constant independent of a, b and z.

Proof. If $b<a$, then we have by [1.5] and [1.6]

$$E_z(e^{-\lambda\rho_{n+1}})\leqq E_z(e^{-\lambda\tau_n})$$

$$=E_z(e^{-\lambda\rho_n}E_{z(\rho_n)}(e^{-\lambda\sigma_b}))$$

$$=E_z(e^{-\lambda\rho_n}E_a^{R,1}(e^{-\lambda\sigma_b}))$$

$$=E_z(e^{-\lambda\rho_n})e^{-\sqrt{2\lambda}(a-b)}.$$

Similarly, if $b>a$, then

$$E_z(e^{-\rho_{n+1}})\leqq E_z(e^{-\lambda\rho_n-\lambda(\rho_{n+1}-\tau_m)})$$

$$=E_z(e^{-\lambda\rho_n}E_{z(\tau_n)}(e^{-\lambda\sigma_a}))$$

$$=E_z(e^{-\lambda\rho_n}E_b^{R,1}(e^{-\lambda\sigma_a}))$$

$$=E_z(e^{-\lambda\rho_n})e^{-\sqrt{2\lambda}(b-a)}.$$

Therefore, in both cases we have by induction

$$E_z(e^{-\lambda\rho_{n+1}})\leqq E_z(e^{-\lambda\sigma_a})e^{-n\sqrt{2\lambda}|b-a|}\qquad (n=0, 1, 2, \cdots) \tag{12.9}$$

and (12.7) is obvious. Since

$$E_z(e^{-\lambda\sigma_a})=E_y^{R,1}(e^{-\lambda\sigma_a})=e^{-\sqrt{2\lambda}(y-a)}\qquad \text{if } y\geqq a,$$

setting $K(\lambda)=\sup_{0<y\leqq 1}\frac{y}{1-e^{-\sqrt{2\lambda y}}}$, we have (12.8).

[12.8] THEOREM. *For any positive* a *and* b *with* $a\neq b$, *let* $\rho_n=\rho_n(a, b, w)$ *and* $\tau_n=\tau_n(a, b, w)$ $(n=0, 1, 2, \cdots)$ *be defined as in* (12.1), $\xi_n=\xi_n(w)$ *and* $\eta_n=\eta_n(w)$ $(n=0, 1, 2, \cdots)$ *be measurable functions on* (W, B) *with* $\rho_n\leqq\xi_n$, $\eta_n\leqq\tau_n$ *and* λ *be any fixed positive number.*

(1) *If* ϕ *is a bounded uniformly continuous function on* R, *then we have*

$$\lim_{b\to a}2|b-a|E_z\Big(\sum_{n=0}^{\infty}e^{-\lambda\xi_n}\phi(x(\eta_n))\Big)=E_z\Big(\int_0^{\infty}e^{-\lambda t}\phi(x(t))dL_a\Big).$$

(2) *If* ϕ *is in* $C_p(R)$, *then we have*

$$\lim_{b\to a}2|b-a|E_{\tilde{m}}\Big(\sum_{n=0}^{\infty}e^{-\lambda\xi_n}\phi(x(\eta_n))\Big)=\frac{1}{\lambda}\int_0^{2\pi}\phi(x)m_P(x, a)dx.$$

we set $\phi(x(t))=0$ *if* $z(t)=\partial$ *and* $E_{\tilde{m}}(\cdot)$ *is defined in* [12.6].

Proof. If (1) holds, then (2) follows from by (12.8), the dominated convergence theorem and [12.6]. Now we shall prove (1).

1° Set $\varepsilon=|b-a|$ and define

$$d(\delta)=\sup_{|\xi-x|<\delta}|\phi(\xi)-\phi(x)|$$

for any positive δ,

$$e(t)=e(t, w)=\sup_{0\leqq s\leqq t}|\phi(x(s))-\phi(x(t))|$$

and

$$p_1(\varepsilon)=\sup_x E_{(x,a)}\{e(\sigma_b(w), w)\}.$$

Then

$$p_1(\varepsilon)\leqq d(\delta)+2\|\phi\|\sup_x P_{(x,a)}(\sup_{0\leqq s\leqq\sigma_b}|x(s)-x(0)|>\delta,\ \sigma_b<\sigma)$$
$$+2\|\phi\|\sup_x P_{(x,a)}(\sigma_b\geqq\sigma).$$

Therefore by [12.5] $\overline{\lim}_{\varepsilon\to 0}p_1(\varepsilon)\leqq d(\delta)$.

Since ϕ is uniformly continuous, $\lim_{\delta\to 0}d(\delta)=0$. We have

(12.10) $$\lim_{\varepsilon\to 0}p_1(\varepsilon)=0.$$

Set $p_2(\varepsilon)=\sup_{\lambda} E_{(x,a)}(1-e^{-\lambda\sigma_b})$. Then

$$p_2(\varepsilon)\leqq\lambda\delta+\sup_x P_{(x,a)}(\sigma_b>\delta)$$

for any positive δ. Therefore by [12.5]

(12.11) $$\lim_{\varepsilon\to 0} p_2(\varepsilon)=0\,.$$

2°

$$J_1(\varepsilon)=2\varepsilon\{E_z(\Sigma e^{-\lambda\xi_n}\phi(x(\eta_n)))-E_z(\Sigma e^{-\lambda\rho_n}\phi(x(\rho_n)))\}\longrightarrow 0 \qquad (\varepsilon\to 0)\,.$$

Proof of 2°.

$$|J_1(\varepsilon)|\leqq I_1(\varepsilon)+I_2(\varepsilon)\,,$$

where

$$I_1(\varepsilon)=2\varepsilon\|\phi\|E_z(\Sigma(e^{-\lambda\rho_n}-e^{-\lambda\tau_n}))$$

and

$$I_2(\varepsilon)=2\varepsilon E_z\Big(\Sigma e^{-\lambda\rho_n}\sup_{\rho_n\leqq s\leqq\tau_n}|\phi(x(s))-\phi(x(\rho_n))|\Big).$$

Then by [1.5] and (12.8)

$$I_1(\varepsilon)=2\varepsilon\|\phi\|E_z\{\Sigma e^{-\lambda\rho_n}E_{z(\rho_n)}(1-e^{-\lambda\sigma_b})\}$$
$$\leqq 2\|\phi\|K(\lambda)p_2(\varepsilon)$$

and

$$I_2(\varepsilon)=2E_z\{\Sigma e^{-\lambda\rho_n}E_{z(\rho_n)}(e(\sigma_b))\}$$
$$\leqq 2K(\lambda)p_1(\varepsilon)\,,$$

where $K(\lambda)$ is defined as in (12.8). Therefore by (12.10) and (12.11)

$$|J_1(\varepsilon)|=I_1(\varepsilon)+I_2(\varepsilon)\longrightarrow 0 \qquad (\varepsilon\to 0)\,.$$

3°

$$J_2(\varepsilon)=2\varepsilon E_z\{\Sigma e^{-\lambda\rho_n}\phi(x(\rho_n))\}-E_z\Big\{\Sigma\phi(x(\rho_n))\int_{\rho_n}^{\tau_n}e^{-\lambda t}dL_a\Big\}\longrightarrow 0 \quad (\varepsilon\to 0).$$

Proof of 3°. By (2) in [12.3]

$$2\varepsilon E_z\{\Sigma e^{-\lambda\rho_n}\phi(x(\rho_n))\}=E_z\{\Sigma e^{-\lambda\rho_n}\phi(x(\rho_n))L_a(\sigma_b)\}$$
$$=E_z\Big\{e^{-\lambda\rho_n}\phi(x(\rho_n))\int_{\rho_n}^{\tau_n}dL_a\Big\}\,.$$

Hence

$$|J_2(\varepsilon)|\leqq E_z\Big\{\Sigma e^{-\lambda\rho_n}|\phi(z(\rho_n))|\int_{\rho_n}^{\tau_n}(1-e^{-\lambda t})dL_a\Big\}$$
$$\leqq\|\phi\|E_z[\Sigma e^{-\lambda\rho_n}E_{z(\rho_n)}\{(1-e^{-\lambda\sigma_b})L_a(\sigma_b)\}]$$
$$\leqq\|\phi\|E_z[\Sigma e^{-\lambda\rho_n}E_{z(\rho_n)}(1-e^{-\lambda\sigma_b})^{1/2}E_{z(\rho_n)}(L_a(\sigma_b)^2)^{1/2}]$$
$$\leqq\|\phi\|E_z(\Sigma e^{-\lambda\rho_n})p_2(\varepsilon)^{1/2}\sqrt{8\varepsilon^2}$$
$$\leqq\|\phi\|K(\lambda)\sqrt{8}\,p_2(\varepsilon)^{1/2}\,.$$

Therefore by (12.11)

$$\lim_{\varepsilon\to 0} J_2(\varepsilon)=0 .$$

4°

$$J_3(\varepsilon)=E_z\left\{\Sigma\phi(x(\rho_n))\int_{\rho_n}^{\tau_n} e^{-\lambda t}dL_a\right\}-E_z\left(\int_0^\infty e^{-\lambda t}\phi(x(t))dL_a\right)\longrightarrow 0 \quad (\varepsilon\to 0)$$

Proof of 4°. Since $L_a(\rho_0)=0$ and $L_a(\tau_n)=L_a(\rho_{n+1})$ by (4) in [12.2],

$$J_3(\varepsilon)=E_z\left\{\Sigma\int_{\rho_n}^{\tau_n} e^{-\lambda t}(\phi(x(\rho_n))-\phi(x(t)))dL_a\right\}.$$

$$|J_3(\varepsilon)|\leqq E_z(\Sigma e^{-\lambda\rho_n})\sup_x E_{(x,a)}(e(\sigma_b)L_a(\sigma_b))$$

$$\leqq E_z(\Sigma e^{-\lambda\rho_n})\sup_x E_{(x,a)}(e(\sigma_b)^2)^{1/2}E_{(x,a)}(L_a(\sigma_b)^2)^{1/2}$$

$$\leqq 4K(\lambda)\|\phi\|^{1/2}p_1(\varepsilon)^{1/2} .$$

Therefore, by (12.10), 4° is proved. From 2°, 3° and 4° we can see that (1) holds.

In the remainder of the section, we shall investigate properties of the last hitting time.

[12.9] DEFINITION. Let a and b be any positive numbers with $a\neq b$. If $z(0, w)\in\partial_a$, set

$$\hat\rho=\hat\rho(a, b, w)=\inf\{t: t\leqq\sigma_b \text{ and } z_s\notin\partial_a \text{ for any } s\in(t, \sigma_b)\}.$$

For general w, set

$$\hat\rho=\hat\rho(a, b, w)=\sigma_a+\hat\rho(\theta_{\sigma_a}w) .$$

This is the last hitting time of ∂_a before reaching ∂_b.

For c with $c\in(a, b)$, set

$$\hat\rho_c=\hat\rho_c(a, b, w)=\hat\rho+\sigma_c(\theta_{\hat\rho}w) . \tag{12.12}$$

The sequence

$$\bar\rho_n=\rho_n(a, c, w) \quad\text{and}\quad \bar\tau_n=\tau_n(a, c, w) \quad (n=0, 1, 2, \cdots)$$

are as given in (12.1). Then we can easily see:

[12.10]
(1) $\hat\rho_c\downarrow\hat\rho$ as $c\to a$.
(2) If $\bar\rho_n<\sigma_a+\sigma_b(\theta_{\sigma_a}w)\leqq\bar\rho_{n+1}$, then $\hat\rho_c=\bar\tau_n$.
(3) Especially, $\hat\rho$ and $\hat\rho_c$ are B-measurable.

[12.11] $\hat{\rho}$ and $\hat{\rho}_c$ are finite except on a set of P_z-measure zero for any positive z in D.

Proof. By [1.6], $\tau=\sigma_a+\sigma_b(\theta_{\sigma_a}w)<\infty$ a.s. P_z. On the other hand $\hat{\rho}$, $\hat{\rho}<\tau$.

[12.12] PROPOSITION. *Let f and g be in $B_b(R)$. For positive a and b with $a\neq b$, set $\hat{\rho}=\hat{\rho}(a, b, w)$ and $\tau=\sigma_a+\sigma_b(\theta_{\sigma_a}w)$. Then for any positive λ it holds that*

$$E_z\{e^{-\lambda\rho}f(x(\hat{\rho}))g(x(\tau))\}$$
$$=|b-a|E_z\{e^{-\lambda\rho}f(x(\hat{\rho}))Q^{|b-a|}g(x(\hat{\rho}))\}, \tag{12.13}$$

where $Q^{|b-a|}g(x)=\int q^{|b-a|}(\xi-x)g(\xi)d\xi$ *is defined in* § 0.8°.

Proof. It is sufficieient to prove (12.13) for f and g in $C_K(R)$. For any c with $c\in(a, b)$, $\hat{\rho}_c$ is defined as in (12.12). Set $\bar{\rho}_n=\rho_n(a, c, w)$ and $\bar{\tau}_n=\tau_n(a, c, w)$. Then

$$g(x(\tau))I_{\{\bar{\rho}_n<\tau<n+1\}}=g(x(\tau))I_{\{\bar{\tau}_n<\tau<\bar{\rho}_{n+1}\}}$$
$$=g(x(\sigma_b(\theta_{\bar{\tau}_n}w),\ \theta_{\bar{\tau}_n}w))I_{\{\bar{\tau}_n<\tau\}}I_{\{\sigma_b(\theta_{\bar{\tau}_n}w)<\sigma_a(\theta_{\bar{\tau}_n}w)\}}\,.$$

Therefore, noting (2) in [12.10] and [1.5], we have

$$E_z(e^{-\lambda\rho_c}f(x(\hat{\rho}_c))g(x(\tau)))$$
$$=E_z\Big(\sum_{n=0}^{\infty}e^{-\lambda\bar{\tau}_n}f(x(\bar{\tau}_n))g(x(\tau))I_{\{\bar{\rho}_n<\tau<\bar{\rho}_{n+1}\}}\Big)$$
$$=E_z\Big(\sum_{n=0}^{\infty}e^{-\lambda\bar{\tau}_n}f(x(\bar{\tau}_n))I_{\{\bar{\tau}_n<\tau\}}E_{z(\bar{\tau}_n)}(g(x(\sigma_b))I_{\{\sigma_b<\sigma_a\}}\Big)$$
$$=E_z\Big\{\sum_{n=0}^{\infty}e^{-\lambda\bar{\tau}_n}f(x(\bar{\tau}_n))I_{\{\bar{\tau}_n<\tau\}}\,{}_a^b\Pi_c^b\,g(x(\bar{\tau}_n))\Big\}.$$

In the same way we get

$$E_z\{e^{-\lambda\rho_c}f(x(\hat{\rho}_c))\,{}_a^b\Pi_c^b\,g(x(\hat{\rho}_c))\}$$
$$=E_z\{e^{-\lambda\rho_c}))\,{}_a^b\Pi_c^b\,g(x(\hat{\rho}_c))1(x(\tau))\}$$
$$=E_z\Big\{\sum_{n=0}^{\infty}e^{-\lambda\bar{\sigma}_n}f(x(\bar{\tau}_n))\,{}_a^b\Pi_c^b\,g(x(\bar{\tau}_n))I_{\{\bar{\tau}_n<\tau\}}\,{}_a^b\Pi_c^b\,1(x(\bar{\tau}_n))\Big\}$$
$$=\frac{c-a}{b-a}E_z\Big\{\sum_{n=0}^{\infty}e^{-\lambda\bar{\sigma}_n}f(x(\bar{\tau}_n))\,{}_a^b\Pi_c^b\,g(x(\bar{\tau}_n))I_{\{\bar{\tau}_n<\tau\}}\Big\}.$$

Therefore

(12.14)
$$E_z(e^{-\lambda\hat{\rho}_c}f(x(\hat{\rho}_c))g(x(\tau)))$$
$$=|b-a|E_z\Big(e^{-\lambda\hat{\rho}_c}f(x(\hat{\rho}_c))\frac{{}_a\Pi_c^b g(x(\hat{\rho}_c))}{|c-a|}\Big).$$

If $c\to a$, then $\hat{\rho}_c\to\hat{\rho}$ by (1) in [12.10]. Therefore $f(x(\hat{\rho}_c))\to f(x(\hat{\rho}))$ and $\frac{{}_a\Pi_c^b g(x(\hat{\rho}_c))}{|c-a|}\to Q^{b-a}g(x(\hat{\rho}))$ boundedly as $c\to a$, since we have assumed that f and g are in $C_K(R)$. By the bounded convergence theorem, (12.13) is obtained from (12.14).

For positive a and b with $a\neq b$, set $\hat{\rho}=\hat{\rho}(a, b, w)$, $\rho_n=\rho_n(a, b, w)$ and $\tau_n=\tau_n(a, b, w)$. We define $\hat{\rho}_n=\hat{\rho}_n(a, b, w)$ by

(12.15)
$$\hat{\rho}_n=\rho_n+\hat{\rho}(\theta_{\rho_n}w) \qquad (n=0, 1, 2, \cdots).$$

For any c in (a, b), set $\bar{\rho}_k=\rho_k(a, c, w)$ and $\bar{\tau}_k=\tau_k(a, c, w)$. We also define

(12.16)
$$\hat{\rho}_{n,c}=\hat{\rho}_n+\sigma_c(\theta_{\hat{\rho}_n}w).$$

Then as a generalization of [12.10], we have:

[12.13]
(1) $\hat{\rho}_{n,c}\downarrow\hat{\rho}_n$ as $c\to a$.
(2) $\bar{\rho}_k<\tau_n<\bar{\rho}_{k+1}$ for some n if and only if $\bar{\rho}_k+\sigma_b(\theta_{\bar{\rho}_k}w)<\bar{\rho}_{k+1}$. In this case, it holds that $\rho_n\leqq\bar{\rho}_k$, $\rho_{n+1}=\bar{\rho}_{k+1}$, $\hat{\rho}_{n,c}=\bar{\tau}_k$ and $\tau_n=\bar{\rho}_k+\sigma_b(\theta_{\bar{\rho}_k}w)=\bar{\tau}_k+\sigma_b(\theta_{\bar{\tau}_k}w)$.

[12.14] PROPOSITION. *For any positive a and b with $a\neq b$, let $\hat{\rho}_n=\hat{\rho}_n(a,b,w)$ and $\tau_n=\tau_n(a, b, w)$ be defined by* (12.15) *and by* (12.1), *respectively. Then for, any positive λ, it holds that:*
(1) *for ϕ, ψ in $B_b(R)$ and z in D*

(12.17)
$$2E_z\Big\{\sum_{n=0}^{\infty}e^{-\lambda\hat{\rho}_n}\phi(x(\hat{\rho}_n))\psi(x(\tau_n))\Big\}$$
$$=E_z\Big\{\int_0^{\infty}e^{-\lambda t}\phi(x(t))Q^{|b-a|}\psi(x(t))dL_a\Big\},$$

and
(2) *for ϕ and ψ in $B_p(R)$*

(12.18)
$$2E_{\tilde{m}}\Big\{\sum_{n=0}^{\infty}e^{-\lambda\hat{\rho}_n}\phi(x(\rho_n))\psi(x(\tau_n))\Big\}$$
$$=\frac{1}{\lambda}\int_0^{2\pi}\phi(x)Q^{|b-a|}\psi(x)m_P(x, a)dx.$$

Proof.
1° The both sides of (12.17) and those of (12.18) consist of integrations

(and sumation) of ϕ and ψ by positive measures and they are finite if $\phi=\psi=1$. Therefore, we may assume that ϕ and ψ are in $C_K(R)$ in (12.17) and in $C_p(R)$ in (12.18), respectively.

2° If (12.17) holds for ϕ and ψ in $C_p(R)$, then, integrating the both sides of (12.7) by $m_P(z)dz$ over $\tilde{D}$, we immediately obtain (12.18) by [12.6].

3° Since by [1.5] and [12.12]

$$\begin{aligned}
&E_z\Big\{\sum_{n=0}^{\infty} e^{-\lambda\hat{\rho}_n}\phi(x(\hat{\rho}_n))\psi(x(\tau_n))\Big\}\\
&=E_z\{\Sigma e^{-\lambda\rho_n}E_{z(\rho_p)}(e^{-\lambda\rho}\phi(x(\hat{\rho}))\psi(x(\tau))\}\\
&=|b-a|E_z\{\Sigma e^{-\lambda\rho_n}E_{z(\rho_n)}(e^{-\lambda\hat{\rho}}\phi(x(\hat{\rho}))Q^{|b-a|}\psi(x(\hat{\rho})))\}\\
&=|b-a|E_z\{\Sigma e^{-\lambda\hat{\rho}_n}\phi(x(\hat{\rho}_n))Q^{|b-a|}\psi(x(\hat{\rho}_n))\}\,.
\end{aligned}$$

If follows from 1°, 2° and 3°, that, in order to prove (12.17), it is sufficient to show

$$\begin{aligned}
(12.19)\qquad &2|b-a|E_z\Big\{\sum_{n=0}^{\infty} e^{-\lambda\hat{\rho}_n}\phi(x(\hat{\rho}_n))\Big\}\\
&=E_z\Big\{\int_0^{\infty} e^{-\lambda t}\phi(x(t))dL^a\Big\}
\end{aligned}$$

for ϕ which is bounded and uniformly continuous.

4° For any c in (a, b), let $\rho_n=\rho_n(a, b, w)$, $\bar{\rho}_k=\rho_k(a, c, w)$ and $\bar{\tau}_k=\tau_k(a, c, w)$ be defined by (12.1) and $\hat{\rho}_{n,c}$ be defined by (12.16). Then by (2) in [12.13]

$$\begin{aligned}
\sum_{n=0}^{\infty} e^{-\lambda\hat{\rho}_{n,c}}\phi(x(\hat{\rho}_{n,c}))&=\sum_{k=0}^{\infty} e^{-\lambda\bar{\tau}_k}\phi(x(\bar{\tau}_k))I_{(\bar{\rho}_k+\sigma_b(\theta_{\bar{\rho}_k}w)<\bar{\rho}_{k+1})}\\
&=\sum_{k=0}^{\infty} e^{-\lambda\bar{\tau}_k}\phi(x(\bar{\tau}_k))I_{(\bar{\tau}_k+\sigma_k(\theta_{\bar{\tau}_k}w)<\bar{\rho}_{k+1})}\,.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(12.20)\qquad &E_z\Big(\sum_{n=0}^{\infty} e^{-\lambda\rho_{n,c}}\phi(x(\hat{\rho}_{n,c}))\Big)\\
&=E_z\Big\{\sum_{k=0}^{\infty} e^{-\lambda\bar{\tau}_k}\phi(x(\bar{\tau}_k))P_{z(\bar{\tau}_k)}(\sigma_b<\sigma_a)\Big\}\\
&=E_z\Big(\sum_{k=0}^{\infty} e^{-\lambda\bar{\tau}_k}\phi(x(\bar{\tau}_k))\Big)\frac{c-a}{b-a}\,.
\end{aligned}$$

By theorem [12.8], the right side of (12.20) converges to

$$\frac{1}{2|b-a|}E_z\Big(\int_0^{\infty} e^{-\lambda t}\phi(x(t))dL_a\Big)\qquad \text{as } c\to a.$$

The left side of (12.20) converges to

$$E_z\Big(\sum_{n=0}^{\infty} e^{-\lambda \hat{\rho}_n}\phi(x(\hat{\rho}_n))\Big) \qquad \text{as } c\to a\,,$$

since $e^{-\lambda \hat{\rho}_{n,c}}\phi(x(\hat{\rho}_{n,c}))\to e^{-\lambda\hat{\rho}_n}\phi(x(\hat{\rho}_n))$ by (1) in [12.13], $|e^{-\lambda\hat{\rho}_{n,c}}\phi(x(\hat{\rho}_{n,c}))|\leqq e^{-\lambda\rho_n}\|\phi\|$ and $E_z\Big(\sum_{n=0}^{\infty} e^{-\lambda\rho_n}\Big)<\infty$ by [12.7]. Therefore (12.20) is proved.

§ 13. A sufficient condition for a process belonging to $\mathcal{P}_c$.

For ρ in $M(R)$, we shall write

$$\rho\in M_i(R) \tag{13.1}$$

if and only if $\rho(U)>0$ for any open set U in R. Set

$$\delta(\rho,\,\varepsilon)=\inf_x \rho((x-\varepsilon,\,x+\varepsilon))\,. \tag{13.2}$$

[13.1] *Remark.* In [11.9], we have seen that, if ρ is in $M_{p,N}(R)$, then ρ is in $M_i(R)$ if and only if $\delta(\rho,\,\varepsilon)>0$ for any positive ε.

[13.2] For ρ in $M_{p,N}(R)$, set $v(z)=\int\pi^y(\xi-x)\rho(d\xi)$. Then $\delta(v(x,\,y)dx,\,\varepsilon)\geqq\delta(\rho,\,\varepsilon)$ holds for any positive ε.

Proof.

$$\begin{aligned}\int_{x-\varepsilon}^{x+\varepsilon} v(t,\,y)dt &= \frac{1}{\pi}\int_{x-\varepsilon}^{x+\varepsilon}dt\int\frac{y\rho(d\xi)}{y^2+(\xi-t^2)}\\ &=\frac{1}{\pi}\int\frac{yd\eta}{y^2+\eta^2}\int_{x-\eta-\varepsilon}^{x-\eta+\varepsilon}\rho(d\xi)\\ &\geqq\delta(\rho,\,\varepsilon)\,.\end{aligned}$$

In this section, we shall fix a process P in $\mathcal{P}$ which satisfies [M] and [V], and $B_P=\{\sigma_P,\,\mu_P,\,k_P,\,p_P\}$, s_P, m_P, u_P, U_P etc. are as defined in chapter III. As a corollary to [13.2], we immediately have:

[13.3]

$$u_P(x+\varepsilon,\,y)-u_P(x-\varepsilon,\,y)=\delta(s_P(x,\,y)dx,\,\varepsilon)\geqq\delta(\sigma_P,\,\varepsilon)\,.$$

[13.4] For any a, b, α and β with $0<b<a$, $0<\beta$ and $0<\alpha\leqq\pi$,

$$H_b^a(x,\,U_{2(\alpha+\beta)}(x)^c)\leqq\frac{8ap_P(a)}{\delta(\mu_P,\,\alpha)\delta(\sigma_P,\,\beta)^2}\,, \tag{13.3}$$

where $p_P(a)=B_P(u_P(\cdot,\,a),\,u_P(\cdot,\,a))$ and $U_\delta(x)=\{\xi\colon|\xi-x|\leqq\delta\}$ in R.

Proof. By (8.7) in [8.5], for any $b<a$

$$B_P(x, d\xi)=(P^{a-b}+Q^{a-b}H_b^a)(x, d\xi).$$

Noting $\phi(x)=\int Q^{a-b}H_b^a(x, d\xi)(u_P(\xi, a)-u_P(x, a))$ is in C_P, we have by [13.3],

$$2p_P(a)\geqq\int_x^{x+2\alpha} m_P(t, a)dt\int_x^{\infty} Q^{a-b}(t, d\eta)\int_{x+2\alpha+2\beta}^{\infty} H_b^a(\eta, d\xi)(u_P(\xi, a)-u_P(t, a))^2$$

$$\geqq\int_x^{x+2\alpha} m_P(t, a)dt\int_x^{\infty} Q^{a-b}(t, d\eta)H_b^a(\eta, [x+2\alpha+2\beta, \infty))\delta(\sigma_P, \beta)^2.$$

We have $H_b^a(\eta, [x+2\alpha+2\beta, \infty))\geqq H_b^a(x, [x+2\alpha+2\beta, \infty))$ if $x\leqq\eta$ by [M] (See also [9.2]), and for $x\leqq t$

$$\int_x^{\infty} Q^{a-b}(t, d\eta)\geqq\int_x^{\infty} q^{a-b}(\eta)d\eta=\frac{1}{2(a-b)}.$$

Using [13.2]

$$2p_P(a)\geqq\frac{\delta(\mu_P, \alpha)\delta(\sigma_P, \beta)^2}{2a}H_b^a(x, [x+2\alpha+2\beta, \infty)).$$

In a similar way, we can show that

$$2p_P(a)\geqq\frac{\delta(\mu_P, \alpha)\delta(\sigma_P, \beta)^2}{2a}H_b^a(x, (-\infty, x-2\alpha-2\beta]).$$

Therefore (13.3) is proved.

By (3) in [10.15] $p_P(a)$ decreases as a decreases. Hence as a corollary to [13.4], the following holds.

[13.5] For positive a and ε, set

$$C_1(a, \varepsilon)=\sup_{x, b; b<a} H_b^a(x, U_\varepsilon(x)^c).$$

If σ_P and μ_P are in $M_i(R)$, then $\lim_{a\to 0} C_1(a, \varepsilon)=0$.

In the following, σ_a $(a>0)$ denotes the hitting time of ∂_a. For $b>0$, $\xi\in R$ and $\varepsilon>0$, set

(13.4) $$D(\xi, b, \varepsilon)=\{z=(x, y);\ y\geqq b \text{ and } |x-\xi|\geqq 4\varepsilon\}$$

and let $\tau(\xi)=\tau(\xi, b, \varepsilon, w)$ be the hitting time of $D(\xi, b, \varepsilon)$.

[13.6] For positive a and ε, set

$$C_2(a, \varepsilon)=\sup_{b; b<a}\int_0^{2\pi} m(x, 2a)dx\int Q^a(x, d\xi)P_{(\xi, a)}(\tau(\xi, b, \varepsilon)\leqq\sigma_{2a}).$$

If P satisfies $[L]$ and σ_P and μ_P are in $M_i(R)$, then

$$\lim_{a\to 0} C_2(a, \varepsilon)=0.$$

Proof. Set $\tau=\tau(\xi, b, \varepsilon)$ and $\sigma=\sigma_{2a}$. Take a_0 so small that $C_1(2a, 2\varepsilon)<1/2$ for $a\leqq a_0$. Since $|x(\sigma)-\xi|\geqq 2\varepsilon$ if both $\tau\leqq\sigma$ and $|x(\sigma)-x(\tau)|<2\varepsilon$ hold, by [1.5]

$$\begin{aligned}P_{(\xi,a)}(\tau\leqq\sigma)&\leqq P_{(\xi,a)}(\tau\leqq\sigma,\ |x(\sigma)-x(\tau)|\geqq 2\varepsilon)\\&\quad+P_{(\xi,a)}(\tau\leqq\sigma,\ |x(\sigma)-x(\tau)|<2\varepsilon)\\&\leqq E_{(\xi,a)}\{\tau\leqq\sigma,\ H^{2a}_{y(\tau)}(x(\tau), U_{2\varepsilon}(x(\tau))^c)\}\\&\quad+P_{(\xi,a)}(|x(\sigma)-\xi|\geqq 2\varepsilon)\\&\leqq C_1(2a, 2\varepsilon)P_{(\xi,a)}(\tau\leqq\sigma)+H^{2a}_a(\xi, U_{2\varepsilon}(\xi)^c).\end{aligned}$$

Therefore, for $a\leqq a_0$

$$P_{(\xi,a)}(\tau\leqq\sigma)\leqq 2H^{2a}_a(\xi, U_{2\varepsilon}(\xi)^c).$$

Now

$$\begin{aligned}&\int_0^{2\pi} m(x, 2a)dx\int Q^a(x, d\xi)P_{(\xi,a)}(\tau\leqq\sigma)\\&\leqq 2\int_0^{2\pi} m(x, 2a)dx\int Q^a(x, d\xi)\int_{|\eta-\xi|\geqq 2\varepsilon} H^{2a}_a(\xi, d\eta)\\&\leqq 2(I_1(a)+I_2(a)),\end{aligned}$$

where

$$I_1(a)=\int_0^{2\pi} m(x, 2a)dx\int_{|\xi-x|\geqq\varepsilon} Q^a(x, d\xi)$$

and

$$I_2(a)=\int_0^{2\pi} m(x, 2a)dx\int Q^a(x, d\xi)\int_{|\eta-x|\geqq\varepsilon} H^a(\xi, d\eta).$$

$$I_1(a)=2\pi\int_{|\xi|\geqq\varepsilon} q^a(\xi)d\xi=\frac{4\pi}{a}\left(1-\tanh\frac{\pi\varepsilon}{2a}\right)$$

and $\lim I_1(a)=0$. Moreover, by (8.7) in [8.5] $B^{2a}_P(x, d\eta)\geqq Q^a H^{2a}_a(x, d\eta)$ and

$$\begin{aligned}I_2(a)&\leqq\int_0^{2\pi} m(x, 2a)\int_{|\xi-x|\geqq\varepsilon} B^{2a}_P(x, d\eta)\\&\leqq \inf_{|\eta-x|\geqq\varepsilon}\frac{1}{(u_P(\xi, 2a)-u|x, 2a))^2}B^a_P(u\,;\varepsilon)\\&\leqq\frac{1}{\delta(\sigma_P, \varepsilon/2)^2}B^a_P(u\,;\varepsilon),\end{aligned}$$

where $B_P^a(u\,;\,\varepsilon)$ is given in [11.4] and the condition [L] implies that $\lim_{a\to 0} B_P^a(u\,;\,\varepsilon) =0$. Thus [13.6] is proved.

For positive a, let $\rho_n=\rho_n(2a,\ a,\ w)$ and $\tau_n=\tau_n(2a,\ a,\ w)$ be defined as in (12.1) ($n=0,\ 1,\ 2,\ \cdots$). For any b with $0<b<a$ and any positive ε, let $\tilde{\tau}(\xi)=\tau(\xi,\ b,\ \varepsilon,\ w)$ be defined as in (13.4). Set

$$\tilde{\tau}_n=\tau_n+\tilde{\tau}(x(\tau_n),\ \theta_{\tau_n}w)) \qquad (n=0,\ 1,\ 2,\ \cdots) \tag{13.5}$$

and for positive T

$$\begin{aligned}\mathfrak{U}(a,\ b,\ \varepsilon,\ T)=\{w: &\text{ there exist } \tau_n \text{ with } \tau_n\leqq T \text{ and } s \text{ in } [\tau_n,\ \rho_{n+1}] \\ &\text{such that both } y_s\geqq b \text{ and } |x(s)-x(\tau_n)|\geqq 4\varepsilon \text{ hold}\} \\ =\{w: &\text{ there exists } n \text{ such that } \tau_n\leqq T \text{ and } \tilde{\tau}_n\leqq\rho_{n+1} \text{ hold.}\}.\end{aligned} \tag{13.6}$$

[13.7] Set

$$C_3(a,\ \varepsilon)=\sup_{T,\,b;\,b<a}\frac{1}{T}P_{\tilde{m}}(\mathfrak{U}(a,\ b,\ \varepsilon,\ T)),$$

where $P_{\tilde{m}}(\cdot)=\int_{\tilde{D}}P_z(\cdot)m_P(z)dz$ and $\tilde{D}=\{z\in D\,;\,0\leqq x<2\pi\}$. If P satisfies [L] and σ_P and μ_P are in $M_i(R)$, then

$$\lim_{a\to 0} C_3(a,\ \varepsilon)=0\,.$$

Proof. For positive λ

$$\begin{aligned}P_{\tilde{m}}(\mathfrak{U}(a,\ b,\ \varepsilon,\ T))&\leqq\sum_{n=0}^{\infty}P_{\tilde{m}}(\tilde{\tau}_n\leqq\rho_{n+1},\ \tau_n\leqq T) \\ &\leqq e^{\lambda T}E_{\tilde{m}}(\Sigma e^{-\lambda\tau_n}I_{(\tilde{\tau}_n<\rho_{p+1})}) \\ &=e^{\lambda T}E_{\tilde{m}}\{\Sigma e^{-\lambda\tau_n}P_{z(\tau_n)}(\tilde{\tau}(x(0))<\sigma_{2a})\}.\end{aligned}$$

Let $\hat{\rho}=\hat{\rho}(2a,\ a,\ w)$ be the last exist time to ∂_{2a} before reaching ∂_a defined in [12.9]. Set $\hat{\rho}_n=\rho_n+\hat{\rho}(\theta_{\rho_n}w)$ and $\phi(x)=P_{(x,\,a)}(\tilde{\tau}(x)<\sigma_{2a})$. Since $\hat{\rho}_n<\tau_n$ and ϕ is in $B_p(R)$ by (p.5), we have, by (12.18) in [12.14],

$$\begin{aligned}&P_{\tilde{m}}(\mathfrak{U}(a,\ b,\ \varepsilon,\ T))\leqq e^{\lambda T}E_{\tilde{m}}(\Sigma e^{-\lambda\hat{\rho}_n}\phi(x(\tau_n))) \\ &=\frac{e^{\lambda T}}{2\lambda}\int_0^{2\pi}Q^a\phi(x)m_P(x,\ 2a)dx \\ &=\frac{e^{\lambda T}}{2\lambda}\int_0^{2\pi}m_P(x,\ 2a)\,dx\int Q^a(x,\ d\xi)P_{(\xi,\,a)}(\tilde{\tau}(\xi)<\sigma_{2a}) \\ &\leqq\frac{e^{\lambda T}}{2\lambda}C_2(a,\ \varepsilon),\end{aligned}$$

where $C_2(a, \varepsilon)$ is defined as in [13.6]. Put $\lambda=1/T$. Then

$$\frac{1}{T}P_{\tilde{m}}(\mathfrak{U}(a, b, \varepsilon, T))\leqq\frac{e}{2}C_2(a, \varepsilon).$$

[13.7] is a consequence of [13.6].

[13.8] PROPOSITION. *If P satisfies [M], [V] and [L], and μ_P and σ_P are iu $M_i(R)$, then P is in $\mathfrak{P}_c$.*

Proof. 1° By [13.7], we can choose a positive sequence $\{a_n\}$ such that $a_{n+1}<a_n$, $\Sigma a_n<\infty$ and $\Sigma C_3(a_n, 1/2^n)<\infty$. Then, for fixed T

$$\sum_{n=0}^{\infty}P_{\tilde{m}}\Big(\mathfrak{U}\Big(a_n, a_{n+1}, \frac{1}{2^n}, T\Big)\Big)\leqq\sum_{n=0}^{\infty}TC_3\Big(a_n, \frac{1}{2^n}\Big)<\infty .$$

Set $\mathfrak{U}(T)=\overline{\lim_{n\to\infty}}\,\mathfrak{U}(a_n, a_{n+1}, 1/2^n, T)$. Then, by Borel-Cantelli's theorem for σ-finite measure $P_{\tilde{m}}$, we have $P_{\tilde{m}}(\mathfrak{U}(T))=0$. Set

$$\mathfrak{U}=\bigcup_{N=1}^{\infty}\mathfrak{U}(N),\quad \mathfrak{U}(T)\uparrow\mathfrak{U}\ (T\uparrow\infty),\quad\text{and}\quad P_{\tilde{m}}(\mathfrak{U})=0 .$$

2° If $z(0, w)\in D^{[a,\infty)}$ and $0<b<a$, then $w\in\mathfrak{U}(N)$ implies $\theta_{\sigma_b}w\in\mathfrak{U}(N)$. For, $\sigma_b<\sigma_{2a_n}=\rho_0(2a_n, a_n)$ if $2a_n<b$. Conversely, if $\theta_{\sigma_b}w\in\mathfrak{U}(N)$ and $M>\sigma_b(w)$, then $w\in\mathfrak{U}(N+M)$. Therefore, $w\in\mathfrak{U}$ if and only if $\theta_{\sigma_b}w\in\mathfrak{U}$ for w with $z(0, w)\in D^{(b,\infty)}$. $P_z(\mathfrak{U})$ is harmonic and therefore continuous in D. Noting that $P_z(\mathfrak{U})$ is in $C_p(D)$, by 1° we have $P_z(\mathfrak{U})=0$ for any z in D.

3° Set $\rho_k(n)=\rho_k(2a_n, a_n, w)$, $\tau_k(n)=\tau_k(2a_n, a_n, w)$ and $W_n=\{w: z(0, w)\in D^{(2a_n,\infty)}\}$ $(k=0, 1, 2, \cdots, n=1, 2, \cdots)$. Define

$$\begin{aligned}\tilde{z}_n(t, w)=&\frac{(\rho_{k+1}(n)-t)z(\tau_k(n))+(t-\tau_k(n))z(\rho_{k+1}(n))}{\rho_{k+1}(n)-\tau_k(n)}\\ &\qquad\text{if } t\in(\tau_k(n), \rho_{k+1}(n))\qquad(k=0, 1, 2, \cdots)\\ =&z(t, w)\qquad\text{if otherwise.}\end{aligned}$$

Then, for $w\in W_n$, $\tilde{z}_n(t, w)$ is a continuous mapping of t in $[0, \infty)$ into $D^{[a_n,\infty)}$.

4° Let n_0 and N be any fixed positive integers. For any fixed w in $W_{n_0}\cap\mathfrak{U}(N)^c$, we shall show that $\tilde{z}_n(t, w)$ converges uniformly in $t\in[0, N]$ by the topology of $\bar{D}$.

Proof of 4°. For a fixed $w\in W_{n_0}\cap\mathfrak{U}(N)^c$, there exists a positive integer $n_1=n_1(w)\geqq n_0$ such that $w\notin\mathfrak{U}(a_n, a_{n+1}, 1/2^n, N)$ for $n\geqq n_1$.

Take any $n\geqq n_1$.

(i) If $t\notin\bigcup_k(\tau_k(n), \rho_{k+1}(n))$, then $z(t, w)\in D^{[a_n,\infty)}$ and $t\notin\bigcup_l(\tau_l(n+1), \rho_{l+1}(n+1))$. Therefore

$$\tilde{z}_n(t)=z(t)=\tilde{z}_{n+1}(t) .$$

(ii) If $t \leqq N$, $t \in (\tau_k(n), \rho_{k+1}(n))$ for some k and $z(t) \in D^{[a_{n+1}, \infty)}$, then $|x(t)-x(\tau_k(n))| < 4/2^n$, since $w \notin \mathfrak{U}(a_n, a_{n+1}, 1/2^n, N)$. Especially

$$|x(\rho_{k+1}(n)-x(\tau_k(n))| \leqq \frac{4}{2^n} \quad \text{and} \quad |\tilde{x}_n(t)-x(\tau_k(n))| > \frac{4}{2^n}.$$

(iii) If $t \leqq N$, $t \in (\tau_k(n), \rho_{k+1}(n))$ for some k and $t \notin \bigcup_l (\tau_l(n+1), \rho_{l+1}(n+1))$, then $\tilde{z}_{n+1}(t)=z(t) \in D^{[a_{n+1}, \infty)}$. Therefore by (ii) $|\tilde{x}_{n+1}(t)-\tilde{x}_n(t)| < 8/2^n$.

(iv) If $t \leqq N$ and $t \in (\tau_k(n), \rho_{k+1}(n)) \cap (\tau_l(n+1), \rho_{l+1}(n+1))$ for some k and l, then $z \in (\tau_l(n+1))$ and $z(\rho_{l+1}(n+1))$ are in $D^{[a_{n+1}, \infty)}$. Therefore, by (ii) we also have

$$|\tilde{x}_{n+1}(t)-\tilde{x}_n(t)| < \frac{8}{2^n}.$$

(v) If $t \leqq N$ and $t \in (\tau_k(n), \rho_{k+1}(n))$ for some k, then $\tilde{z}_n(t)$ and $\tilde{z}_{n+1}(t)$ are in D^{2a_n} and $|\tilde{y}_{n+1}(t)-\tilde{y}_n(t)| \leqq 2a_n$. In this case, by (iii) and (iv) we have seen $|\tilde{x}_{n+1}(t)-\tilde{x}_n(t)| \leqq 8/2^n$, and therefore $|\tilde{z}_{n+1}(t)-\tilde{z}_n(t)| \leqq 8/2^n+4a_n$.

Since $\Sigma(8/2^n+4a_n) < \infty$, 4° is proved by (i) and (v).

5° Set $W_\infty = \bigcup W_n = \{w; z(0, w) \in D\}$ and $W_0 = W_\infty \cap \mathfrak{U}^c$. Noting 2° and [1.2], we have $P_z(W_0)=1$ for any z in D. Let $w \in W_0$ be given. Then, for any positive integer N, there exists n such that $w \in W_n \cap \mathfrak{U}(N)^c$. Therefore $\tilde{z}_n(t, w)$ converges uniformly in $t \in [0, N]$ for any N. Set $\tilde{z}(t, w) = \lim \tilde{z}_n(t, w)$. Then $\tilde{z}(t, w)$ is a continuous function of $t \in [0, \infty)$ into $\bar{D}$. Define a mapping ϕ from W_0 into $\overline{W}$ by

$$z(t, \phi(w)) = \tilde{z}(t, w) \qquad (0 \leqq t < \infty).$$

Measurability of the mapping ϕ is obvious by definition. Therefore, by proposition [1.11], we can see that P is in $\mathcal{P}_c$. Proposition [13.8] is proved.

§ 14. Necessity of the conditions given in § 13.

In the following, we shall use the identical notation σ_a $(a \geqq 0)$ for the hitting time of ∂_a for paths in W and in $\overline{W}$. Here $\sigma_0(w)$ for w in W denotes the hitting time of ∂. For $0 \leqq a$, b and $a \neq b$,

$$\rho_n(a, b) = \rho_n(a, b, w) \quad \text{or} \quad \rho_n(a, b, \bar{w})$$

$$\tau_n(a, b) = \tau_n(a, b, w) \quad \text{or} \quad \tau_n(a, b, \bar{w})$$

are definen as in (12.1), and

$$\hat{\rho}(a, b) = \hat{\rho}(a, b, w) \quad \text{or} \quad \hat{\rho}(a, b, \bar{w})$$

as in [12.9] also far paths in W or $\overline{W}$.

Note that $\sigma_a = \rho_0(a, b)$ and

$$\tau(a, b) = \tau_0(a, b) = \sigma_a + \sigma_b \cdot \theta_{\sigma_a}.$$

Then if holds that

$$(14.1)\quad \begin{cases} \sigma_a(\bar{w})=\sigma_a(\iota\bar{w}), \quad \rho_n(a, b, \bar{w})=\rho_n(a, b, \iota\bar{w}), \\ \tau_n(a, b, \bar{w})=\tau_n(a, b, \iota\bar{w}) \text{ and } \hat{\rho}(a, b, \bar{w})=\hat{\rho}(a, b, \iota\bar{w}) \end{cases}$$

where ι is the injection defined by (1.6).

[14.1] Let P be in $\mathfrak{P}$ and $\bar{P}$ be in $\bar{\mathfrak{P}}$.

(1) Set

$$W_r=\{w\in W\,;\, z(r, w)\in D \text{ for any rational } r\}$$

and

$$\bar{W}_r=\{w\in\bar{W}\,;\, z(r, \bar{w})\in D \text{ for any rational } r\}.$$

Then $P_z(W_r)=1$ and $\bar{P}_z(\bar{W}_r)=1$ for any z in D.

(2) Let γ be any random time and σ_b^* $(b>0)$ be the hitting time to $D^{[b,\infty)}$. Set $\gamma_b=\gamma+\sigma_b^*\circ\theta_\gamma$. Then $\gamma_b\downarrow\gamma$ as $b\downarrow 0$ a.s. P_z (or a.s. $\bar{P}_z$) for any z in D.

(3) It holds that $\sigma_0\leqq\hat{\rho}(0, b)<\tau(0, b)$ for $b>0$, and

$$\tau(0, b)\downarrow\sigma_0 \quad \text{as } b\downarrow 0 \text{ a.s. } P_z \text{ (or a.s. } \bar{P}_z)$$

for any z in D.

(4) Fix $b>0$. If $\tau(0, b)<\infty$, then there exists $a_1=a_1(b, w)$ or $a_1(b, \bar{w})$ such that $\hat{\rho}(0, b)<\hat{\rho}(a, b)$ for $a\leqq a_1$, and

$$(14.2)\qquad \lim_{a\to 0}\hat{\rho}(a, b)=\hat{\rho}(0, b).$$

Proof. (1) is a consequence of (p.2) in [1.1] (or ($\bar{\text{p}}$.2) in [1.8]). (2) and (3) follow from (1). If $\hat{\rho}(a_n, b)<\hat{\rho}(0, b)$ holds for some sequence $\{a_n\}$ with $a_n\downarrow 0$, then $\sigma_{a_p}\leqq\hat{\rho}(a_n, b)<\tau(a_n, b)\leqq\sigma_0$ and $\sigma_{a_n}\uparrow\sigma_0$, which contradict the continuity of $z(t)$. The first part of (4) is proved. For a with $0<a<\min\{a_1, b\}$, $\sigma_0\leqq\hat{\rho}(0, b)<\hat{\rho}(a, b)<\tau(0, b)$ and $\hat{\rho}(a, b)$ decreases as a decreases. Therefore $z(\lim_{a\to 0}\hat{\rho}(a, b))=\lim_{a\to 0}z(\hat{\rho}(a, b))=\partial$ (or $\in\partial_0$), which implies that (14.2) holds.

In the remainder of this section, we shall fix a process P in $\mathfrak{P}$ which satisfies [V] and [M].

[14.2] Assume $\sigma_P((c_1, c_2))=0$ for some c_1 and c_2 with $c_1<c_2$, and $\phi=H^\alpha f$ for f in $B_b(R)$. Then the boundary function of ϕ on ∂_0 is constant on (c_1, c_2), that is, for $\zeta=(\xi, 0)$ with ξ in (c_1, c_2)

$$(14.3)\qquad \lim_{z\to\zeta}\phi(z)=k\,.$$

Proof. Let $\bar{J}$ be a closed interval contained in (c_1, c_2). Then $s_P(z)=\int\pi^y(\xi-x)\sigma_P(d\xi)\to 0$ as $z\to(\xi, 0)$ uniformly in $\xi\in\bar{J}$. Therefore, by (3) in [9.9], $\phi_x(z)\to 0$ as $z\to(\xi, 0)$ uniformly in $\xi\in J$, and (14.3) is easily proved.

[14.3] PROPOSITION. *If P in $\mathfrak{P}_c$ satisfies [M] and [V], then σ_P is in $M_i(R)$.*

Proof. Since P is in $\mathfrak{P}_c$, $P=\iota\bar{P}$ for some $\bar{P}$ in $\bar{\mathfrak{P}}$. Assuming $\sigma((c_1, c_2))=0$ for some c_1 and c_2 with $c_1<c_2$, we shall show a contradiction.

1° Let J be a fixed non-empty open interval with $\bar{J}\subset(c_1, c_2)$. For any positive a, set $\phi_a(z)=H^a I_J(z)$, where I_J is the indicator of J. Then by [14.2] $\phi_a(z)\to k_a=k_a(J)$ as $z\to(\xi, 0)$ for ξ in (c_1, c_2). Since $0\leqq k_a\leqq 1$, we can choose a sequence $\{a_n\}$ such that $a_n\to 0$ and $k_{a_n}\to k$ as $n\to\infty$. Set $\phi_n=\phi_{a_n}$, $k_n=k_{a_n}$ and $\tau_n=\tau(0, a_n)$.

2° Let K be another non-empty open interval with $\bar{K}\subset(c_1, c_2)$. Then by [1.5], for any m and n with $m<n$, and $z(t)=(x(t), y(t))$,

$$\begin{aligned}(14.4)\qquad \bar{P}_z(x(\tau_n)\in K,\ x(\tau_m)\in J)&=P_z(x(\tau_n)\in K,\ x(\tau_m)\in J)\\ &=E_z(\phi_m(z(\tau_n))I_{\{x(\tau_n)\in K\}})\\ &=\bar{E}_z(\phi_m(z(\tau_n))I_{\{x(\tau_n)\in K\}}).\end{aligned}$$

Set $K=J$ in (14.4). Since $\tau_n\downarrow\sigma_0$ as $n\to\infty$ by (3) in [14.1], we have, for path's in $\bar{W}$,

$$\{x(\sigma_0)\in J\}\subset \varliminf_{m\to\infty}\varliminf_{n\to\infty}\{x(\tau_n)\in J,\ x(\tau_m)\in J\}$$

and

$$k_m I_{\{x(\sigma_0)\in \bar{J}\}}\geqq \varlimsup_{n\to\infty}\phi_m(z(\tau_n))I_{\{x(\tau_n)\in J\}}.$$

Therefore

$$\begin{aligned}\bar{P}_z(x(\sigma_0)\in J)&\leqq \varliminf_{m\to\infty}\varliminf_{n\to\infty}\bar{E}_z\{\phi_m(z(\tau_n))I_{\{x(\tau_n)\in J\}}\}\\ &\leqq \lim_{m\to\infty}k_m\bar{P}_z(x(\sigma_0)\in\bar{J})\\ &=k\bar{P}_z(x(\sigma_0)\in\bar{J}).\end{aligned}$$

By (p.4) in [1.8]

$$\bar{P}_z(z(\sigma_0)\in J)=P_z^{B,2}(z(\sigma_0)\in J)>0$$

and

$$\bar{P}_z(z(\sigma_0)\in\bar{J})=P_z^{B,2}(z(\sigma_0)\in\bar{J})=P_z^{B,2}(z(\sigma_0)\in J).$$

Hence we have $k=1$.

3° Take a non-empty K with $\bar{J}\cap\bar{K}=\emptyset$. Then, for paths in $\bar{W}$

$$\phi=\{x(\sigma_0)\in\bar{J}\cap\bar{K}\}\supset \varlimsup_{m\to\infty}\varlimsup_{n\to\infty}\{x(\tau_n)\in K,\ x(\tau_m)\in J\}$$

and

$$k_m I_{\{x(\sigma_0)\in K\}}\leqq \varliminf_{n\to\infty}\phi_m(z(\tau_n))I_{\{x(\tau_n)\in K\}}.$$

By (14.4), we have

$$0 \geqq k\bar{P}_z(x(\sigma_0) \in K).$$

Since $\bar{P}_z(x(\sigma_0) \in K) = P_z^{B,2}(x(\sigma_0) \in K) > 0$, we have $k=0$, which is a contradiction.

[14.4] PROPOSITION. *If P in $\mathcal{P}_c$ satisfies [V], then μ_P is in $M_i(R)$.*

Proof. Let $P = \iota\bar{P}$ for $\bar{P}$ in $\bar{\mathcal{P}}$. Assume μ_P is not in $M_i(R)$. Then there exist c_1 and c_2 with $0 < c_1 < c_2 < 2\pi$ such that $\mu_P((c_1, c_2)) = 0$. We shall show a contradiction. Take a non-empty open interval J with $\bar{J} \subset (c_1, c_2)$. Set $\tilde{J} = \bigcup_{n=-\infty}^{\infty} (J + 2n\pi)$ and for $0 < a < b$

$$F(a, b, T) = \bar{P}_{\tilde{m}}(\sigma_a \leqq T,\ x(\tau(a, b)) \in J).$$

Then by [12.14] for a fixed positive λ

$$F(a, b, T) \leqq e^{\lambda T} E_{\tilde{m}}\left\{\sum_{n=0}^{\infty} e^{-\lambda \hat{\rho}_n(a,b)} I_{\tilde{J}}(x(\tau_n(a, b)))\right\}$$

$$= \frac{e^{\lambda T}}{2\lambda}\int_0^{2\pi} m_P(x, a) Q^{b-a} I_{\tilde{J}}(x) dx.$$

Since $\sigma_a \uparrow \sigma_0$, $Q^{b-a} I_{\tilde{J}}(x) \to Q^b I_{\tilde{J}}(x)$ uniformly in x and $m_P(x, a)dx \to \mu_P(dx)$ weakly as $a \to 0$. By

$$\tau(a, b) = \tau(0, b) \qquad \text{if } a < b \text{ and } \hat{\rho}(0, b) < \hat{\rho}(a, b),$$

and by (4) in [14.1], we have

$$F(b, T) = \bar{P}_{\tilde{m}}(\sigma_0 \leqq T,\ x(\tau(0, b)) \in J)$$

$$\leqq \lim_{a \to 0} F(a, b, T)$$

$$= \frac{e^{\lambda T}}{2\lambda}\int_0^{2\pi} Q^b I_{\tilde{J}}(x) \mu_P(dx)$$

$$\leqq \frac{\pi e^{\lambda T}}{\lambda} Q^b(0, U_\varepsilon(0)^c),$$

where $\varepsilon = \inf\{|x - \xi| : x \in \tilde{J},\ \xi \in (0, 2\pi) - (c_1, c_2\}\}$. Therefore, by (2) in [14.1],

$$\bar{P}_{\tilde{m}}(\sigma_0 \leqq T,\ x(\sigma_0) \in J) \leqq \varliminf_{b \to 0} F(b, T) = 0.$$

On the other hand, for $T > 0$

$$\bar{P}_{\tilde{m}}(\sigma_0 \leqq T,\ x(\sigma_0) \in J) = P_{\tilde{m}}^{B,2}(\sigma_0 \leqq T,\ x(\sigma_0) \in J) > 0,$$

which is a contradiction.

[14.5] Let f be in $B_b(R)$ and a be a positive number. Then for a fixed positive ε

$$\lim_{b\uparrow a}\int_{|\xi-x|\geqq\varepsilon}\frac{H_b^a(x, d\xi)f(\xi)}{a-b}=\int_{|\xi-x|\geqq\varepsilon}B_P^a(x, d\xi)f(\xi)$$

where the left side converges boundedly in x.

Proof. By ($\bar{\text{h}}$.3) in [2.2] and (8.7) in [8.5], we can easily see for a fixed c with $0<c<b<a$

$$\frac{1}{a-b}\int_{|\xi-x|\geqq\varepsilon}H_b^a(x, d\xi)f(\xi)$$

$$=\frac{1}{a-b}\left\{\int_{|\xi-x|\geqq\varepsilon}{}^{a-c}\pi^{a-b}(\xi)f(x+\xi)d\xi\right.$$

$$\left.+\int_{|\xi-x|\geqq\varepsilon}{}^{a-c}\pi^{b-c}(\eta)d\eta\int H_c^a(\eta, d\xi)f(\xi)\right\}$$

is bounded in b and x for $b\in[a+c/2, a)$, and converges to $\int_{|\xi-x|\geqq\varepsilon}B_P^a(x, d\xi)f(\xi)$ as $b\uparrow a$.

For any positive ε, set

(14.5) $$\gamma_\varepsilon(w)=\inf\{t: |x(t)-x(0)|>\varepsilon \text{ and } z(t)\in D\}$$

for w in W with $z(0, w)\in D$, and

(14.6) $$\gamma_\varepsilon(\bar{w})=\inf\{t: |x(t)-x(0)|>\varepsilon\}$$

for $\bar{w}$ in $\bar{W}$. Then, by (1) in [14.1] it is easily seen that for any z in D

(14.7) $$\gamma_\varepsilon(\bar{w})=\gamma_\varepsilon(\iota\bar{w}) \qquad \text{a.s. } \bar{P}_z.$$

[14.6] Let P in $\mathfrak{P}_c$ satisfy [V] and [M]. Set $\gamma=\gamma_{\alpha+5\varepsilon}$ for positive α and ε with $0<\varepsilon\leqq\pi$. Then, there exists a positive constant $a_0=a_0(\varepsilon, P)$ such that

(14.8) $$\int_0^{2\pi}m(x, a)\varlimsup_{y\uparrow a}\frac{P_z(\gamma<\sigma_a)}{a-y}dx\leqq\frac{2p_P(a)}{\delta(\sigma_P, \alpha)^2}$$

for any $a\leqq a_0$.

Proof. By proposition [14.3] and [14.4], we have seen that σ_P and μ_P are in $M_i(R)$ and therefore $\delta(\sigma_P, \varepsilon)$, $\delta(\mu_P, \varepsilon)$ and $\delta(\sigma_P, \alpha)$ are positive. Set

$$a_0=\text{Min}\left\{\frac{\delta(\sigma_P, \varepsilon)^2\delta(\mu_P, \varepsilon)}{16p_P(1)}, 1\right\}$$

and for $0<b<a$ $\gamma_b=\gamma+\sigma_b^*\circ\theta_\gamma$, where σ_b^* is the hitting time of $D^{[b,\infty)}$. Then

$$P_z(\gamma_b<\sigma_a)\leqq J_1+J_2+J_3$$

where

$$J_1=P_z(\gamma_b<\sigma_{a'},\ |x(\sigma_a)-x|<\alpha,\ |x(\gamma_b)-x|\geqq\alpha+4\varepsilon),$$

$$J_2=P_z(\gamma_b<\sigma_{a'},\ |x(\sigma_a)-x|<\alpha,\ |x(\gamma_b)-x|<\alpha+4\varepsilon),$$

$$J_3=P_z(\gamma_b<\sigma_a,\ |x(\sigma_a)-x|\geqq\alpha).$$

Since $p_P(a)\leqq p_P(1)$ if $a\leqq a_0\leqq 1$ by (2) in [10.15], for $a\leqq a_0$ by [1.5] and [13.4]

$$\begin{aligned} J_1&\leqq P_z(\gamma_b<\sigma_a,\ |x(\sigma_a)-x(\gamma_b)|\geqq 4\varepsilon)\\ &=E_z(H^a_{y(\gamma_b)}(x(\gamma_b),\ U_{4\varepsilon}(x(\gamma_b))^c)I_{\{\gamma_b<\sigma_a\}})\\ &\leqq\frac{8a\,p_P(a)}{\delta(\sigma_P,\ \varepsilon)^2\delta(\mu_P,\ \varepsilon)}P_z(\gamma_b<\sigma_a)\leqq\frac{1}{2}P_z(\gamma_b<\sigma_a),\end{aligned}$$

and

$$J_3\leqq P_z(|x(\sigma_a)-x|\geqq\alpha)=H^a(z,\ U_\alpha(x)^c).$$

Therefore,

$$P_z(\gamma_b<\sigma_a)\leqq 2J_2+2H^a(z,\ U_\alpha(x)^c).$$

Since $\gamma_b\downarrow\gamma$ as $b\downarrow 0$ by (2) in [14.1] and

$$|x(\gamma,\ \bar{w})-x(0,\ \bar{w})|=\alpha+5\varepsilon \qquad \text{if } \gamma(\bar{w})<\infty \text{ for } \bar{w} \text{ in } \overline{W},$$

$$J_2\leqq\bar{P}_z(|x(\gamma_b)-x|<\alpha+4\varepsilon,\ \gamma_b<\infty)$$

and

$$\varlimsup_{b\downarrow 0} J_2\leqq\bar{P}_z(|x(\gamma)-x|\leqq\alpha+4\varepsilon,\ \gamma<\infty)=0,$$

where $P=\iota\bar{P}$ for $\bar{P}$ in $\bar{\mathcal{P}}$. Therefore we have for $a\leqq a_0$

$$P_z(\gamma<\sigma_a)=\lim_{b\to 0}P_z(\gamma_b<\sigma_a)\leqq 2H^a(z,\ U_\alpha(x)^c).$$

and by [14.5]

$$\begin{aligned}\int_0^{2\pi}m_P(x,\ a)\varlimsup_{y\uparrow a}\frac{P_z(\gamma<\sigma_a)}{a-y}dx&\leqq 2\int_0^{2\pi}m_P(x,\ a)\lim_{y\uparrow a}\frac{H^a(z,\ U_\alpha(x)^c)}{a-y}dx\\ &\leqq 2\int_0^{2\pi}m_P(x,\ a)B^a_P(x,\ U_\alpha(x)^c)dx\end{aligned}$$

Since $|u_P(\xi,\ a)-u_P(x,\ a)|\geqq\delta(\sigma_P,\ \alpha)$ if $|\xi-x|\geqq\alpha$ by [13.3], we have for $a\leqq a_0$

$$\begin{aligned}&\int_0^{2\pi}m_P(x,\ a)\varlimsup_{y\uparrow a}\frac{P_z(\gamma<\sigma_a)}{a-y}dx\\ &\leqq\frac{2}{\delta(\sigma_P,\ \alpha)^2}\int_0^{2\pi}m_P(x,\ a)\int B^a_P(x,\ a)(u_P(\xi,\ a)-u_P(x,\ a))^2\\ &=\frac{2p_P(a)}{\delta(\sigma_P,\ \alpha)^2},\end{aligned}$$

which completes the proof.

[14.7] Let P in $\mathcal{P}_c$ satisfy [V] and [M]. Then for any positive α and ε with $0<\varepsilon\leqq\pi$,

$$\int_0^{2\pi} m_P(x, a)B_P^a(x, U_{3\alpha+8\varepsilon}(x)^c)dx \leqq \frac{16a p_P(a)^2}{\delta(\mu_P, \varepsilon)\delta(\sigma_P, \alpha)^4} \tag{14.9}$$

for $a\leqq a_0$, where a_0 is the constant given in [14.6] and $U_\varepsilon(x)=\{\xi\in R;\ |\xi-x|<\delta\}$.

Proof. Let $P=\iota\bar{P}$ for $\bar{P}$ in $\mathcal{P}$. Set $\gamma=\gamma_{\alpha+5\varepsilon}$ and $\gamma_b=\gamma+\sigma_b^*\circ\theta_\gamma$ where $\gamma_{\alpha+5\varepsilon}$ is defined by (14.5) and σ_b is the hitting time to $D^{[b,\infty)}$ ($b>0$). Since $|x(\gamma, \bar{w})-x(0, \bar{w})|=\alpha+5\varepsilon$ if $\gamma(\bar{w})<\infty$ for $\bar{w}$ in $\bar{W}$, by [13.4]

$$\begin{aligned}
H^a(z, U_{3\alpha+8\varepsilon}(x)^c &= \bar{P}_z(|x(\sigma_a)-x|\geqq 3\alpha+8\varepsilon) \\
&\leqq \bar{P}_z(\gamma<\sigma_a, |x(\sigma_a)-x(\gamma)|\geqq 2\alpha+3\varepsilon) \\
&\leqq \varliminf_{b\to 0} \bar{P}_z(\gamma_b<\sigma_a,\ |x(\sigma_a)-x(\gamma_b)|\geqq 2(\alpha+\varepsilon)) \\
&= \varliminf_{b\to 0} \bar{E}_z\{H^a_{y(\gamma_b)}(x(\gamma_b), U_{2(\alpha+\varepsilon)}(x(\gamma_b))^c)I_{(\gamma_b<\sigma_a)}\} \\
&\leqq \frac{8a p_P(a)}{\delta(\mu_P, \varepsilon)\delta(\sigma_P, \alpha)^2} \lim_{b\to 0} \bar{P}_z(\gamma_b<\sigma_a) \\
&= \frac{8a p_P(a)}{\delta(\mu_P, \varepsilon)\delta(\sigma_P, \alpha)^2} \bar{P}_z(\gamma<\sigma_a).
\end{aligned}$$

Therefore, by [14.5] and [14.6], for $a\leqq a_0$

$$\begin{aligned}
&\int_0^{2\pi} m_P(x, a)B_P^a(x, U_{3\alpha+8\varepsilon}(x)^c dx \\
&= \int_0^{2\pi} m_P(x, a)\lim_{y\uparrow a} \frac{H^a(z, U_{3\alpha+8\varepsilon}(x)^c)}{a-y} dx \\
&\leqq \frac{16a p_P(a)^2}{\delta(\mu_P, \varepsilon)\delta(\sigma_P, \alpha)^4},
\end{aligned}$$

which completes the proof.

[14.8] PROPOSITION. *Let P in $\mathcal{P}_c$ satisfy* [V] *and* [M], *then P satisfies* [L^*] *and therefore* [L].

Proof. By [11.10] it is sufficient to prove [L^*]. Take $\varepsilon=\pi$ and $\alpha=N\pi$ in (14.9). Then $\delta(\mu_P, \pi)=2\pi$ and $\delta(\sigma_P, N\pi)=2N\pi$ and

$$\int_0^{2\pi} m_P(x, a)B_P^a(x, U_{(8+3N)\pi}(x)^c)dx \leqq \frac{a p_P(a)^2}{2N^4\pi^5}$$

for $a \leqq a_0$ with positive a_0. Therefore

$$\int_0^{2\pi} m_P(x,\ a)dx \int_{|\xi-x|\geqq 11\pi} B_P^a(x,\ d\xi) \qquad (\xi-x)^2$$

$$\leqq \frac{a p_P(a)^2}{2\pi^5} \sum_{N=1}^{\infty} \frac{(11\pi+8N\pi)^2}{N^4}.$$

Take $\alpha=\varepsilon$ and $\delta=\varepsilon$ in (14.9), for $a \leqq a_0(\varepsilon)$

$$\int_0^{2\pi} m_P(x,\ a)dx \int_{11\pi>|\xi-x|\geqq 11\varepsilon} B_P^a(x,\ d\xi)(\xi-x)^2$$

$$\leqq \frac{16(11\pi)^2 a p_P(a)^2}{\delta(\mu_P,\ \varepsilon)\delta(\sigma_P,\ \varepsilon)^4}.$$

Therefore, for a fixed positive ε and $a \leqq a_0(\varepsilon)$

$$B_P^a(11\varepsilon)=\int_0^{2\pi} m_P(x,\ a)dx \int_{|\xi-x|\geqq 11\varepsilon} B_P^a(x,\ d\xi)(\xi-x)^2 \leqq K a p_P(a)^2.$$

Since $p_P(a)$ decreases as a decreases by (3) in [10.15], we have

$$\lim_{a\to 0} B_P^a(11\varepsilon)=0.$$

[14.8] is proved, for ε is arbitrary.

From propositions [13.8], [14.3], [14.4] and [14.8], we have the following theorem.

[14.9] THEOREM. *Let P in $\mathfrak{P}$ satisfy [V] and [M]. Then P is in $\mathfrak{P}_c$ if and only if P satisfies [L] and μ_P and σ_P are in $M_i(R)$.*

Combining theorem [14.9] with theorem [11.7], we also have:

[14.10] COROLLARY. *Let P in $\mathfrak{P}_c$ satisfy [V] and [M], then P is B_P-process with μ_P and σ_P in $M_i(R)$.*

§ 15. Processes which satisfy the condition [H.C].

[15.1] Let P in $\mathfrak{P}$ satisfy [V] and [M]. Set

$$M(a,\ b)=\sup_x \int H_b^a(x,\ d\xi)(\xi-x)^2,$$

$$m(a,\ b)=\inf_x \int H_b^a(x,\ d\xi)(\xi-x)^2$$

for $0<b<a$. Then

$$M(a, b) \leqq 2m(a, b) + 24\pi^2 .$$

Proof. For fixed a and b with $0<b<a$, set

$$M^+(x) = \int_{\xi \geqq x} H_b^a(x, d\xi)(\xi - x)^2 \qquad \text{and}$$

$$M^-(x) = \int_{\xi \leqq x} H_b^a(x, d\xi)(\xi - x)^2 .$$

Then $M(a, b) = \sup_x \{M^+(x) + M^-(x)\}$ and $m(a, b) = \inf_x \{M^+(x) + M^-(x)\}$.

By [M], $\phi(t) = \int_{\xi \geqq x} H_b^a(t, d\xi)(\xi - x)^2$ is nondecreasing in t. For $x<y<x+2\pi$,

$$\begin{aligned} M^+(x) &= \int_{\xi \geqq y} H_b^a(x, d\xi)(\xi - x)^2 + \int_{y>\xi \geqq x} H_b^a(\xi - x)(\xi - x)^2 \\ &\leqq 2\int_{\xi \geqq y} H_b^a(x, d\xi)(\xi - y)^2 + 2\int_{\xi \geqq Y} H_b^a(x, d\xi)(y - x)^2 + (2\pi)^2 \\ &\leqq 2M^+(y) + 12\pi^2 . \end{aligned}$$

By (p.5) in [1.1], $M^+(x)$ is periodic with period 2π. Therefore

$$M^+(x) \leqq 2M^+(y) + 12\pi^2$$

for any x and y. Similarly we have for any x and y

$$M^-(x) \leqq 2M^-(y) + 12\pi^2 .$$

We have

$$\sup_x (M^+(x) + M^-(x)) \leqq 2 \inf_x (M^+(x) + M^-(x)) + 24\pi^2 .$$

[15.2] Let P in $\mathcal{P}$ satisfy [V] and [M] and c be a fixed positive number. Then for any a and b with $0<b<a\leqq c$, $M(a, b) \leqq K$, where $K=K(c)$ is a constant independent of a and b.

Proof. By § 0, 8°, we can see for $0<s<r$

$$\int^r \pi^s(x) x^2 dx \leqq Cr^2 ,$$

where $C = \dfrac{1}{2\pi^3}\displaystyle\int \frac{u^2}{\cosh u - 1} du$ is an absolute constant. For $b \in \left(\frac{c}{2}, c\right)$, by $(\bar{h}, 3)$ in [2.2]

$$M(c,\, b) \leqq \sup_x \int_{c/2}^{c} \Pi_b^c(x,\, d\xi)(\xi - x)^2$$

$$+2\int_{c/2}^{c} \Pi_b^{c/2}(x,\, d\eta) H_{c/2}^c(\eta,\, d\xi)\{(\xi-\eta)^2+(\eta-x)^2\}$$

$$\leqq C\left(\frac{c}{2}\right)^2 + 2M\left(c,\, \frac{c}{2}\right) + 2C\left(\frac{c}{2}\right)^2 = C_1 .$$

For $b \in (0,\, c/2)$, again by $(\bar{h}.3)$

$$2M\left(c,\, \frac{c}{2}\right) \geqq 2\int_b^c \Pi_{c/2}^b(x,\, d\eta) H_b^c(\eta,\, d\xi)(\xi - x)^2$$

$$\geqq \int_b^c \Pi_{c/2}^b(x,\, d\eta) H_b^c(\eta,\, d\xi)\{(\xi-\eta)^2 - 2(\eta - x)^2\}$$

$$\geqq \frac{1}{2} m(c,\, b) - 2C(c-b)^2 .$$

Therefore by [15.1]

$$M(c,\, b) \leqq 2m(c,\, b) + 24\pi^2$$

$$\leqq 8M\left(c,\, \frac{c}{2}\right) + 8Cc^2 + 24\pi^2 = C_2 .$$

For $0<b<a<c$, by $(\bar{h}.2)$ in [2.2]

$$2M(c,\, b) \geqq 2\int H_b^a(x,\, d\eta) H_a^c(\eta,\, d\xi)(\xi - x)^2$$

$$\geqq \int H_b^a(x,\, d\eta) H_a^c(\eta,\, d\xi)\{(\eta - x)^2 - 2(\xi-\eta)^2\}$$

$$\geqq m(a,\, b) - 2M(c,\, a) .$$

By [15.1]

$$M(a,\, b) \leqq 4(M(c,\, b) + M(c,\, a)) + 24\pi^2$$

$$\leqq 8\, \mathrm{Max}\, \{C_1,\, C_2\} + 24\pi^2 = K ,$$

whicn completes the proof.

[15.3] PROPOSITION. *Let P in $\mathfrak{P}$ satisfy $[V]$ and $[M]$. Then P satisfies $[H.C]$ if and only if σ_P has no discrete mass.*

Proof. Since $\dfrac{d}{dx} u_P(z) = s_P(z) = \dfrac{1}{\pi}\displaystyle\int \frac{y}{(\xi - x)^2 + y^2} \sigma_P(d\xi)$, u_P has a continuous boundary function on ∂_0 in $\bar{D}$ if and only if σ_P has no discrete mass. Assume that P satisfies $[H.C]$. For $a>0$, set

$$f_N(x)=\begin{cases} u_P(N, a) & \text{if } x\geqq N, \\ u_P(x, a) & \text{if } |x|<N, \\ u_P(-N, a) & \text{if } x\leqq -N \end{cases}$$

and $\phi_N(z)=H^a f_N(z)$ for z in D^a $(N=1, 2, \cdots)$. By the assumption, $\phi_N(z)$ can be extended to a continuous function in $D^{[0,a]}=\bar{D}^a$. On the other hand, $|u_P(x, a)-u_P(\xi, a)\leqq C+|x-\xi|$. Therefore, for z in $D_r^a=\{0<y<a, |x|\leqq r\}$ and $N>r$

$$|u_P(z)-\phi_N(z)|\leqq\int_{|\xi|\geqq N} H_y^a(x, d\xi)|u_P(\xi, a)-f_N(\xi)|d\xi$$

$$\leqq\frac{C+2N}{(N-r)^2}\int H_y^a(x, d\xi)(\xi-x)^2\leqq\frac{C+2N}{(N-r)^2}K$$

by [15.2]. The function $u_P(z)$ can be approximated by $\phi_N(z)$ uniformly in D_r^a. Since r is arbitrary, u_P can be extended to a continuous function on $\bar{D}^a$. Conversely, assume that σ_P has no discrete mass. Let f be any function in $C_K(R)$ and a be any positive number. Set $\phi(z)=H^a f(z)$ for z in D^a. Then, by (3) in [9.9], for a fixed $b<a$ and z in D^b

$$(15.1) \qquad |\phi_x(z)|\leqq Ks_P(z).$$

Therefore, $\phi(z)$ has a continuous boundary function $\phi_0(x)=\phi_0(0)+\int_0^x g(t)\sigma_P(dt)$ on ∂_0 with $\|g\|\leqq K$. Thus (1) in the condition $[H.C]$ in [3.3] is proved. Note that by (2) in [9.8], the constant K appearing in (15.1) can be taken so as

$$K=\sup_x\frac{|\phi_x(x, b)|}{s_P(x, b)}\leqq C\|\phi\|=C\|f\|,$$

where $C=C(P, a, b)$ is a constant independent of ϕ. Let f_N $(N=1, 2, \cdots)$ be in $C_K(R)$ with $f_N\uparrow 1$ as $N\to\infty$, and set $\phi_N=H^a f_N$. We may assume that ϕ_N is continuous in $\bar{D}^b=D^{[0,b]}$. Then, by the above remark, the boundary functions of ϕ_N's $(N=1, 2, \cdots)$ on ∂_0 and on ∂_b are equicontinuous. They are also equicontinuous in $\bar{D}^b$. Since $\phi_N(z)\uparrow 1$ for z in D^b, we have $\phi_N(x, 0)\uparrow 1$ $(N\to\infty)$. Hence (2) in the condition $[H.C]$ is proved.

Let P be in $\mathfrak{T}_c$ and $P=\iota\bar{P}$ for $\bar{P}$ in $\tilde{\mathfrak{P}}$, and P satisfy the condition $[H.C]$. For f in $C_b(R)$, set $\phi=H^a f$ $(a>0)$. Then by $[H.C]$ and [3.5] we may assume that ϕ is in $C_b(\bar{D}^a)$. Set $A(\beta)=\{z\in\bar{D}^a; \phi>\beta\}$ for any real β and

$$(15.2) \qquad \begin{cases} \rho_\beta(w)=\inf\{t: z(t)\in A(\beta)\cap D\} & \text{for } w\in W, \\ \rho_\beta(\bar{w})=\inf\{t: z(t)\in A(\beta)\} & \text{for } w\in\bar{W}. \end{cases}$$

Then, by (1) in [14.1], for any z in D

$$\rho_\beta(\bar{w})=\rho_\beta(\iota\bar{w}) \qquad \text{a. s. } \bar{P}_z.$$

For any open set U in R, define $\mathfrak{U}$ in B by

$$\mathfrak{U}=\{w: \lim_{a\to 0} x(\sigma_a)\in U \text{ and } x(0)\in D\}, \tag{15.3}$$

where σ_a is the hitting time of ∂_a $(a\geqq 0)$. Then $\mathfrak{U}$ is in B_{σ_0} and

$$\iota^{-1}\mathfrak{U}=\{\bar{w}: x(\sigma_0)\in U \text{ and } x(0)\in D\}.$$

[15.4] Under the above assumptions and notations, set $\tau_a=\sigma_0+\sigma_a\circ\theta_{\sigma_0}$. If there exists an open set U such that $\phi(x, 0)<\alpha$ for any x in U, then, for any $\beta>\alpha$ and z in D,

$$\bar{P}_z\{x(\sigma_0)\in U,\ \phi(z(s))\leqq\beta \text{ for any } s\in(\sigma_0, \tau_a)\}>0\,.$$

Proof. Set $\rho=\sigma_0+\rho_\beta\circ\theta_{\sigma_0}$, where ρ_β is defined in (15.2). Assuming

$$\bar{P}_z\{x(\sigma_0)\in U,\ \phi(z(s))\leqq\beta \qquad \text{for any } s\in(\sigma_0, \tau_a)\}$$
$$=\bar{P}_z(x(\sigma_0)\in U\,\rho\geqq\tau_a)=0\,,$$

we shall show a contradiction. For $b<a$ set

$$\rho_b=\rho+\sigma_b\circ\theta_p$$

and

$$\tau_b=\sigma_0+\sigma_b\circ\theta_{\sigma_0}\,,$$

where σ_b is the hitting time of $D^{[b,\infty)}$. By (2), (3) in [14.1] $\rho_b\downarrow\rho$ and $\tau_b\downarrow\sigma_0$ as $b\downarrow 0$.

1° Using [1.5], we have

$$\begin{aligned}
&\bar{E}_z(f(x(\tau_a))I_{\{z(\sigma_0)\in U\}})\\
&=\bar{E}_z(f(x(\tau_a))I_{\{z(\sigma_0)\in U,\,\rho<\tau_a\}})\\
&=\lim_{b\to 0}\bar{E}_z(f(x(\tau_a))I_{\{\rho_b<\tau_a,\,x(\sigma_0)\in U\}})\\
&=\lim_{b\to 0}E_zE_{z(\rho_b)}(f(x(\tau_a)))I_{\{\rho_b<\tau_a\}\cap\mathfrak{U}})\\
&=\lim_{b\to 0}\bar{E}_z(\phi(z(\rho_c))I_{\{\rho_b<\tau_a,\,x(\sigma_0)\in U\}})\\
&=\bar{E}_z(\phi(z(\rho))I_{\{z(\sigma_0)\in U,\,\rho<\tau_a\}})\\
&\geqq\beta P_z(z(\sigma_0)\in U)\,.
\end{aligned}$$

2° Similarly, we obtain

$$\begin{aligned}&\bar{E}_z(f(x(\tau_a))I_{\{z(\sigma_0)\in U\}})\\&=E_z(f(x(\tau_a))I_{\mathfrak{U}})\\&=\lim_{b\to 0} E_z(\phi(z(\tau_b))I_{\mathfrak{U}})\\&=\lim_{b\to 0} \bar{E}_z(\phi(z(\tau_b))I_{\{z(\sigma_0)\in U\}})\\&=\bar{E}_z(\phi(z(\sigma_0))I_{\{z(\sigma_0)\in U\}})\\&\leqq \alpha\bar{P}_z(z(\sigma_0)\in U).\end{aligned}$$

Since $\bar{P}_z(z(\sigma_0)\in U)=P_z^{B,2}(z(\sigma_0)\in U)>0$, by 1° and 2° we have a contradiction.

[15.5] *Remark.* Replacing ϕ by $-\phi$ in [15.4], we also obtain: If there exists an open set U such that $\phi(x, 0)>\alpha$ for any x in U, then, for any $\beta<\alpha$ and z in D,

$$\bar{P}_z\{z(\sigma_0)\in U,\ \phi(z(s))\geqq\beta \text{ for any } s\in(\sigma_0, \tau_a)\}>0.$$

[15.6] PROPOSITION. *Let P in $\mathcal{P}_c$ satisfy [H. C], then P satisfies [M].*

Proof. Let f in $C_b(R)$ be any nondecreasing function and set $\phi=H^a f$ ($a>0$). We may assume that ϕ is in $C_b(\bar{D}^a)$ by [H.C]. Assume that there exist x_1 and x_2 in R such that $\phi(x_1, 0)>\phi(x_2, 0)$ and $x_1<x_2$. Then there exist open intervals J_1 and J_2 with $J_i\in x_i$ ($i=1, 2$) and $J_1\cap J_2=\emptyset$ and α and β with $\alpha<\beta$ such that $\phi(x, 0)>\beta$ for x in J_1 and $\phi(x, 0)<\alpha$ for x in J_2. Take $\bar{\alpha}$ and $\bar{\beta}$ such that $\alpha<\bar{\alpha}<\bar{\beta}<\beta$. Then by [15.4] and [15.5]

$$A_1=\{\bar{w} : z(\sigma_0)\in J_1,\ \phi(z(s))\geqq\bar{\beta} \text{ for any } s\in(\sigma_0, \tau_a)\}$$

and

$$A_2=\{\bar{w} : z(\sigma_0)\in J_2,\ \phi(z(s))\leqq\bar{\alpha} \text{ for any } s\in(\sigma_0, \tau_a)\}$$

have positive probabilities ($\bar{P}_z$, $z\in D$). Especially they are non-empty sets. Take $\bar{w}_1$ from A_1 and $\bar{w}_2$ from A_2. Then curves

$$C_1=\{z(s, \bar{w}_1): \sigma_0(\bar{w}_1)\leqq s\leqq\tau_a(\bar{w}_1)\}$$

and

$$C_2=\{z(s, \bar{w}_2): \sigma_0(\bar{w}_2)\leqq s\leqq\tau_a(\bar{w}_2)\}$$

in $\bar{D}^a$ both start from ∂_0 and end on ∂_a and they can not intersect. On the other hand, by construction of J_1 and J_2,

$$x(\sigma_0(\bar{w}_1), \bar{w}_1)<x(\sigma_0(\bar{w}_2), \bar{w}_2) \quad\text{and}\quad x(\tau_a(\bar{w}_1), \bar{w}_1)>x(\tau_a(\bar{w}_2), \bar{w}_2),$$

since

$$f(x(\tau_a(\bar{w}_1), \bar{w}_1))\geqq\bar{\beta}>\bar{\alpha}\geqq f(x(\tau_a(\bar{w}_2), \bar{w}_2)).$$

This is impossible. Therefore $\phi(x, 0)$ is nondecreasing. Then

$$\phi(z)={}_0^a\Pi_y^a f(x)+{}_0^a\Pi_y^0\phi(\cdot,0)(x)$$

is also nondecreasing, which completes the proof.

[15.7] Let P in $\mathscr{P}$ satisfy the condition $[M]$. Then for any fixed positive a

$$\lim_{\alpha\to\infty}\sup_{z\in D^a} H^a(z, U_\alpha(x)^c)=0\,, \tag{15.3}$$

where $U_\alpha(x)=\{\xi\in R: |\xi-x|<\alpha\}$ and $z=(x, y)$.

Proof. Set $H(z, \alpha)=H^a(z, [\alpha, \infty))$, then $H(z, \alpha)$ is increasing in x by $[M]$ and $H(\cdot, \alpha)$ is bounded harmonic in D^a with $0\leqq H(z, \alpha)\leqq 1$. Therefore $H(\cdot, \alpha)$ has a monotone bounded boundary function $H_0(x, \alpha)=H((x, 0), \alpha)$ such that

$$H((x, y), \alpha)={}_0^a\Pi_y^a(x, [\alpha, \infty))+\int_0^a \Pi_y^0(x, d\xi)H_0(\xi\,'\alpha)\,. \tag{15.5}$$

We may assume that $H_0(x, \alpha)$ is right continuous in x. Since $H(z, \alpha)$ $(0\leqq y<a)$ is increasing in x, decreasing in α and $H(z+2\pi, \alpha+2\pi)=H(z, \alpha)$, we have

$$H((0, y), \alpha+2\pi)\leqq H((x, y), x+\alpha)\leqq H((0, y), \alpha-2\pi)\,. \tag{15.6}$$

Also, by (15.5), $\lim_{\alpha\to\infty} H_0(0, \alpha)=0$ holds, for $\lim_{\alpha\to\infty} H(z, \alpha)=0$ holds for $z\in D^a$. By (15.5) and (15.6)

$$H((0, y), \alpha)\leqq {}_0^a\Pi_y^0(0, [\alpha, \infty))+{}_0^a\Pi_y^0\Big(0, \Big[\frac{\alpha}{2}, \infty\Big)\Big)+H_0\Big(\frac{\alpha}{2}, \alpha\Big)$$

$$\leqq 2\int_{\alpha/2}^{\infty}\frac{d\xi}{\cosh(\pi\xi/a)-1}+H_0\Big(0, \frac{\alpha}{2}-2\pi\Big)=k(\alpha)$$

and $\lim_{\alpha\to\infty} k(\alpha)=0$. Therefore, by using (15.6) again, we have

$$0\leqq \lim_{\alpha\to\infty}\sup_{z\in D^a} H(z, \alpha)\leqq \lim_{\alpha\to\infty}\sup_{0<y<a} H((0, y), \alpha-2\pi)$$

$$\leqq \lim_{\alpha\to\infty} k(\alpha-2\pi)=0\,.$$

In a similar way we can show

$$\lim_{\alpha\to\infty}\sup_{z\in D^a} H^a(z, (-\infty, -\alpha))=0\,.$$

[15.8] Let P in $\mathscr{P}_c$ satisfy the condition $[H.C]$. Set

$$\gamma_\alpha(\overline{w})=\inf\{t: |x(t)-x(0)|\geqq\alpha\}.$$

Then $\lim_{\alpha\to\infty}\sup_{z\in D^a}\overline{P}_z(\gamma_\alpha<\sigma_a)=0$.

Proof. Set $\gamma_{\alpha,b}=\gamma_\alpha+\sigma_b^*\circ\theta_{\gamma_\alpha}$ where σ_b^* is the hitting time of $D^{[b,\infty)}$ $(b<a)$.

$$\bar{P}_z(\gamma_\alpha<\sigma_a)\leqq\bar{P}_z\left(|x(\sigma_a)-x|\geqq\frac{\alpha}{3}\right)+\bar{P}_z\left(\gamma_\alpha<\sigma_a,\ |x(\sigma_a)-x|<\frac{\alpha}{3}\right).$$

Since $|x(\gamma_\alpha)-x(0)|=\alpha$ if $\gamma_\alpha<\infty$ in $\overline{W}$, noting $\gamma_{\alpha,b}\downarrow\gamma_\alpha$ as $b\downarrow 0$, we have by [1.5]

$$\bar{P}_z\left(\gamma_\alpha<\sigma_a,\ |x(\sigma_a)-x|<\frac{\alpha}{3}\right)$$

$$\leqq\varliminf_{b\to 0}\bar{P}_z\left(\gamma_{\alpha,b}<\sigma_a,\ |x(\sigma_a)-x|<\frac{\alpha}{3},\ |x(\gamma_{\alpha,b})-x|>\frac{2}{3}\alpha\right)$$

$$\leqq\varliminf_{b\to 0}\bar{P}_z\left(\gamma_{\alpha,b}<\sigma_a,\ |x(\sigma_a)-x(\gamma_{\alpha,b})|\geqq\frac{\alpha}{3}\right)$$

$$=\varliminf_{b\to 0}\bar{E}_z\left\{I_{(\gamma_{\alpha,b}<\sigma_a)}P_{z(\gamma_{\alpha,b})}\left(|x(\sigma_a)-x(0)|\geqq\frac{\alpha}{3}\right)\right\}$$

$$\leqq\sup_{z\in D^a}H^a(z,\ U_{\alpha/3}(x)^c).$$

Therefore

$$\bar{P}_z(\gamma_\alpha<\sigma_a)\leqq 2\sup_{z\in D^a}H^a(z,\ U_{\alpha/3}(x)^c).$$

[15.8] follows from [15.7], for P satisfies condition [M].

[15.9] PROPOSITION. *Let P in $\mathcal{P}_c$ satisfy [H. C]. Then P satisfies [V_r]* ($r=1, 2, \cdots$).

Proof. Define γ_α and $\gamma_{\alpha,b}$ as in [15.8]. By [15.8] we can take α so large that $\sup_{z\in D^a}P_z(\gamma_\alpha<\sigma_a)<1/2$. Then, by [1.5],

$$\bar{P}_z(\gamma_{2(n+1)\alpha}<\sigma_a)$$

$$\leqq\varliminf_{b\to 0}\bar{P}_z\{\gamma_{2n\alpha,b}<\gamma_{2(n+1)\alpha}<\sigma_a,\ |x(\gamma_{2n\alpha,b})-x|<(2n+1)\alpha\}$$

$$\leqq\varliminf_{b\to 0}\bar{P}_z\{\gamma_{2n\alpha,b}<\gamma_{2n\alpha,b}+\gamma_\alpha\circ\theta_{\gamma_{2n\alpha,b}}<\sigma_a\}$$

$$=\lim_{b\to 0}E_z\{I_{(\gamma_{2n\alpha,b}<\sigma_a)}P_{z(\gamma_{2n\alpha,b})}(\gamma_\alpha<\sigma_a)\}$$

$$\leqq\varliminf_{b\to 0}\frac{1}{2}\bar{P}_z(\gamma_{2n\alpha,b}<\sigma_a)$$

$$=\frac{1}{2}\bar{P}_z(\gamma_{2n\alpha}<\sigma_a).$$

By induction we have

$$\sup_{z\in D^a} \bar{P}_z(\gamma_{2n\alpha}<\sigma_a)<\frac{1}{2^n}.$$

Since

$$P_z(|x(\sigma_a)-x|>2n\alpha)\leqq \bar{P}_z(\gamma_{2n\alpha}<\sigma_a),$$

we have

$$\sup_{z\in D^a}\int H^a(z, d\xi)(\xi-x)^{2r}\leqq \sum_{n=0}^{\infty}\{2(n+1)\alpha\}^{2r}\frac{1}{2^n}<\infty.$$

Combining [15.3], [15.6] and [15.9] with theorem [14.9], we have proved the following theorem.

[15.10] THEOREM. *Let P be in $\mathcal{P}$. Then, P is in $\mathcal{P}_c$ and satisfies [H. C] if and only if P satisfies [M], [V] and [L], μ_P and σ_P are in $M_i(R)$ and σ_P has no discrete mass. In this case, P is a B_P-process.*

By theorem [3.12] and [4.10], we also have:

[15.11] PROPOSITION. *If P in $\mathcal{P}$ is a Feller process on $\bar{D}$ with continuous path functions in the sense that P is in $\mathcal{P}_c$ and satisfies [C], then P is B_P-process for which μ_P and σ_P are in $M_i(R)$ and σ_P has no discrete mass.*

V Construction of B-processes.

§ 16. Construction of processes $P_{\alpha,\beta}$

We begin by giving several notations and lemmas. Set

$$C_r=\left\{f\in C(R):\ \sup_x\frac{|f(x)|}{1+|x|^r}<\infty\right\},$$

$$C_r^*=\left\{f\in C_r:\ \lim_{|x|\to\infty}\frac{f(x)}{1+|x|^r}=0\right\}$$

and set $\|f_r\|=\sup_x\frac{|f(x)|}{1+|x|^r}$ $(r=0, 1, 2, \cdots)$. Then C_r and C_r^* are Banach spaces with $\|\ \|_r$-norm.

[16.1] $C_r^*\subset C_r\subset C_{r+1}^*$,

$$C_K(R) \text{ is dense in } C_r^*,$$

$$C_0=C_b(R) \text{ and } \|\ \|_0=\frac{1}{2}\|\ \|.$$

By an operator A on C_r (or C_r^*), we shall mean a linear operator A from C_r into C_r (or from C_r^* into C_r^*). Set

$$\|A\|_r=\sup_{f\neq 0}\frac{\|Af\|_r}{\|f\|_r} \quad\text{and}\quad \|A\|=\|A\|_0 .$$

We shall say:
A is monotone if Af is nondecreasing for any nondecreasing f.
A is positive if Af is nonnegative for any nonnegative f.
A is periodic (with period 2π) if $Af_{2\pi}(x+2\pi)=f(x)$, where $f_{2\pi}(x)=f(x-2\pi)$.

[16.2] Let $Q(x, d\xi)$ be a positive kernel on $R\times\mathcal{B}(R)$ with $\|Q\|=\sup_x Q(x,R)<\infty$. If $\sup_x\int Q(x, d\xi)|\xi-x|^r=k<\infty$ for $r\geqq 1$, then $Qf(x)=\int Q(x, d\xi)f(\xi)$ is well-defined for f in C_r and $\|Qf\|_r\leqq 2^{r-1}(\|Q\|+k)\|f\|_r$ holds. Moreover Q is an operator on C_0^*.

Proof. If f is in C_r

$$\frac{|Qf(x)|}{1+|x|^r}\leqq\|f\|_r\int Q(x, d\xi)\frac{1+|\xi|^r}{1+|x|^r}$$

$$\leqq 2^{r-1}\|f\|_r\int Q(x, d\xi)\frac{1+|x|^r+|\xi-x|^r}{1+x^r}$$

$$\leqq 2^{r-1}(\|Q\|+k)\|f\|_r .$$

If f is in C_0^*, then

$$|Qf(x)|\leqq\|f\|\int_{|\xi-x|\geqq N}Q(x, d\xi)+\sup_{|\xi-x|<N}|f(\xi)|\,\|Q\|$$

$$\leqq\frac{k}{N^r}\|f\|\int Q(x, d\xi)|\xi-x|^r+\sup_{|\xi-x|<N}|f(\xi)|\,\|Q\|$$

and $\overline{\lim}_{|x|\to\infty}|Qf(x)|\leqq\frac{k}{N^r}\|f\|\int Q(x, d\xi)|\xi-x|^r$. Since $r\geqq 1$ and N is arbitrary, Qf is in C_0^*.

[16.3] For $r\geqq 0$, let A be an operator on C_r with $\|A\|_r<\infty$. If $Af\geqq 0$ for any nonnegative f in $C_K(R)$, then there exists a unique positive kernel $Q(x, d\xi)$ on $R\times\mathfrak{B}(R)$ for which

$$Af(x)=\int Q(x, d\xi)f(\xi) \tag{16.1}$$

for f in C_r^*. If, moreover, A is periodic, then Q is periodic (that is, $Q(x+2\pi, d\xi+2\pi)=Q(x, d\xi)$),

$$\sup_x\int Q(x, d\xi)|\xi-x|^r<2^{r-1}\pi^r(1+\pi^r)\|A\|_r$$

and A is an operator on C_0^*.

Proof. It is obvious that there exists a unique positive kernel $Q(x, d\xi)$ with $\|Q\|=\sup_x Q(x, R)<\infty$ for which (16.1) holds for f in C_0^*. Set $\phi_N(x)=\frac{N(1+|x|^r)}{N+|x|^{r+1}}$. Then ϕ_N is in C_0^* and

$$\int Q(x, d\xi)(1+|\xi|^r)=\lim_{N\to\infty}\int Q(x, d\xi)\phi_N(\xi) \tag{16.2}$$
$$\leqq \lim_{N\to\infty} A\phi_N(x)$$
$$\leqq (1+|x|^r)\|A\|_r<\infty .$$

Therefore, approximating any function in C_r^* by functions in C_0^* in $\|\ \|_r$-norm, we can see that (16.1) holds for any f in C_r^*. If A is periodic, then Q is obviously periodic and by (16.2)

$$\sup_x \int Q(x, d\xi)|\xi-x|^r=\sup_{|x|\leqq\pi}\int Q(x, d\xi)|\xi-x|^r$$
$$\leqq 2^{r-1}\sup_{|x|\leqq\pi}\int Q(x, d\xi)(|\xi|^r+\pi^r)$$
$$\leqq 2^{r-1}\pi^r(1+\pi^r)\|A\|_r .$$

By [16.2] A is an operator on C_0^*.

[16.4] Let Q and S be positive kernels on $R\times\mathfrak{B}(R)$ with $\|Q\|=\sup_x Q(x,R)<\infty$ and $\|S\|=\sup_x S(x, R)<\infty$. If

$$\sup_x \int Q(x, d\xi)|\xi-x|^r=k_Q<\infty \quad\text{and}\quad \sup_x \int S(x, d\xi)|\xi-x|^r=k_S<\infty$$

for some $r\geqq 1$, then

$$\int QS(x, d\xi)|\xi-x|^r\leqq 2^{r-1}(k_Q\|S\|+k_S\|Q\|) \tag{16.3}$$

and

$$\int Q^n(x, d\xi)|\xi-x|^r\leqq n^r k_Q\|Q\|^{n-1} . \tag{16.4}$$

Proof. We have

$$\int QS(x, d\xi)|\xi-x|^r\leqq 2^{r-1}\int Q(x, d\eta)S(\eta, d\xi)(|\eta-x|^r+|\xi-\eta|^r)$$
$$\leqq 2^{r-1}(k_Q\|S\|+k_S\|Q\|)$$

and

$$\int Q^n(x, d\xi)|\xi-x|^r$$

$$\leqq n^{r-1}\int Q(x, d\xi_1)Q(\xi_1, d\xi_2)\cdots Q(\xi_{n-1}, d\xi_n)\Big(\sum_{k=1}^{n}|\xi_k-\xi_{k-1}|^r\Big)$$

$$\leqq n^{r-1}\cdot nk_Q\|Q\|^{n-1} \qquad (\xi_0=x).$$

For f in $C(R)$, set

$$\|f\|_{U_p(x)}=\sup_{\xi\in U_p(x)}|f(\xi)|, \tag{16.5}$$

where $U_P(x)=\{\xi\in R: |\xi-x|<p\}$.

[16,5] Let A and B be bounded operators on C_0. For given $x\in R$ and $\varepsilon>0$, assume that

$$\|Af\|_{U_p(x)}\leqq\gamma_A\|f\|_{U_{p+\varepsilon}(x)}+\delta_A\|f\|$$

and

$$\|Bf\|_{U_p(x)}\leqq\gamma_B\|f\|_{U_{p+\varepsilon}(x)}+\delta_B\|f\|$$

for any $p>0$ and f in C_0. Then,

$$\|ABf\|_{U_p(x)}\leqq\gamma\|f\|_{U_{p+2\varepsilon}(x)}+\delta\|f\|, \tag{16.6}$$

where $\gamma=\gamma_A\gamma_B$ and $\delta=\gamma_A\delta_B+\delta_A\|B\|$, and

$$\|A^nf\|_{U_p(x)}\leqq\gamma_n\|f\|_{P_{p+n\varepsilon}(x)}+\delta_n\|f\|, \tag{16.7}$$

where $\gamma_n=\gamma_A^n$ and

$$\delta_n=(\gamma_A^{n-1}+\gamma_A^{n-2}\|A\|+\cdots+\gamma_A\|A\|^{n-2}+\|A\|^{n-1})\delta_A.$$

Proof. Since

$$\|ABf\|_{U_p(x)}\leqq\gamma_A\|Bf\|_{U_{p+\varepsilon}(x)}+\delta_A\|Bf\|$$
$$\leqq\gamma_A(\gamma_B\|f\|_{U_{p+2\varepsilon}(x)}+\delta_B\|f\|)+\delta_A\|B\|\|f\|$$
$$\leqq\gamma_A\gamma_B\|f\|_{U_{p+2\varepsilon}(x)}+(\gamma_A\delta_B+\delta_A\|B\|)\|f\|,$$

(16.6) is proved. (16.7) is obtained by induction.

[16.6] Let f be in $C^2(R)$. Then for any $K\neq 0$

$$|f'(x)|\leqq\frac{2}{|K|}\sup_{\xi\in[x,x+K]}|f(\xi)|+\frac{|K|}{2}\sup_{\xi\in[x,x+K]}|f''(\xi)|,$$

where $[x, x+K]$ is replaced by $[x+K, x]$ if $K<0$.

Proof. Since $f(x+K)=f(x)+Kf'(x)+(1/2)K^2f''(\xi)$ for some $\xi\in[x, x+K]$, [16.6] is obvious.

In the following, C_k's $(k=1, 2, \cdots)$ stand for absolute positive constants and $C_k(x)$'s for positive functions which depend only on x. Set for $a>0$

$$\tilde{g}^a(x)=\int_0^\infty e^{-t/a}\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dt=\sqrt{a/2}\,e^{-\sqrt{2/a}|x|}. \tag{16.8}$$

By § 0, 8° and (16.8), we can easily obtain:

[16.7]

(1) $\int {}^a\pi^b(x)|x|^r dx \leqq C_1(r)a^r \qquad (0<b<a,\ 0\leqq r),$

(2) $\int q^a(x)|x|^r dx \leqq C_1(r)a^{r-1} \qquad (0<a,\ 0\leqq r),$

(3) $\int p^a(x)|x|^r dx \leqq C_1(r)a^{r-1} \qquad (0<a,\ 2\leqq r),$

(4) $\int \tilde{g}^a(x)|x|^r dx \leqq C_1(r)a^{(r/2)+1} \qquad (0<a,\ 0\leqq r),$

For positive ε

(5) $\int_{|x|\geqq\varepsilon} {}^a\pi^b(x)dx \leqq C_2(\varepsilon, a) \qquad (0<b<a),$

(6) $\int_{|x|\geqq\varepsilon} q^a(x)dx \leqq C_2(\varepsilon, a) \qquad (0<a),$

(7) $\int_{|x|\geqq\varepsilon} p^a(x)x^2 dx \leqq C_2(\varepsilon, a) \qquad (0<a),$

(8) $\int_{|x|\geqq\varepsilon} \tilde{g}^a(x)dx \leqq C_2(\varepsilon, a) \qquad (0<a),$

where $\lim_{a\to 0}\frac{C_2(\varepsilon, a)}{a^s}=0$ for any $s>0$.

For positive a and $x\in R$ set

$$\tilde{G}^a f(x)=\int \tilde{g}^a(\xi-x)f(\xi)d\xi=E_x^{B,1}\left(\int_0^\infty e^{-t/a}f(x(t))dt\right), \tag{16.9}$$

where $(P_x^{B,1}, x(t))$ is the one-dimensional Brownian motion starting at x. $P^a f$ and $Q^a f$ are defined as in (8.3) and (8.4).

[16.8] For f in C_r

(1) $\|{}^a_0\Pi^0_b f\|_r,\ \|{}^a_0\Pi^a_b f\|_r \leqq C_3(r)(1+a^r)\|f\|_r \qquad (0<b<a,\ 0\leqq r),$

(2) $\|Q^a f\|_r \leqq C_3(r)\frac{1}{a}(1+a^r)\|f\|_r \qquad (0<a,\ 0\leqq r)$,

(3) $\|P^a f\|_r \leqq C_3(r)a(1+a^r)\|f''\|_r \qquad (0<a,\ 0\leqq r,\ f''\in C_r)$,

(4) $\|\tilde{G}^a f\|_r \leqq C_3(r)a(1+a^{r/2})\|f\|_r \qquad (0<a,\ 0\leqq r)$.

For f in C_0 and positive p and ε

(5) $\|{}_0^a\Pi_b^0 f\|_{U_p(x)},\ \|{}_0^a\Pi_b^a f\|_{U_p(x)} \leqq C_4\|f\|_{U_{p+\varepsilon}(x)} + C_5(\varepsilon, a)\|f\| \qquad (0<b<a)$,

(6) $\|Q^a f\|_{U_p(x)} \leqq \frac{1}{a}(C_4\|f\|_{U_{p+\varepsilon}(x)} + C_5(\varepsilon, a)\|f\|) \qquad (a>0)$,

(7) $\|P^a f\|_{U_p(x)} \leqq a(C_4\|f''\|_{U_{p+\varepsilon}(x)} + C_5(\varepsilon, a)\|f''\|) \qquad (a>0,\ f''\in C_0)$,

(8) $\|\tilde{G}^a f\|_{U_p(x)} \leqq a(C_4\|f\|_{U_{p+\varepsilon}(x)} + C_5(\varepsilon, a)\|f\|) \qquad (a>0)$,

where $\lim_{a\to 0}\frac{C_5(\varepsilon, a)}{a^s}=0$ for any $s>0$.

Proof. We shall prove (3) and (7). The rest are easy to prove. By (3) in [16.7], we have

$$\begin{aligned}|P^a f(x)| &= \left|\int_{[x]}^{*} P^a(x, d\xi)(f(\xi)-f(x)-(\xi-x)f'(x))\right| \\ &\leqq \int P^a(x, d\xi)\sup_{y\in(x,\xi)}|f''(y)|\frac{(x-\xi)^2}{2} \\ &\leqq C'(r)\|f''\|_r\int P^a(x, d\xi)\frac{(\xi-x)^2}{2}\{1+|x|^r+|\xi-x\}^r\} \\ &\leqq C'(r)\{C_2(2)a(1+|x|^r)+C_2(r+2)a^{r+1}\}\|f''\|_r.\end{aligned}$$

Similarly by (3) and (7) in [16.7]

$$\begin{aligned}\|P^a f\|_{U_p(x)} &\leqq \|f''\|_{U_{p+\varepsilon}(x)}\int P^a(x, d\xi)\frac{(x-\xi)^2}{2} + \|f''\|\int_{|x|\geqq\varepsilon} p^a(x)\frac{x^2}{2}dx \\ &\leqq a\left(C_1(2)\|f''\|_{U_{p+\varepsilon}(x)} + \frac{1}{a}C_2(\varepsilon, a)\|f''\|\right).\end{aligned}$$

[16.9]

(1) For f in C_r and $0<b<a$

$$Q^a f = Q^b\,{}_0^a\Pi_b^a f. \tag{16.10}$$

(2) For f in $C^2(R)$ with $f''\in C_r$ and $0<b<a$

$$P^a f = P^b f + Q^b\,{}_0^a\Pi_b^0 f + \left(\frac{1}{a}-\frac{1}{b}\right)f. \tag{16.11}$$

Proof. By [16.1] and (2) and (3) in [16.8], it is sufficient to prove (16.10) for f in $C_K(R)$ and (16.11) for f in $C_K^2(R)$. (16.10) is a consequence of the relation

$$ {}^a_0\Pi^a_c = {}^b_0\Pi^b_c \, {}^a_0\Pi^a_b \qquad \text{for } 0<c<b<a . $$

For f in $C_K^2(R)$ and $0<c<b<a$

$$ \int {}^a_0\Pi^0_c(x, d\xi)(f(\xi)-f(x)) $$

$$ = {}^a_0\Pi^0_c f(x) - \frac{a-c}{a} f(x) $$

$$ = {}^b_0\Pi^0_c f(x) + {}^b_0\Pi^b_c \, {}^a_0\Pi^0_b f(x) - \frac{a-c}{a} f(x) $$

$$ = \int {}^b_0\Pi^0_c(x, d\xi)(f(\xi)-f(x)) + {}^b_0\Pi^b_c \, {}^a_0\Pi^0_b f(x) + \Big(\frac{c}{a}-\frac{c}{b}\Big) f(x). $$

Therefore

$$ P^a f(x) = \lim_{c\to 0} \frac{1}{c} \int {}^a_0\Pi^0_c(x, d\xi)(f(\xi)-f(x)) $$

$$ = P^b f(x) + Q^b \, {}^a_0\Pi^0_b f(x) + \Big(\frac{1}{a}-\frac{1}{b}\Big) f(x). $$

In the following assume that functions $\alpha(x)$ and $\beta(x)$ in $C_p^2(R)$ with $\alpha(x)>0$ are given and fixed. Set $\alpha^* = \sup_x \alpha(x)$ and $\alpha_*(x) = \inf_x \alpha(x)$. Then α_* is positive. Hereafter K_j's ($j=1, 2, \cdots$) stand for positive constants which depend only on α^*, α_* and $\|\beta\|$, and $K_j(x)$'s ($j=1, 2, \cdots$) for positive functions of x which depend only on α^*, α_* and $\|\beta\|$. Define for $a>0$

$$ G^a f(x) = E_x^{B,1}\Big[\int_0^\infty \exp\Big\{-\int_0^t \frac{ds}{a\alpha(x(s))}\Big\} \frac{f(x(t))}{\alpha(x(t))} dt\Big]. \tag{16.12} $$

Then by Kac's theorem we immediately have:

[16.10] For f in C_0 and positive a, $G^a f$ is in $C^2(R)\cap C_0$ and it holds that

$$ \Big(\frac{1}{a} - \alpha \frac{d^2}{dx^2}\Big) G^a f = f. \tag{16.13} $$

[16.11] For any $r \geqq 0$ and f in C_r, $G^a f$ is in $C^2(R)\cap C_0$ and for $0<a\leqq 1$

(1) $\|G^a f\|_r \leqq a K_1(r) \|f\|_r$,

(2) $\|(G^a f)'\|_r \leqq \sqrt{a}\, K_1(r) \|f\|_r$,

(3) $\|(G^a f)''\|_r \leqq K_1(r) \|f\|_r$.

For any f in C_0, any $p>0$, $\varepsilon>0$ and $0<a\leqq 1$,

(4) $\|G^a f\|_{U_p(x)}\leqq aK_2\|f\|_{U_{p+\varepsilon}(x)}+K_3(\varepsilon, a)\|f\|$,

(5) $\|(G^a f)'\|_{U_p(x)}\leqq \sqrt{a}\, K_4(\varepsilon)\|f\|_{U_{p+\varepsilon}(x)}+K_3(\varepsilon, a)\|f\|$,

(6) $\|(G^a f)''\|_{U_p(x)}\leqq K_2\|f\|_{U_{p+\varepsilon}(x)}+K_3(\varepsilon, a)\|f\|$,

where $\lim_{a\to 0}\dfrac{K_3(\varepsilon, a)}{a^s}=0$ for any $s>0$.

Proof. Since

$$|G^a f(x)|\leqq G^a|f|(x)\leqq \frac{1}{\alpha_*}\tilde{G}^{a\alpha_*}|f|(x), \tag{16.14}$$

$G^a f$ is well-defined for f in C_r and (1) holds for $0<a\leqq 1$ by (4) in [16.8]. If f is in C_0, then by (16.13)

$$|(G^a f)''(x)|\leqq \frac{1}{\alpha_*}\Big(\frac{1}{a}|G^a f(x)|+|f(x)|\Big) \tag{16.15}$$

and (3) is an immediate consequence of (1). Taking $K=\sqrt{a}$ in [16.6], we get

$$|(G^a f)'(x)|\leqq \frac{2}{\sqrt{a}}\sup_{\xi\in[x,\, x+\sqrt{a}]}|G^a f(\xi)|+\frac{\sqrt{a}}{2}\sup_{\xi\in[x,\, x+\sqrt{a}]}|(G^a f)''(\xi)|. \tag{16.16}$$

Hence (2) follows to (1) and (3). For f in C_r, take a sequence $\{f_n\}$ in C_0 such that $f_n\to f$ in C_{r+1}. Replacing r by $r+1$ in the above argument, we can see that $G^a f_n\to G^a f$ in C_{r+1} and $\{(G^a f_n)'\}$ and $\{(G^a f_n)''\}$ converge in C_{r+1}. Therefore $G^a f$ is in $C^2(R)$ and (16.15) and (16.16) hold for f in C_r. (2) and (3) can be easily proved for f in C_r. (4) is a consequence of (16.14) and (8) in [16.8]. (6) is proved by (4) and (16.15). For f in C_0 and $a\leqq(\varepsilon/2)^2$ we have by (16.16),

$$\|(G^a f)'\|_{U_p(x)}\leqq \frac{2}{\sqrt{a}}\|G^a f\|_{U_{p+\varepsilon/2}(x)}+\frac{\sqrt{a}}{2}\|(G^a f)''\|_{U_{p+\varepsilon/2}}.$$

Therefore (5) is obtained from (4) and (6).

[16.12] *Remark.* In a way similar to the proof of [16.11], we can show (16.13) also holds for f in C_r.

[16.13] Set $F^a=P^a+\beta(x)(d/dx)$. Then for $0<a\leqq 1$, $r\geqq 0$ and f in C_r

(1) $\|F^a G^a f\|_r\leqq \sqrt{a}\, K_5(r)\|f\|_r$.

For $0<a\leqq 1$, $p>0$, $\varepsilon>0$ and f in C_0

(2) $\|F^a G^a f\|_{U_p(x)}\leqq \sqrt{a}\, K_6(\varepsilon)\|f\|_{U_{p+\varepsilon}(x)}+K_7(\varepsilon, a)\|f\|$.

Proof. (1) is a consequence of (3) in [16.3] and (2) and (3) in [16.11].

Applying [16.5], we have, by (7) in [16.8] and (6) in [16.11],

$$\|P^a G^a f\|_{U_p(x)} \leqq a C_4 K_2 \|f\|_{U_{p+\varepsilon}(x)}$$
$$+\Big(a C_4 K_3\Big(\frac{\varepsilon}{2}, a\Big) + a^2 C_5\Big(\frac{\varepsilon}{2}, a\Big) K_1(0)\Big)\|f\| .$$

Combining this with (5) in [16.11] we can prove (2).

[16.14] For any $r \geqq 0$, there exists $K_8(r)$ such that for $0 < a \leqq K_8(r)$

$$\sum_{n=0}^{\infty} \|F^a G^a\|_r^n < \infty . \tag{16.17}$$

Set $L^a f = \sum_{n=0}^{\infty} (F^a G^a)^n f$ for f in C_r and $0 < a \leqq K_8(r)$. Then

(1) $\|L^a f\|_r \leqq K_9(r) \|f\|_r$,

(2) $\|G^a L^a f\|_r \leqq a K_9(r) \|f\|_r$,

(3) $\|G^a L^a f\|_{U_p(x)} \leqq a K_{10}(\varepsilon)(\|f\|_{U_{p+\varepsilon}(x)} + a^{3/2}\|f\|)$.

(4) $G^a L^a f$ is in $C^2(R) \cap C_r$ and satisfies

$$\Big(\alpha(x)\frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + P^a - \frac{1}{a}\Big) G^a L^a f = -f . \tag{16.18}$$

Proof. Take $K_8(r) = \mathrm{Min}(1, 1/2K_5(r)^2)$. By (1) in [16.13], (16.17) and (1) are obvious. (2) is a consequence of (1) and (1) in [16.11]. By [16.5], [16.11] and [16.13], for f in C_0

$$\|G^a L^a f\|_{U_p(x)} \leqq \sum_{n=0}^{2} \|G^a (F^a G^a)^n f\|_{U_p(x)} + \|G^a (F^a G^a)^3 L^a f\|$$
$$\leqq a K_2 \|f\|_{U_{p+\varepsilon}(x)} + a^{3/2} K_2 K_6(\varepsilon) \|f\|_{U_{p+2\varepsilon}(x)}$$
$$+ a^2 K_2 K_6(\varepsilon)^2 \|f\|_{U_{p+3\varepsilon}(x)} + (K'(\varepsilon, a) + a^{5/2} K_1(0) K_5(0)^3 K_9(0)) \|f\| ,$$

where $\lim_{a \to 0} (K'(\varepsilon, a)/a^s) = 0$ for any $s > 0$. Thus (3) is proved. Since $L^a f$ is in C_r, $G^a L^a f$ is in $C^2(R)$ and by remark [16.12]

$$\Big(\frac{1}{2} - \alpha \frac{d^2}{dx^2}\Big) G^a L^a f = L^a f = f + F^a G^a L^a f$$
$$= f + P^a (G^a L^a f) + \beta \frac{d}{dx}(G^a L^a f).$$

(16.18) is proved.

By construction it is easily seen:

[16.15] $G^a L^a$ is periodic as an operator on C_r ($r \geqq 0$, $a \leqq K_8(r)$).

[16.16] For any positive a there exists a positive kernel $H_0^a(x, d\xi)$ on $R \times \mathfrak{B}(R)$ with the following properties:

(1) H_0^a is a periodic probability kernel.
(2) H_0^a is monotone.
(3) $\sup_x \int H_0^a(x, d\xi)|\xi - x|^r < \infty \qquad (r=1, 2, \cdots)$.
(4) H_0^a maps C_r into C_r ($r=0, 1, 2, \cdots$) and C_0^* into C_0^*.
(5) For f in C_r, $\phi = H_0^a f$ is in $C^2(R)$ and satisfies

$$\alpha(x)\phi''(x) + \beta(x)\phi'(x) + P^a\phi(x) + Q^a f(x) - \frac{1}{a}\phi(x) = 0. \tag{16.19}$$

(6) For any positive ε

$$\int_{|\xi - x| \geqq \varepsilon} H_0^a(x, d\xi) \leqq a^{3/2} K_{11}(\varepsilon).$$

Moreover,

(7) A kernel $H_0^a(x, d\xi)$ is uuiquely determined by the properties that H_0^a maps C_0^* into $C_0^* \cap C^2(R)$ and $\phi = H_0^a f$ satisfies (16.19).

Proof. 1° *Uniqueness* Suppose that there exist two kernels H_{0i}^a ($i=1, 2$) satisfying conditions in (7). For f in C_0^*, set $\psi = H_{01}^a f - H_{02}^a f$. Then ψ is in $C_0^* \cap C^2(R)$ and satisfies

$$\alpha\psi'' + \beta\psi' + P^a\psi - \frac{1}{a}\psi = 0. \tag{16.20}$$

Therefore, ψ can not take positive maximum nor negative minimum, and hence $\psi = 0$. (7) is proved.

2° For any given r ($r=0, 1, 2, \cdots$) take $K'(r) = \operatorname{Min}_{s \leqq r+1} K_8(s)$, where $K_8(s)$ is given in [16.14]. For $a \leqq K'(r)$ set $\tilde{H}f = G^a L^a Q^a f$. Then, by (2) in [16.8] and (2) in [16.14], $\|\tilde{H}f\|_s \leqq K''(r)\|f\|_s$ for f in C_s ($s=0, 1, 2, \cdots, r+1$). Moreover, by (4) in [16.14] $\tilde{H}f$ is in $C^2(R)$ and satisfies (16.19) for f in C_{r+1} and by [16.15] $\tilde{H}$ is periodic as an operator on C_{r+1}. If f is in $\bigcup_{N=1}^{\infty} C_{p,N}(R) \subset C_0$ and nonnegative, then $\phi = \tilde{H}f$ is in $\bigcup_{N=1}^{\infty} C_{p,N}$ and satisfies

$$\alpha\phi'' + \beta\phi' + P^a\phi - \frac{1}{a}\phi = -Q^a f \leqq 0.$$

Therefore ϕ can not take negative minimum and $\tilde{H}f \geqq 0$. Since any function in $C_K(R)$ can be approximated by functions in $\bigcup_{N=1}^{\infty} C_{p,N}$ in C_{r+1}^*-topology ($r \geqq 0$), we have $\tilde{H}f \geqq 0$ if f is in $C_K(R)$. Now, applying [16.3] to $\tilde{H}$ (where r is replaced by $r+1$), we see that there exists a positive periodic kernel $\tilde{H}_0^a(x, d\xi)$ such that

$$\tilde{H}f(x) = \tilde{H}_0^a f(x) \qquad \text{for } f \in C_r \subset C_{r+1}^*,$$

$$\sup_x \int \tilde{H}_0^a(x, d\xi)|\xi - x|^r < K^{(3)}(r) \tag{16.21}$$

and $\tilde{H}_0^a = \tilde{H}$ maps C_0^* into C_0^* by [16.2]. The function $\phi = \tilde{H}_0^a 1 - 1$ is a solution of (16.20) and in $C_p(R)$. Therefore by maximum principle $\tilde{H}_0^a 1 = 1$, or $\tilde{H}_0^a$ is a probability kernel. Now for $K'(0) \geqq K'(1) \geqq \cdots \geqq K'(r) \geqq \cdots > 0$ we have contructed kernels $\tilde{H}_0^a(x, d\xi)$ $(0 < a \leqq K'(r))$ which satisfy (1), (3), (4) and (5) for fixed r. By (7) they are independent of r if defined.

3° Using [16.5], we have, by (2) and (6) in [16.8] and (3) in [16.14],

$$\|\tilde{H}_0^a f\|_{U_p(x)} = \|G^a L^a Q^a f\|_{U_p(x)}$$
$$\leqq K_{10}(\varepsilon')\{C_4 \|f\|_{U_{p+2\varepsilon'}(x)} + (C_5(\varepsilon', a) + 2a^{3/2}C_3(0))\|f\|\}$$

for any f in C_0. Take $p = \varepsilon'$, $\varepsilon = 4\varepsilon'$ and f in C_0 with

$$f = \begin{cases} 0 & \text{in } U_{3\varepsilon'}(x), \\ 1 & \text{in } U_\varepsilon(x)^c. \end{cases}$$

Then

$$\int_{|\xi - x| \geqq \varepsilon} \tilde{H}_0^a(x, d\xi) \leqq K^{(4)}(\varepsilon) a^{3/2} \quad (a \leqq K'(0))).$$

Thus (6) is proved.

4° We shall prove (2) for small a. Let f be in $C_b^1(R)$ and nondecreasing. For a fixed a with $0 < a \leqq K'(1)$, set $\phi = \tilde{H}_0^a f$. We shall show that $\lim_{|x| \to \infty} \phi'(x) = 0$. There exists $\mu = \lim_{x \to \infty} f(x)$ and

$$|\phi(x) - \mu| \leqq \int_{|\xi - x| \leqq K} \tilde{H}_0^a(x, d\xi)|f(\xi) - \mu| + 2\|f\| \int_{|\xi - x| > K} \tilde{H}_0^a(x, d\xi).$$

Therefore, for any positive K

$$\overline{\lim_{x \to \infty}} |\phi(x) - \mu| \leqq 2\|f\| \frac{1}{K} \int \tilde{H}_0^a(x, d\xi)|\xi - x|,$$

and $\lim_{x \to \infty} \phi(x) = \mu$. Similarly we have $\lim_{x \to -\infty} \phi(x) = \lim_{x \to -\infty} f(x)$. Noting (3) in [16.11], we have

$$\|\phi''\| = \|(G^a L^a Q^v f)''\| \leqq K_1(0)\|L^a\|\|Q^a\|\|f\| < \infty$$

and $|\phi'(x) - 1/\varepsilon(\phi(x+\varepsilon) - \phi(x))| \leqq \varepsilon\|\phi''\|$. Therefore, $\overline{\lim_{|x| \to \infty}} |\phi'(x)| \leqq \varepsilon\|\phi''\|$ for any positive ε, and $\lim_{|x| \to \infty} \phi'(x) = 0$. Since by (16.19).

$$\phi''=\frac{1}{\alpha}\Big(\frac{1}{a}\phi-\beta\phi'-P^a\phi-Q^a f\Big),$$

ϕ is in $C^3(R)$. Differentiating (16.19), we also have

$$\alpha\phi'''+(\beta+\alpha')\phi''+\Big(\beta'-\frac{1}{a}\Big)\phi'+P^a\phi'=-Q^a f'\leqq 0\,.$$

Take $a\leqq\text{Min}\,\{K'(1), (1/(1+\|\beta'\|)\}$, then ϕ' can not take negative minimum.. Since we have seen that ϕ' is in C_0^*, $\phi'\geqq 0$ or ϕ is nondecreasing, (2) is proved for

$$0<a\leqq\tilde{K}=\text{Min}\Big\{K'(1), \frac{1}{1+\|\beta'\|}\Big\}\,.$$

5° Let a be any positive number. For a fixed r ($r=1, 2, \cdots$) take b so small as $b<\text{Min}\{a, \tilde{K}, K'(r)\}$, and set

$$H_0^a=\sum_{n=0}^{\infty}(\tilde{H}_0^b\,{}_0^a\Pi_b^0)^n\tilde{H}_0^b\,{}_0^a\Pi_b^a\,.$$

Since $\tilde{H}_a^b\,{}_0^a\Pi_b^0(x, R)=(a-b/a)<1$ and $\tilde{H}_0^b\,{}_0^a\Pi_b^a(x, R)=b/a$, H_0^a is well-defined as a periodic probability kernel. Using [16.4], we have by (1) in [16.7] and (16.21)

$$\sup_x\int H_0^a(x, d\xi)|\xi-x|^r<\infty\,.$$

Noting [16.2], we see that $\tilde{H}_0^a$ satisfies (1), (3) and (4). (2) is obvious, since $\tilde{H}_0^b$, ${}_0^a\Pi_b^0$ and ${}_0^a\Pi_b^a$ are monotone. Set $\phi=H_0^a f$ for f in C_r. Then $\phi=\tilde{H}_0^b({}_0^a\Pi_b^0\phi+{}_0^a\Pi_b^a f)$. Since we have already seen that $\tilde{H}_0^b$ satisfies (16.19), ϕ satisfies

$$\alpha\phi''+\beta\phi'+P^b\phi+Q^b({}_0^a\Pi_b^0\phi+{}_0^a\Pi_b^a f)-\frac{1}{b}\phi=0$$

and by [16.9] ϕ itself satisfies (16.19). Hence (5) is proved. By uniqueness, we see that H_0^a is independent of b and $H_0^a=\tilde{H}_0^a$ if the right side is defined. (6) is trivial, since it holds for $a\leqq K'(0)$ by 3° and $H_0^a(x, R)=1$ for any a.

[16.17] *Remark.* By (16.21) it holds that for $0<a\leqq K_{12}(r)$

$$\sup_x\int H_0^a(x, d\xi)|\xi-x|^r\leqq K_{13}(r),$$

where the right side is independent of a.

By the explicit form of ${}^r\pi^s(x)$ in § 0.8° and the definitions of P^r and Q^r in (8.3) and (8.4), we can easily show:

[16.18] Let f be in C_r and g be in $C_r\cap C^2(R)$. Set $u(z)={}_0^a\Pi_y^a f(x)+{}_0^a\Pi_y^0 g(x)$ for z in D^a. Then u is well-defined and harmonic in D^a and u, u_x, u_{xx} and

u_y are in $C(D^{[0,a)})$. Moreover, $u(x, 0)=g(x)$, $u_x(x, 0)=g'(x)$, $u_{xx}(x, 0)=g''(x)$ and

$$u_y(x, 0)=P^a g-\frac{1}{a}g+Q^a f\,.$$

[16.19] THEOREM. *Let* α *and* β *in* $C_p^2(R)$ *with* $\alpha>0$ *be given, and* H_0^a *be the kenel given in* [16.16]. *For any positive a and b with* $0<b<a$ *set*

$$H_b^a={}_0^a\Pi_b^a+{}_0^a\Pi_b^0 H_0^a \tag{16.22}$$

and

$$H^a(z, d\xi)=H_y^a(x, d\xi) \quad \text{for } z \text{ in } D^a\,. \tag{16.23}$$

Then $H=\{H^a(x, d\xi)\}$ belongs to $\mathscr{H}$. $P=P(H)$ satisfies $[M]$, $[V_r]$ ($r=1, 2, \cdots$) and $[L^*]$ (and therefore $[L]$). Moreover H satisfies:

(1) For any f in C_r ($r=1, 2, \cdots$) set $u(z)=H^a f(z)=\int H^a(z, d\xi)f(\xi)$. Then u, u_x. u_{xx} and u_y are in $C(D^{[0,a)})$ and u satisfies

$$\alpha(x)u_{xx}(x, 0)+\beta(x)u_x(x, 0)+u_y(x, 0)=0 \tag{16.24}$$

on ∂_0.

H in $\mathscr{H}$ is uniquely determined if (1) is satisfied for any f in $C_b(R)$.

Proof.

1° Let H satisfy (1) for f in $C_b(R)$. For f in $C_{p,N}(R)$, $u=H^a f$ is harmonic in D^a and $C_{p,N}(R)$ ($N=1, 2, \cdots$). Since $u=f$ on ∂_a and u satisfies (16.24), we can easily show, by maximum principle of harmonic function, that u is uniquely determined. Probability kernels $H^a(z, d\xi)'s$ ($a>0$, $z\in D^a$) are also determined, since f is arbitrary in $\bigcup_N C_{p,N}(R)$.

2° In the following, let $H=\{H^a(z, d\xi)\}$ be defined by (16.23). Then by definition and [16.16], H satisfies (h.1), (h.3) and (h.4) in [2.1]. For f in C_0^∞, set $u=H^a f$, $\phi=H_0^a f$, $\tilde{u}=H^b H_b^a f$ and $\tilde{\phi}=H_0^b H_b^a f=H_0^b({}_0^a\Pi_b^0\phi+{}_0^a\Pi_b^a f)$ ($b>a$). Then u and $\tilde{u}$ are harmonic in D^b, $u(x, b)=H^a f(x, b)=\tilde{u}(x, b)$ on ∂_b and $u=\phi$ and $\tilde{u}=\tilde{\phi}$ on ∂_0. By (5) in [16.16] $\tilde{\phi}$ and ϕ satisfy

$$\alpha\tilde{\phi}''+\beta\tilde{\phi}'+P^b\tilde{\phi}+Q^b({}_0^a\Pi_b^0\phi+{}_0^a\Pi_b^a f)-\frac{1}{b}\tilde{\phi}=0\,. \tag{16.25}$$

$$\alpha\phi''+\beta\phi'+P^a\phi+Q^a f-\frac{1}{a}\phi=0\,. \tag{16.26}$$

By [16.9], (16.26) is transformed into

$$\alpha\phi''+\beta\phi'+P^b\phi+Q^b({}_0^a\Pi_b^0\phi+{}_0^a\Pi_b^a f)-\frac{1}{b}\phi=0 \tag{16.27}$$

By (16.25) and (16.27)

$$\alpha(\phi''-\tilde{\phi}'')+\beta(\phi'-\tilde{\phi}')+P^b(\phi-\tilde{\phi})-\frac{1}{b}(\tilde{\phi}-\phi)=0\,.$$

Since $\phi-\tilde{\phi}$ is in C_0^* by (4) in [16.16], we can show $\phi=\tilde{\phi}$ by maximum principle. Therefore $u=\tilde{u}$ and $H^a=H^bH^a$ in D^b. Hence (h.2) is proved.

3° For f in C_r set $u=H^af$ and $\phi=H_0^af$. Then $u(z)={}_0^a\Pi_y^af(x)+{}_0^a\Pi_y^0\phi(x)$. By (4) and (5) in [16.16] ϕ is in $C^2(R)\cap C_r$ and satisfies (16.19). On the other hand, by [16.18], u, u_x, u_{xx} and u_y are in $C(D^{[0,a)})$ and $u=\phi$, $u_x=\phi'$, $u_{xx}=\phi''$ and $u_y=P^a\phi+Q^af-(1/a)\phi$ on ∂_0. (16.24) is a consequence of (16.19).

4° Since H_0^a, ${}_0^a\Pi_b^a$ and ${}_0^a\Pi_b^0$ are monotone, H satisfies [M]. Using [16.4], we can see by (1) in [16.7] and (3) in [16.16] that H satisfies [V_r] ($r=1, 2\cdots$). Especially by [16.17], we have

$$\sup_x \int H^a(x, d\xi)|\xi-x|^r \leqq K_{14}(r) \qquad \text{for } 0<a\leqq K_{12}(r). \tag{16.28}$$

On the other hand, by (5) in [16.7] and (6) in [16.16]

$$\int_{|\xi-x|\geqq\varepsilon} {}_0^{2a}\Pi_a^{2a}(x, d\xi)\leqq C_2(\varepsilon, 2a),$$

$$\int_{|\xi-x|\geqq\varepsilon} {}_0^{2a}\Pi_a^0(x, d\eta)H_0^{2a}(\eta, d\xi)$$

$$\leqq\Big(\int_{|\eta-x|\geqq\varepsilon/2}+\int_{|\xi-\eta|\geqq\varepsilon/2}\Big){}_0^{2a}\Pi_a^0(x, d\eta)H_0^{2a}(\eta, d\xi)$$

$$\leqq C_2\Big(\frac{\varepsilon}{2}, 2a\Big)+K_{11}\Big(\frac{\varepsilon}{2}\Big)(2a)^{3/2},$$

where ε is a fixed positive number and $\lim_{a\to 0}(C_2(\varepsilon, a)/a^s)=0$ for any $s>0$. Therefore we have

$$\int_{|\xi-x|\geqq\varepsilon} H_a^{2a}(x, d\xi)\leqq K'(\varepsilon)a^{3/2}.$$

For $a\leqq K_{12}(9)$

$$\int_{|\xi-x|\geqq\varepsilon} H_a^{2a}(x, d\xi)(\xi-x)^2$$

$$\leqq\Big(\int_{a^{-1/6}>|\xi-x|\geqq\varepsilon}+\int_{|\xi-x|\geqq a^{-1/6}}\Big)H_a^{2a}(x, d\xi)(\xi-x)^2$$

$$\leqq a^{3/2-1/3}K'(\varepsilon)+a^{7/6}\int H_a^{2a}(x, d\xi)(\xi-x)^9$$

$$\leqq a^{7/6}(K'(\varepsilon)+K_{14}(r))\,.$$

Hence $\lim_{a\to 0}\sup_x \frac{1}{a}\int H_a^{2a}(x, d\xi)(\xi-x)^2=0$. By proposition [11.11] H satisfies [L^*].

[16.20] DEFINITION. Let α and β be in $C_p^2(R)$ with $\alpha>0$. $P_{\alpha,\beta}$ is the process such that $H_{\alpha,\beta}=H(P_{\alpha,\beta})$ satisfies condition (1) in [16.19]. Combining theorem [16.19] with theorem [11.7], we have:

[16.21] COROLLARY. *$P_{\alpha,\beta}$ is a B_P-process.*

§ 17. Existence of B-process (1): Smooth case.

Let σ and μ be in $M_p(R)$ with $\sigma(dx)=s_0(x)dx$ and $\mu(dx)=m_0(x)dx$. We shall assume s_0 and m_0 are $C_p^\infty(R)$ and positive. For any constant k, set for z in D

$$(17.1)\qquad \begin{cases} m(z)=\int_0^{2\pi}\tilde{h}^{\xi}(z)m_0(\xi)d\xi\,, \\ l(z)=\int_0^{2\pi}\tilde{k}_{\xi}(z)m_0(\xi)d\xi-k\,, \\ s(z)=\int_0^{2\pi}\tilde{h}_{\xi}(z)s_0(\xi)d\xi\,, \\ t(z)=\int_0^{2\pi}\tilde{k}_{\xi}(z)s_0(\xi)d\xi+k\,. \end{cases}$$

Then, they are in $C^\infty(\bar{D})$, and m_0 and s_0 are boundary functions of m and s on ∂_0, respectively. Let l_0 and t_0 be boundary functions of l and t on ∂_0, respectively. Since $\{\sigma, \mu\}$ satisfies the condition [P] in [5.11], there exists a non-negative minimum solution $U=U^0$ in D of

$$(17.2)\qquad \begin{cases} U_x=mt+ls\,, \\ U_y=ms-lt\,. \end{cases}$$

Set, $p_0=p_0(\sigma, \mu, k)$, that is,

$$(17.3)\qquad 2\pi p_0=\int_0^{2\pi}U^0(x, 0)s_0(x)dx=\inf_{y>0}\int U^0(x, y)s(x, y)dx\,.$$

Take any positive p with $p>p_0$. Then by definition [4.19] $B=\{\sigma, \mu, k, p\}$ is in $\boldsymbol{B}$. In this section we shall construct B-process for this B.

Set $U_B=p-p_0+U^0$. Then U_B is a solution of (17.2) with

$$2\pi p=\inf_{y>0}\int_0^{2\pi}U_B(x, y)s(x, y)dx$$

Obviously, U is in $C_p^\infty(\bar{D})$ by (17.2) and $U_B>0$ in $\bar{D}$ for $p>p_0$. Define α and β in $C_p^\infty(R)$ by

(17.4)
$$\begin{cases} \alpha(x)=\dfrac{1}{s_0(x)m_0(x)}U_B(x,\,0)\,, \\ \beta(x)=\dfrac{1}{s_0(x)}(t_0(x)-\alpha(x)s_0'(x))\,. \end{cases}$$

Then α and β are in $C_P^\infty(R)$ with $\alpha>0$. By theorem [16.18] we can construct $P=P_{\alpha,\beta}$. Since P satisfies $[M]$, $[V]$ and $[L]$, $B_P=\{\sigma_P, \mu_P, k_P, p_P\}$ is well-defined and belongs to B. Moreover P is B_P-process (c.f. [16.21]). In this section, we shall show that $B=B_P$. Set $H=H(P_{\alpha,\beta})=\{H^\alpha(z,\,d\xi)\}$.

[17.1] For f in $C_q(R)$, set $\phi=H^\alpha f$. Then ϕ, ϕ_x, ϕ_{xx} and ϕ_y are in $C(R)$ and it holds that

(17.5)
$$(\alpha\ m_0\ \phi_x)_x+m_0\phi_y-l_0\phi_x=0 \qquad \text{on } \partial_0\,.$$

Proof. By theorem [16.19] ϕ, ϕ_x, ϕ_{xx} and ϕ_y are in $C^2(R)$ and

(17.6)
$$\alpha\phi_{xx}+\beta\phi_x+\phi_y=0$$

holds on ∂_0. By (17.2) and (17.4)

$$(\alpha m_0 s_0)'=U_{B,x}(x,\,0)=m_0t_0+l_0s_0$$

and

$$\alpha s_0'+\beta s_0-t_0=0\,.$$

Eliminating t_0, we have

(17.7)
$$(\alpha m_0)'-\beta m_0-l_0=0\,.$$

Eliminating β from (17.6) and (17.7), we have (17.5).

[17.2]
$$\mu=\mu_P,\ k=k_P,\ m=m_P \ \text{ and } \ l=l_P\,,$$

Proof. For f in $C_q^2(R)$ set $\phi=H^\alpha f$. By [8.7], Green's formula and [17.1]

$$\int_0^{2\pi}(m(x,\,a)B_P^\alpha f(x)+l(x,\,a)f'(x))dx$$

$$=\int_0^{2\pi}(-m(x,\,a)\phi_y(x,\,a)+l(x,\,a)\phi_x(x,\,a))dx$$

$$=\int_0^{2\pi}(-m_0(x)\phi_y(x,\,0)+l_0(x)\phi_x(x,\,0)dx$$

$$=\int_0^{2\pi}(\alpha m_0\phi_x(x,\,0))_x dx=0\,.$$

Therefore by (3) in [8.17], we can see

$$m=m_P \quad \text{and} \quad l=l_P ,$$

and therefore $\mu=\mu_P$ and $k=k_P$.

[17.3]

$$\sigma=\sigma_P, \quad s=s_P \quad \text{and} \quad t=t_P .$$

Proof. Define u by $u_x=s$, $u_y=-t$ and $u(0, 1)=0$. Then u is harmonic in D, $u(x+2\pi, y)-u(x, y)=\int_0^{2\pi} s(x, y)dx=2\pi$ and $u_x=s>0$ in $\bar{D}$. By (17.4)

$$\alpha u_{xx}(x, 0)+\beta u_x(x, 0)+u_y(x, 0)$$
$$=\alpha s_0'+\beta s_0-t_0=0 .$$

Set $v=H^a u(\cdot, a)$. Since $v(\cdot, a)$ is in C_1, v, v_x, v_{xx} and v_y is in $C(\bar{D})$ and

$$\alpha v_{xx}(x, 0)+\beta v_x(x, 0)+v_y(x, 0)=0$$

holhs by theorem [16.19]. Since $w=u-v$ is harmonic in D^a and belongs to $C_p(\bar{D}^a)$ and $w=0$ on ∂_a, we have $w=0$ or $u=v$ by maximum principle. That is, u is in H_q. We have $u=u_P$ by theorem [9.5]. Therefore $s=s_P$, $t=t_P$ and $\sigma=\sigma_P$.

[17.4]

$$U_B=U_P \quad \text{and} \quad p=p_P .$$

Proof. Since U_P is a solution of (17.2), we have

$$U_P=U_B+C$$

for some constant C. Therefore U_P is in $C_P^\infty(\bar{D})$ and

$$U_P(x, 0)=\alpha m_0 s_0+C .$$

Set $\phi=H^a f$ for f in $C_p(R)$, and let V be any solution of

$$\text{(17.8)} \qquad \begin{cases} V_x=-m\phi_y+l\phi_x , \\ V_y=m\phi_x+l\phi_y . \end{cases}$$

Then V is in $C^1(\bar{D})$, and by [17.1]

$$V_x(x, 0)=-m_0\phi_y(x, 0)+l_0\phi_x(x, 0)$$
$$=(\alpha m_0\phi_x(x, 0))_x .$$

Therefore, for some constant C_1

$$V(x, 0)=\alpha m_0\phi_x(x, 0)+C_1 .$$

Since P is B_P-process, choosing a suitable constant C_1, we have by (7.1)

$$V(x, 0)s_0(x)=U_P(x, 0)\phi_x(x, 0)$$

or

$$C_1 s_0(x)=C\phi_x(x, 0). \tag{17.9}$$

Integrating the both sides from 0 to 2π, we have

$$2\pi C_1=0 \quad\text{and}\quad C\phi_x(x, 0)=0.$$

If $\phi_x(x, 0)\equiv 0$, then by (16.24) in theorem [16.19] $\phi_y(x, 0)\equiv 0$ and ϕ is a constant function. Therefore, choosing nonconstant f in $C_p(R)$, we may assume $\phi_x(x_0, 0)\neq 0$ for some point x_0. Then $C=0$. Therefore we have

$$U_B=U_P \quad\text{and}\quad p=p_P.$$

By [17.2], [17.3] and [17.4] we have proved $B=B_P$. Therefore we have the following theorem.

[17.5] THEOREM. *Let* $B=\{\sigma, \mu, k, p\}$ *in* $\mathfrak{B}$ *with the following properties be given:* $\sigma(dx)=s_0(x)dx$ *and* $\mu(dx)=m_0(x)dx$, s_0 *and* m_0 *are in* $C_p^\infty(R)$ *and positive and* $p>p_0(\sigma, \mu, k)$, *where* $p_0(\sigma, \mu, k)$ *is given by* (4.14). *Then, there exists a unique B-process P. Moreover* $P=P_{\alpha,\beta}$, *where* α *and* β *are defined by* (17.4).

[17.6] COROLLARY. *The B-process given in theorem* [17.5] *is in* $\mathfrak{P}_c$ *and satisfies* $[M]$, $[V_r]$ $(r=1, 2, \cdots)$, $[L]$ *and* $[C]$.

Proof. By theorem [16.19], $P=P_{\alpha,\beta}$ satisfies $[M]$, $[V_r]$ $(r=1, 2, \cdots)$ and $[L]$. Since $B=B_P$, σ and μ are in $M_i(R)$ and σ has no discrete mass, we see that P is in $\mathfrak{P}_c$ and satisfies $[C]$ (and $[H. C]$) by theorem [15.10].

§ 18. Existance of *B*-process (2): Case when σ and μ are in $M_i(R)$.

For P in $\mathfrak{P}$, set

$$M(a, b)=\sup_x \int H_b^a(x, d\xi)(\xi-x)^2 \tag{18.1}$$

as in § 15. The following lemma gıves another bound for $M(a, b)$ (cf. [15.2]).

[18.1] Let P in $\mathfrak{P}$ satisfy $[M]$ and $[V]$. Then fore $0<b<a$

$$M(a, b)\leqq C_1(a)p_P(a)+C_2(a),$$

where $C_1(a)$ and $C_2(a)$ are constants depending only on a and $p_P(a)$ is given in [10.14].

Proof.

$$s_P(x, a)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\sinh a}{\cosh a-\cos(\xi-x)}\sigma_P(d\xi)\geqq\frac{1}{2}\tanh a$$

and

$$|u_P(\xi, a)-u_P(x. a)|\geqq\tanh a|\xi-x|.$$

By [8.5] and theorem [10.12]

$$\begin{aligned}2\pi p_P(a)&=B_P^a(u(\cdot, a), u(\cdot, a))\\ &\geqq\int_0^{2\pi}m_P(x, a)dx\int Q^{a-b}H_b^a(x, d\xi)(u(\xi, a)-u(x, a))^2\\ &\geqq\frac{1}{4}(\tanh a)^2\int_0^{2\pi}m_P(x, a)dx\int Q^{a-b}H_b^a(x, d\xi)(\xi-x)^2\\ &\geqq\frac{1}{4}(\tanh a)^2\int_0^{2\pi}m_P(x, a)dx\int Q^{a-b}(x, d\eta)H_b^a(\eta, d\xi)\\ &\qquad\times\left\{\frac{1}{2}(\xi-\eta)^2-(\eta-x)^2\right\}\\ &\geqq\frac{1}{2}\pi(\tanh a)^2\left\{\frac{m(a, b)}{2(a-b)}-C_1(a-b)\right\},\end{aligned}$$

where $m(a, b)=\inf_x\int H_b^a(x, d\xi)(\xi-x)^2$ and C_1 is an absolute constant given in [16.7], (2). Therefore

$$m(a, b)\leqq a(\coth a)^2(4p_P(a)+C_2(a).$$

By [15.1], [18.1] is proved.

[18.2] Let $P_{(n)}$ $(n=1, 2, \cdots)$ in $\mathcal{P}$ satisfy [M] and [V]. Assume that $p_{P(n)}(a)\leqq k(a)<\infty$ for each $a>0$. Then there exist a subsequence $\{P(n')\}$ and P in $\mathcal{P}$ such that $P(n')\rightarrow P$ $(n'\rightarrow\infty)$. Moreover P satisfies [M] and [V].

Proof. Set ${}^nH=H(P(n))$. By [18.1], for $0<b<a$

$${}^nH_b^a(x: |\xi-x|\geqq N)\leqq\frac{1}{N^2}M(a, b)\leqq\frac{1}{N^2}(C_1(a)k(a)+C_2(a)).$$

Therefore, by proposition [2.8] we can find a subsequence $\{P(n')\}$ which converges to some P in $\mathcal{P}$. By definition of convergence in $\mathcal{P}$, P obviously satisfies [M]. Since

$$\begin{aligned}\int H_{Pb}^{a}(x, d\xi)\operatorname{Min}\{(\xi-x)^2, K\}&\leqq\lim_{n'\rightarrow\infty}\int{}^{n'}H_b^a(x, d\xi)(\xi-x)^2\\ &\leqq C_1(a)k(a)+C_2(a)\end{aligned}$$

for any positive K, P also satisfies [V].

As a corollary to [18.2], we have:

[18.3] Let $P(n)$ $(n=1, 2, \cdots)$ in $\mathcal{P}$ satisfy [M] and [V] with $p_{P(n)}(a) \leqq k(a) < \infty$. If $P(n) \to P$, then P satisfies [M] and [V].

[18.4] Let $P(n)$ $(n=1, 2, \cdots)$ and P be in $\mathcal{P}$, and assume $P(n) \to P$ $(n \to \infty)$. Set

$$^nB^a(x, d\xi) = B^a_{P(n)}(x, d\xi) \quad \text{and} \quad B^a(x, d\xi) = B^a_P(x, d\xi)$$

(cf. definition [8.12]).

(1) For $\phi(x, \xi)$ in $C_b(R \times R)$ with $|\phi(x, \xi)| \leqq K(\xi - x)^2$,

$$\int {}^nB^a(x, d\xi)\phi(x, \xi) \longrightarrow \int B^a(x, d\xi)\phi(x, \xi) \quad (n \to \infty) \tag{18.2}$$

boundedly in x for any fixed $a > 0$.

(2) For f in $C_b^2(R)$

$$^nB^a f(x) \longrightarrow B^a f(x) \quad (n \to \infty) \tag{18.3}$$

boundedly in x for any fixed $a > 0$.

(3) The measures $^nB^a(x, d\xi)$ $(n=1, 2, \cdots)$ converge to $B^a(x, d\xi)$ weakly on $R - \{x\}$.

Proof. For ϕ in $C_b(R \times R)$ with $|\phi(x, \xi)| \leqq K(\xi - x)^2$, by (8.7) in [8.5]

$$\int {}^nB^a(x, d\xi)|\phi(x, \xi)| \leqq K\int P^{a-c}(x, d\xi)(\xi - x)^2 + \|\phi\| Q^{a-c}(x, R)$$

$$\leqq K(a, c) < \infty$$

where c is some constant less than a. Therefore

$$\int {}^nB^a(x, d\xi)\phi(x, \xi) \quad (n=1, 2, \cdots)$$

are well-defined and bounded in n and x. Using (8.7) again, we have

$$\int {}^nB^a(x, d\xi)\phi(x, \xi) - \int B^a(x, d\xi)\phi(x, \xi)$$

$$= \int Q^{a-c}(x, d\eta)\int ({}^nH_c^a(\eta, d\xi) - H_c^a(\eta, d\xi))\phi(x, \xi),$$

where $^nH = H(P(n))$ and $H = H(P)$. Since $P(n) \to P$,

$$\int {}^nH_c^a(\eta, d\xi)\phi(x, \xi) \longrightarrow \int H_c^a(\eta, d\xi)\phi(x, \xi) \quad (n \to \infty)$$

boundedly in η. Hence (18.2) is proved. (18.3) can be proved in a similar way. (3) is obvious by (8.7).

Now, we shall define convergence in the space $\mathcal{L}$ of boundary conditions defined in § 4.

[18.5] DEFINITION. Let $\boldsymbol{B}(n)=\{\sigma_n, \mu_n, k_n, p_n\}$ $(n=0, 1, 2, \cdots)$ be in $\mathcal{B}$. We shall write

$$\boldsymbol{B}(n) \longrightarrow B(0) \qquad (n\rightarrow\infty)$$

if and only if:

(1) $\sigma_n\rightarrow\sigma_0$ and $\mu_n\rightarrow\mu_0$ in the weak sense as measures on the torus $R/(2\pi)$.

(2) $k_n\rightarrow k_0$, $p_n\rightarrow p_0$ and $p_n(a)\rightarrow p_0(a)$ for any $a>0$, where

$$p_n(a)=p(B(n))(a)=\int_0^{2\pi} U(B(n))(x, a)\, s(B(n))(x, a)dx\,.$$

[18.6] If $B(n)\rightarrow B$ $(n\rightarrow\infty)$, then

$$s(B(n)) \longrightarrow s(B), \quad t(B(n)) \longrightarrow t(B), \quad l(B(n)) \longrightarrow l(B),$$

$$m(B(n)) \longrightarrow m(B) \quad \text{and} \quad u(B(n)) \longrightarrow u(B) \qquad (n\rightarrow\infty)$$

uniformly in $D^{[b,a]}$ for any $0<b<a$.

Proof. Noting that $s(B(n))$, $t(B(n))$, $l(B(n))$ and $m(B(n))$ $(n=1, 2, \cdots)$ are harmonic functions in $C_p(D)$, and $u(B(n))$ $(n=1, 2, \cdots)$ are harmonic functions in $C_q(D)$ with $u(B(n))(z+2\pi)-u(B(n))(z)=2\pi$, we can easily show [18.] by definitions.

[18.7] Let P in $\mathcal{P}$ satisfy [M] and [V]. Then

$$\int_0^{2\pi} m_P(x, a)dx\int B_P^a(x, d\xi)(\xi-x)^2 \leqq 4(\coth a)^2 p_P(a). \tag{18.4}$$

Moreover, if P is in $\mathcal{P}_c$ for any $M>11\pi$

$$\int_0^{2\pi} m_P(x, a)dx\int_{|\xi-x|\geqq M} B_P^a(x, d\xi)(\xi-x)^2 \leqq \frac{C a p_P(a)^2}{M}, \tag{18.5}$$

where C is an absolute constant.

Proof. Since $s_P(x, a)\geqq \operatorname{Min}_x h_\xi(x, a)\geqq (1/2)\tanh a$,

$$\int_0^{2\pi} m_P(x, a)dx\int B^a(x, d\xi)(\xi-x)^2$$

$$\leqq \frac{1}{\operatorname{Min}_x s_P(x, a)^2}\int_0^{2\pi} m_P(x, a)dx\int B^a(x, d\xi)(u_P(\xi, a)-u_P(x, a))^2$$

$$\leqq 4(\coth a)^2 p_P(a)\,.$$

If P is in $\mathcal{P}_c$, set $\varepsilon=\pi$ and $\alpha=N\pi$ in [14.7]. Then

$$\int_0^{2\pi} m_P(x,\ a)dx\int_{|\xi-x|\geqq(3N+8)\pi} B_P^a(x,\ d\xi)\leqq\frac{a p_P(a)^2}{2\pi^5 N^4}.$$

Therefore, for $(3N+8)\pi<M\leqq(3N+11)\pi$ ($N=1, 2, \cdots$)

$$\int_0^{2\pi} m_P(x,\ a)dx\int_{|\xi-x|\geqq M} B_P^a(x,\ d\xi)(\xi-x)^2\leqq C' a p_P(a)^2\sum_{k\geqq N}\frac{(3k+11\pi)^2}{k^4}$$

$$\leqq\frac{C''}{N}a p_P(a)^2\leqq\frac{C}{M}a p_P(a)^2.$$

[18.8] Let $P(n)$ ($n=1, 2, \cdots$) in $\mathcal{P}_c$ satisfy [M] and [V]. Set $m_n=m_{p(n)}$ and ${}^nB^a(x,\ d\xi)=B_{p(n)}^a(x,\ d\xi)$. Assume that $P(n)\to P$ in $\mathcal{P}$, $m_n\to m_P$ and $\{p_{p(n)}(a)\}$ converges ($n\to\infty$). If ϕ in $C(R\times R)$, which is not necessarily bounded, satisfies

$$|\phi(x,\ \xi)|\leqq K(\xi-x)^2, \tag{18.6}$$

then for $a>0$ it holds that

$$\int_0^{2\pi} m_n(x,\ a)dx\int {}^nB^a(x,\ d\xi)\phi(x,\ \xi)\longrightarrow \tag{18.7}$$

$$\int_0^{2\pi} m_P(x,\ a)dx\int B_P^a(x,\ d\xi)\phi(x,\ \xi)\qquad(n\to\infty).$$

Proof. If ϕ is bounded, then (18.7) is obvious by [18.4], since $m_n(x,\ a)\to m_P(x,\ a)$ uniformly in x for fixed a. For general ϕ, we may assume ϕ is non-negative. Set

$$\phi_M=\mathrm{Min}\ \{KM^2,\ \phi\}$$

for positive M with $M>11\pi$. By (18.6), we can see

$$\phi_M(x,\ \xi)=\phi(x,\ \xi)\qquad\text{if }|\xi-x|\leqq M.$$

Therefore by [18.7]

$$\int_0^{2\pi} m_n(x,\ a)dx\int {}^nB^a(x,\ d\xi)(\phi-\phi_M)(x,\ \xi)$$

$$\leqq K\int_0^{2\pi} m_n(x,\ a)dx\int_{|\xi-x|>M} {}^nB^a(x,\ d\xi)(\xi-x)^2$$

$$\leqq\frac{KC_a k(a)^2}{M},$$

where $k(a)=\sup_n p_{p(n)}(a)$ is finite since $\{p_{p(n)}(a)\}$ converges. Therefore

$$\lim_{n\to\infty}\int m_n(x,\ a)dx\int {}^nB^a(x,\ d\xi)\phi_M(x,\ \xi)$$

$$=\int m_P(x,\ a)dx\int B_P^a(x,\ d\xi)\phi_M(x,\ \xi)$$

$$\leqq \varliminf_{n\to\infty}\int m_n(x,\ a)dx\int {}^nB^a(x,\ d\xi)\phi(x,\ \xi)$$

$$\leqq \varlimsup_{n\to\infty}\int m_n(x,\ a)dx\int {}^nB^a(x,\ d\xi)\phi(x,\ \xi)$$

$$\leqq \lim_{n\to\infty}\int m_n(x,\ a)dx\int {}^nB^a(x,\ d\xi)\phi_M(x,\ \xi)+\frac{KC_a k(a)}{M}.$$

Since we can take M arbitrarily large, [18.8] is proved.

[18.9] Under the same assumption as in [18.8], let f_n and g_n $(n=0,1,2,\cdots)$ in $C^1(R)$ satisfy

$$\|f_n'\|\leqq K, \qquad \|g_n'\|\leqq K \tag{18.8}$$

and

$$\|f_n'-f_0'\| \longrightarrow 0, \qquad \|g_n'-g_0'\| \longrightarrow 0 \qquad (n\to\infty) \tag{18.9}$$

Then $B_{p(n)}^a(f_n,\ g_n)\to B_P^a(f_0,\ g_0)$ $(n\to\infty)$ (See notation [10.2]).

Proof. Set $p_n(a)=p_{P(n)}(a)$, $m_n=m_{P(n)}$, ${}^nB^a(x,\ d\xi)=B_{P(n)}^a(x,\ d\xi)$ and

$$\phi_n(x,\ \xi)=\rho_{f_n,g_n}(x,\ \xi)=\int_x^\xi g_n'(t)dt\int_x^t f_n'(s)ds\,.$$

Then

$$|\phi_n(x,\ \xi)-\phi_0(x,\ \xi)|\leqq\frac{K}{2}(\|f_n'-f_0'\|+\|g_n'-g_0'\|)(\xi-x)^2\,.$$

Therefore, by (18.4) in [18.7]

$$|B_{p(n)}^a(f_n,\ g_n)-B_{p(n)}^a(f_0,\ g_0)|$$

$$=\left|\int_0^{2\pi} m_n(x,\ a)dx\int {}^nB^a(x,\ d\xi)(\phi_n-\phi_0)(x,\ \xi)\right|$$

$$\leqq 2K(\|f_n'-f_0'\|+\|g_n'-g_0'\|)(\coth a)^2 p_n(a)\,.$$

Since $\{p_n(a)\}$ converges, the right side of the above inequality converges to zero. On the other hand, since $|\phi_0(x,\ \xi)|\leqq(K^2/2)(\xi-x)^2$, by [18.8]

$$\lim B^{a}_{P(n)}(f_0, g_0) = \lim \int_0^{2\pi} m_n(x, a)dx \int {}^nB^a(x, d\xi)\phi_0(x, \xi)$$

$$= \int_0^{2\pi} m_P(x, a) \int B^a_P(x, d\xi)\phi_0(x, \xi)$$

$$= B^a_P(f_0, g_0).$$

Hence [18.9] is proved.

[18.10] LEMMA. *Let $P(n)$ $(n=1, 2, \cdots)$ in $\mathcal{P}_c$ satisfy [M] and [V]. Assume $B_{P(n)} \to B$ in $\mathcal{P}$ and $P(n) \to P$ in $\mathcal{P}$. Then $B = B_P$.*

Proof. Since $p_n(a) = p_{P(n)}(a) \to p_B(a)$, it holds that $k(a) = \sup_n p_n(a) < \infty$. Therefore by [8.3] P satisfies [M] and [V].

1° Set ${}^nH = H(P(n))$, $H = H(P)$, $u_n = u_{p(n)}$ and $u = u(B)$. Since by [18.1]

$$\int {}^nH^a_b(x, d\xi)(\xi - x)^2 \leqq C_1(a)k(a) + C_2(a) < \infty$$

for $0 < b < a$ and by [18.6] $\{u_n(x, a)\}$ converges to $u(x, a)$ uniformly in x,

$$u(x, b) = \lim u_n(x, b) = \lim {}^nH^a_b u_n(\cdot, a)(x) = H^a_b u(\cdot, a)(x).$$

It is obviou that $u(0, 1) = 0$ and $u(z + 2\pi) - u(z) = 2\pi$. By theorem [9.5] we have $u = u_P$. Therefore $s(B) = s_P$, $t(B) = t_P$, $\sigma_B = \sigma_P$ and $k_B = k_P$ also hold by definition.

2° Set $m_n = m_{P(n)}$ and $m = m(B)$. By [8.12] for any f in $C^2_p(R)$

$$\int^{2\pi} m_n(x, a)(P + {}^nB^a)f(x)dx = 0,$$

where ${}^nB^a(x, d\xi) = B^a_{P(n)}(x, d\xi)$. Since by [18.6] $\{m_n(x, a)\}$ converges to $m(x, a)$ uniformly in x and by [18.4] $\{{}^nB^a f(x)\}$ converges to $B^a_P f(x)$ boundedly in x, we have

$$\int_0^{2\pi} m(x, a)(P + B^a_P)f(x)dx = 0.$$

It is clear that $\int_0^{2\pi} m(x, a)dx = 2\pi$. By [18.12] we have $m_B = m_P$ and $\mu_B = \mu_P$.

3° Set $s_n = s_{P(n)}$ and $s = s(B)$.

$$\|u'_n(\cdot, a)\| = \|s_n(\cdot, a)\| \leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{\sinh a}{\cosh a - 1}\sigma_{p(n)}(dx)$$

$$\leqq \frac{\sinh a}{\cosh a - 1},$$

and by [18.6]

$$\|u_n'(\cdot, a)-u'(\cdot, a)\| \leqq \|s_n(\cdot, a)-s(\cdot, a)\| \longrightarrow 0 \qquad (n\to\infty).$$

Therefore, by [18.9]

$$p_B(a)=\lim_{n\to\infty} p_n(a)=\lim B^a_{P(n)}(u_n(\cdot, a), u_n(\cdot, a))$$
$$=B^a_P(u(\cdot, a), u(\cdot, a))=p_P(a)$$

and $p_B=\inf_{a>0} p_B(a)=\operatorname{idf}_{a>0} p_P(a)=p_P$. By 1°, 2° and 3° we have proved that $B=B_P$.

[18.11] PROPOSITION. *Let $P(n)$ $(n=1, 2, \cdots)$ in $\mathfrak{P}_c$ satisfy [M] and [V]. Assume that $B_{P(n)}\to B$ $(n\to\infty)$ for some $B=\{\sigma, \mu, k, p\}$ in $\mathfrak{B}$ with σ and μ in $M_i(R)$. Then $P(n)\to P$ $(n\to\infty)$ for some P in $\mathfrak{P}$. P is a B-process and $B=B_P$. P satisfies [M], [V] and [L].*

Proof. 1° Since $p_n(a)=p_{P(n)}(a)\to p_B(a)$ $(n\to\infty)$, it holds that $k(a)=\sup_n p_n(a)<\infty$. Therefore, by [8.2], for any subsequence of $\{P(n)\}$, we can choose a subsequence $\{P(n_r)\}$ such that $P(n_r)\to P$ as $r\to\infty$ for some P in $\mathfrak{P}$ and P satisfies [M] and [V]. By [18.10], $B=B_P$. In abbreviation, we shall write $P(r)=P(n_r)$, $\sigma_r=\sigma_{P(r)}$, $\mu_r=\mu_{P(r)}$, $m_r=m_{P(r)}$, $m=m_P$, ${}^rB^a(x, d\xi)=B^a_{P(r)}(x, d\xi)$, $B^a(x, d\xi)=B^a_P(x, d\xi)$ and $p_r(a)=p_{P(r)}(a)$.

2° For ρ in $M_p(R)$, set $\delta(\rho, \varepsilon)=\sup_x \rho((x-\varepsilon, x+\varepsilon))$. Since σ and μ are in $M_i(R)$ and $\mu_r\to\mu$ and $\sigma_r\to\sigma$ weakly, we have for any $\varepsilon>0$

$$\varliminf_{r\to\infty} \delta(\mu_r, \varepsilon)\geqq\delta\left(\sigma, \frac{\varepsilon}{2}\right)>0$$

and

$$\varliminf_{r\to\infty} \delta(\sigma_r, \varepsilon)\geqq\delta\left(\sigma, \frac{\varepsilon}{2}\right)>0.$$

Therefore we may assume

$$\delta(\sigma_r, \varepsilon), \quad \delta(\mu_r, \varepsilon)\geqq\delta_0=\delta_0(\varepsilon)>0.$$

Therefore by [14.7]

$$\int_0^{2\pi} m_r(x, a)^r B^a(x, U^c_{1\varepsilon}(x))dx\leqq\frac{16ap_r(a)^2}{\delta_0^5},$$

and by (3) in [18.4]

$$\varliminf_{r\to\infty} {}^rB^a(x, U^c_{1\varepsilon}(x))\geqq B^a_P(x, U^c_{2\varepsilon}(x)).$$

$$\int_0^{2\pi} m(x, a)B^a_P(x, U^c_{2\varepsilon}(x))dx$$

$$\leqq\varliminf_{r\to\infty}\int_0^{2\pi} m_r(x, a)^r B^a(x, U^c_{1\varepsilon}(x))\leqq\frac{16ap_P(a)^2}{\delta_0^5}$$

for $p_P(a)=p(B)(a)=\lim p_r(a)$. Since $p_P(a)$ is an increasing function in a, we have

$$\lim_{a\to 0}\int_0^{2\pi} m(x,\ a)B_P^a(x,\ U_{12\varepsilon}^c(x))=0.$$

On the other hand, by [18.8] for $M>12\pi$

$$\int_0^{2\pi} m(x,\ a)dx\int_{|\xi-x|\geqq M} B^a(x,\ d\xi)(\xi-x)^2$$

$$\leqq \lim_{r\to\infty}\int_0^{2\pi} m_r(x,\ a)dx\int_{|\xi-x|\geqq M-\pi} {}^rB^a(x,\ d\xi)(\xi-x)^2.$$

Since $P(r)$ $(r=1,\ 2,\ \cdots)$ are in $\mathfrak{P}_c$, by [18.7]

$$\int_0^{2\pi} m_r(x,\ a)dx\int_{|\xi-x|\geqq M-\pi} {}^rB^a(x,\ d\xi)(\xi-x)^2\leqq \frac{Cap_r(a)^2}{M-\pi}$$

and therefore

$$\int_0^{2\pi} m(x,\ a)dx\int_{|\xi-x|\geqq M} B^a(x,\ d\xi)(\xi-x)^2\leqq \frac{Cap_P(a)^2}{M-\pi}$$

and the right side converges to 0 as $a\to 0$. Finally we have

$$\lim_{a\to 0}\int_0^{2\pi} m(x,\ a)\int_{|\xi-x|\geqq 12\varepsilon} B^a(x,\ d\xi)(\xi-x)^2=0$$

for any positive ε and P satisfies [L*].

3° Since P satisfies [M], [V] and [L] and moreover $B\to B_P$ holds, P is B-process by theorem [11.7]. Therefore by uniqueness of B-process (cf. theorem [7.7]) P is independent of the subsequence $\{P(r)\}=\{P(n_r)\}$. Hence

$$P(n)\longrightarrow P \qquad (n\to\infty).$$

Proposition [18.11] is proved.

[18.12] THEOREM. *Let $B=\{\sigma,\ \mu,\ k,\ p\}$ in $\mathfrak{B}$ be given. If σ and μ are in $M_i(R)$, there exists a unique B-process P such that P satisfies* [M], [V] *and* [L] *and $B=B_P$. Moreover P is in $\mathfrak{P}_c$.*

Proof. Set $s=s(B)$, $t=t(B)$, $m=m(B)$, $l=l(B)$ and $U=U(B)$. Define $\sigma_a(dx)=s(x,\ a)dx$, $\mu_a(dx)=m(x,\ a)dx$, $k_a=k$ and

$$p_a=\frac{1}{2\pi}\int_0^{2\pi} s(x,\ a)U(x,\ a)dx=p_B(a).$$

Then, $U_a(z)=U(x,\ y+a)$ is a positive solution of

$$\text{(18.10)}\qquad \begin{cases} (U_a)_x=m_at_a+l_as_a \\ (U_a)_y=m_as_a-l_at_a \end{cases}$$

in D, where

$$m_a(z)=m(x,\ y+a)=\int_{[0,2\pi)}\tilde{h}_\xi(z)\mu_a(d\xi)\ ,$$

$$l_a(z)=l(x,\ y+a)=\int_{[0,2\pi)}\tilde{k}_\xi(z)\mu_a(d\xi)-k\ ,$$

$$s_a(z)=s(x,\ y+a)=\int_{[0,2\pi)}\tilde{h}_\xi(z)\sigma_a(d\xi)\ ,$$

$$t_a(z)=t(x,\ y+a)=\int_{[0,2\pi)}\tilde{k}_\xi(z)\sigma_a(d\xi)+k\ ,$$

Noting $p_a(b)=\frac{1}{2\pi}\int_0^{2\pi}s_a(x,\ b)U_a(x,\ b)dx=p_B(a+b)$ for $b>0$ and $p_a=\inf_{b>0}p_a(b)$, we can see $B_a=\{\sigma_a,\ \mu_a,\ k_a,\ p_a\}$ is in $\mathcal{B}$. By representation of U in [5.9] and [5.10] we have $\lim_{y\to\infty}U(z)=\infty$, therefore $\operatorname{Inf}_{z\in D}U_a(z)=\inf_{y=a}U(x,\ a)>0$ and U_a is greater than the minimum nonnegative solution of (18.10) or $p_a>p(\sigma_a,\ \mu_a,\ k_a)$. Hence B_a satisfies the conditions in theorem [17.5] and there exists a process P_a with $B_{P_a}=B_a$. By [17.6] P_a satisfies $[M]$ and $[V]$ and is in $\mathcal{P}_c$. Noting $p_{P_a}(b)=p_a(b)=p_B(a+b)$, we can easily show $B_a\to B$ as $a\to 0$. Therefore, by proposition [18.11], we can show existence of B-process P which satisfies $[M]$, $[V]$ and $[L]$, since μ and σ are in $M_i(R)$. Uniqueness is obvious by theorem [7.7]. By theorem [14.9] we can see P is in $\mathcal{P}_c$.

§ 19. Existence of B-process (3): General case.

Let σ_i and μ_j $(i,\ j=0,\ 1)$ be in $M_p(R)$ and k be a constant. Assume that $B_{ij}=\{\sigma_i,\ \mu_j,\ k,\ p_{ij}\}$ is in $\mathcal{L}$. Set, for $0\leqq\lambda\leqq 1$,

$$\mu_\lambda=(1-\lambda)\mu_0+\lambda\mu_1\ ,\qquad \sigma_\lambda=(1-\lambda)\sigma_0+\lambda\sigma_1\ ,$$

$$s_\lambda=\int_{[0,2\pi)}\tilde{h}_\xi(z)\sigma_\lambda(d\xi),$$

$$t_\lambda=\int_{[0,2\pi)}\tilde{k}_\xi(z)\sigma_\lambda(d\xi)+k\ ,$$

$$m_\lambda=\int_{[0,2\pi)}\tilde{h}_\xi(z)\mu_\lambda(d\xi)$$

and

$$l_\lambda=\int_{[0,2\pi)}\tilde{k}_\xi(z)\mu_\lambda(d\xi)+k$$

Set

(19.1) $$U^\lambda\equiv U(\lambda:\ B_{ij})=(1-\lambda)^2U_{00}+\lambda(1-\lambda)(U_{01}+U_{10})+\lambda^2U_{11}\ .$$

where $U_{ij}=U(B_{ij})$. Then U^λ is a nonnegative solution of

$$\begin{cases} U_x^\lambda=m_\lambda t_\lambda+l_\lambda s_\lambda\,, \\ U_y^\lambda=m_\lambda s_\lambda-l_\lambda t_\lambda\,. \end{cases} \tag{19.2}$$

Therefore

$$B^\lambda=B(\lambda\,;B_{ij})=\{\sigma_\lambda,\ \mu_\lambda,\ k,\ p_\lambda\} \tag{19.3}$$

is in $\mathcal{L}$ $(0\leqq\lambda\leqq1)$, where $p_\lambda=\inf_{a>0}\int U^\lambda(x,\ a)s_\lambda(x,\ a)dx$, and $U^\lambda=U(B^\lambda)$.

In the following, we shall choose $r\in[0,\ 2\pi)$ so that

$$\sigma_i(\{r\})=\mu_j(\{r\})=0 \qquad (i,\ j=0,\ 1)\,. \tag{19.4}$$

Set $I(r)=[r,\ r+2\pi]$ and

$$F_r(x,\ \alpha)=\int_{I(r)}F(x,\ \xi)\alpha(d\xi) \tag{19.5}$$

$$F_r(\alpha,\ \beta)=\int_{I(r)^2}F(x,\ \xi)\alpha(dx)\beta(d\xi) \tag{19.6}$$

for locally bounded signed measures α and β on R, where $F(x,\ \xi)$ is defined by (5.3). Since

$$\int_{[0,2\pi)}F(x,\ \xi)\rho(d\xi)-\int_{[r,r+2\pi]}F(x,\ \xi+r)\rho(d\xi)=\rho([0,\ r))$$

for any periodic measure ρ, the representation of U^λ given in [5.13] and [5.14] has the following form;

$$U^\lambda(z)=\int_0^{2\pi}\tilde{h}_\xi(z)U_0^\lambda(\xi)d\xi+(1+k^2)y\,, \tag{19.7}$$

where

$$\begin{cases} U_0^\lambda(z)=-T_0(x,\ \sigma_\lambda,\ \mu_\lambda)+kF_r(x,\ \mu_\lambda-\sigma_\lambda)+C_{r\lambda}\,, \\ T_0(x,\ \sigma_\lambda,\ \mu_\lambda)=\int_{I(r)^2}T_0^*(x,\ \xi,\ \eta)\sigma_\lambda(d\xi)\mu_\lambda(d\eta)\,, \\ T_0^*(x,\ \xi,\ \eta)=\begin{cases} T_0(x,\ \xi,\ \eta) & \text{if } \xi\neq\eta\,, \\ 0 & \text{if } \xi=\eta \end{cases} \end{cases} \tag{19.8}$$

and $T_0(x,\ \xi,\ \eta)$ is given by (5) in [5.5]. Noting [5.14], $T_0(x,\ \sigma_i,\ \mu_j)$ and $F_r(x,\ \mu_j-\sigma_i)$ are bounded in x $(i,\ j=1,\ 2)$. Therefore we can easily see:

[19.1]
$$T_0(x,\ \sigma_\lambda,\ \mu_\lambda)\longrightarrow T_0(x,\ \sigma_0,\ \mu_0)\,,$$
$$F_r(x,\ \sigma_\lambda,\ \mu_\lambda)\longrightarrow F_r(x,\ \sigma_0,\ \mu_0)\,,$$
$$U_0^\lambda(x)\longrightarrow U_0^0(x)=(U_{00})_0(x)$$

as $\lambda \to 0$ uniformly in x.

We shall note the following elementary lemma without proof.

[19.2] LEMMA. *Let K be a compact space in R^d and let a and α_n be bounded measures on K, and β and β_n be signed measures on K with $d|\beta_n| \leqq C d\alpha_n$ ($n=1, 2, \cdots$). Assume that $\alpha_n \to \alpha$ and $\beta_n \to \beta$ in the weak sense, then $d|\beta| \leqq C d\alpha$. Moreover, let A be a closed subset of K with $\sigma(A)=0$ and g be a bounded measurable function on K which is continuous except at point in A. Then*

$$\int g d\rho_n \longrightarrow \int g d\rho \qquad (n \to \infty).$$

[19.3] Let α_λ, β_λ and γ_λ ($0 \leqq \lambda \leqq 1$) be signed periodic measures on R with $|\alpha_\lambda|(dx)$, $|\beta_\lambda|(dx) \leqq K\sigma_\lambda(dx)$ and $|\gamma_\lambda|(dx) \leqq K\mu_\lambda(dx)$ ($0<K<\infty$). Assume that $\alpha_\lambda \to \alpha_0$, $\beta_\lambda \to \beta_0$ and $\gamma_\lambda \to \gamma_0$ in the weak sense as $\lambda \to 0$. Then, for any f in $C_p(R)$,

(1) $T_0(f\cdot\alpha_\lambda, \beta_\lambda, \gamma_\lambda)+T_0(f\cdot\beta_\lambda, \alpha_\lambda, \gamma_\lambda) \longrightarrow$

$$T_0(f\cdot\alpha_0, \beta_0, \gamma_0)+T_0(f\cdot\beta_0, \alpha_0, \gamma_0),$$

(2) $F_r(\alpha_\lambda, \gamma_\lambda) \longrightarrow F_r(\alpha_0, \gamma_0)$,

(3) $F_r(f\cdot\alpha_\lambda, \beta_\lambda)+F_r(f\cdot\beta_\lambda, \alpha_\lambda) \longrightarrow F_r(f\cdot\alpha_0, \beta_0)+F_r(f\cdot\beta_0, \alpha_0)$

as $\alpha \to 0$. Where

$$T_0(\alpha, \beta, \gamma)=\iiint_{x, \xi, \eta \in [0, 2\pi)} T_0^*(x, \xi, \eta)\alpha(dx)\beta(d\xi)\gamma(d\eta),$$

$$F_r(\alpha, \beta)=\iint_{x, \xi \in [r, r+2\pi)} F(x, \xi)\alpha(dx)\beta(d\xi).$$

Proof. Set

$$\tilde{T}_0(x, \xi, \eta)=T_0^*(x, \xi, \eta)+T_0^*(\xi, x, \eta).$$

$$T_0^N(x, \xi, \eta)=\mathrm{Min}\{N, T_0^*(x, \xi, \eta)\},$$

$$\tilde{T}_0^N(x, \xi, \eta)=\mathrm{Min}\{N, \tilde{T}_0(x, \xi, \eta)\}.$$

Then, by definition (cf. [5.3] and [5.5]), it holds that for x, ξ, η in $(r, r+2\pi)$

(i) $T_0^N(x, \xi, \eta)$ is bounded and continuous except $\{x=\xi\} \cup \{x=\eta\}$, and

(ii) $\tilde{T}_0^N(x, \xi, \eta)$ is bounded and continuous except $\{x=\eta\} \cup \{\xi=\eta\}$.

Set $I(r)=[r, r+2\pi]$ and $\rho_\lambda(dx, d\xi, d\eta)=\alpha_\lambda(dx)\beta_\lambda(d\xi)\gamma_\lambda(d\eta)$. Since $T_0(x, \xi, \eta)$ is periodic in x, ξ and η, by (19.4)

$$J_\lambda = T_0(f\cdot\alpha_\lambda, \beta_\lambda, \gamma_\lambda) + T_0(f\cdot\beta_\lambda, \alpha_\lambda, \gamma_\lambda)$$

$$= \int_{I(r)^3} \tilde{T}_0(x, \xi, \eta) f(\xi) d\rho_\lambda + \int_{I(r)^3} T_0(x, \xi, \eta)(f(x)-f(\xi)) d\rho_\lambda = J_\lambda^N + C_\lambda^N$$

where $\alpha\rho_\lambda = \alpha_\lambda(dx)\beta_\lambda(d\xi)\gamma_\lambda(d\eta)$

$$J_\lambda^N = \int_{I(r)^3} \tilde{T}_0^N(x, \xi, \eta) f(\xi) d\rho_\lambda + \int_{I(r)^3} T_0^N(x, \xi, \eta)(f(x)-f(\xi)) d\rho_\lambda$$

and

$$C_\lambda^N = \int_{I(r)^3} (\tilde{T}_0 - \tilde{T}_0^N)(x, \xi, \eta) f(\xi) d\rho_\lambda$$

$$+ \int_{I(r)^3} (T_0 - T_0^N)(x, \xi, \eta)(f(x)-f(\xi)) d\rho_\lambda .$$

By assumption and condition [P] in [5.11] γ_λ has no common mass with α_λ and β_λ. Therefore by (i) and (ii), using [19.2], we have

$$J_\lambda^N \longrightarrow J_0^N \qquad \text{as } \lambda \to 0 .$$

On the other hand by assumption

$$|C_N(\lambda)| \leqq 4\|f\| K^3(T_0 - T_0^{N/2})(\sigma_\lambda, \sigma_\lambda, \mu_\lambda)$$

and therefore by [19.1]

$$\overline{\lim_{\lambda\to 0}} |C_N(\lambda) \leqq 4\|f\| K^3(T_0 - T_0^{N/2})(\sigma_0, \sigma_0, \mu_0) .$$

Since $T_0^{N/2} \uparrow T_0$, we have proved (1). For x and ξ in $(r, r+2\pi)$ it holds that

(iii) $F(x, \xi)$ is bounded and continuous except $\{x=\xi\}$.

(iv) $F(x, \xi) + F(\xi, x) = 1$.

Then

$$F_r(f\cdot\alpha_\lambda, \beta_\lambda) + F_r(f\cdot\beta_\lambda, \alpha_\lambda)$$

$$= \int_{I(r)^2} f(\xi)\alpha_\lambda(dx)\beta_\lambda(d\xi)$$

$$+ \int_{I(r)^2} F(x, \xi)(f(x)-f(\xi))\alpha_\lambda(dx)\beta_\lambda(d\xi) .$$

In a way similar to (1), we can easily show (2) and (3).

To proceed from [19.5] to [19.10], we shall impose the following temporary assumption.

[19.4] Assumption. For a positive sequence λ_n with $\lambda_n \to 0$, f in $C_{p,N}(R)$ and a positive constant a, assume:

(1) For each n, $B_N^{\lambda_n}$-solution ϕ_{λ_n} for f in D^a exists.

(2) $\|\phi_{\lambda_n}\| \leqq K_1$ and $\lim_{n\to\infty} \phi_{\lambda_n}(z)=\phi_0(z)$ exists.

(3) $|\sigma_{\phi_{\lambda_n}}|(x) \leqq K_2 \sigma_{\lambda_n}(dx)$.

Here K_1 and K_2 are positive constants independent of n and $\sigma_{\phi_{\lambda_n}}$ is the boundary measures of ϕ_{λ_n} defined in [4.15].

We shall write $B^n=B^{\lambda_n}$, $\sigma_n=\sigma_{\lambda_n}$, $\mu_n=\mu_{\lambda_n}$, $U^n=U^{\lambda_n}$, $\phi_n=\phi_{\lambda_n}$ and etc. Noting $l_n=l_{\lambda_n}\to l_0$ and $m_n=m_{\lambda_n}\to m_0$ $(n\to\infty)$, we can easily have:

[19.5] Under [19.4], $\phi_0(z)$ in (2) belongs to $D^a_{p,N}(B^0)$ which is defined in [4.13]. The boundary measure σ_{ϕ_0} of $\phi_0(z)$ satisfies that $\sigma_{\phi_n}\to\sigma_{\phi_0}$ $(n\to\infty)$ in the weak sense and $|\sigma_{\phi_0}|(dx) \leqq K_2\sigma_0(dx)$.

[19.6] Let f be in $C_p(R)$ and assume [19.4] for $N=1$. As in [5.17], set

$$\varphi_n(z)=(\phi_n)_y(z)+\int_{[0,2\pi)} \tilde{k}_\xi(z)\sigma_{\phi_n}(d\xi) \qquad (n=1, 2, \cdots)$$

and

$$\varphi_0(z)=(\phi_0)_y(z)+\int_{[0,2\pi)} \tilde{k}_\xi(z)\sigma_{\phi_0}(d\xi).$$

Let φ_n^0 $(n=1, 2, \cdots)$ and φ_0^0 be their boundary functions on ∂_0. Then

$$\varphi_n^0(x) \longrightarrow \varphi_0^0(x) \text{ nuiformly in } x.$$

Proof. $\lim_{n\to\infty} \varphi_n(z)=\varphi_0(z)$ in D^a. Set

$$g_n(z)=(\phi_n)_x(z)-\int_{[0,2\pi)} \tilde{h}_\xi(z)\sigma_{\phi_n}(d\xi).$$

Then g_n is a harmonic conjugate of φ_n and can be extended to the harmonic function $\tilde{g}_n$ on $\{z=(x, y): -a<y<a\}$. Moreover $g_n(z)$ also converges in D^a and $\tilde{g}_n$ $(n=1, 2, \cdots)$ are uniformly bounded in $\{z=(x, y): -b>y<b\}$ for any fixed b with $0<b<a$. Noting $\varphi_n(z)$ is periodic in x, we can easily show [19.6].

[19.7] Under the same assumption as in [19.6], it holds that, for any g in $C_p(R)$,

(1) $T_0(g\cdot\sigma_n, \sigma_{\phi_n}, \mu_n) \longrightarrow T_0(g\cdot\sigma_0, \sigma_{\phi_0}, \mu_0)$

(2) $F_r(g\cdot\sigma_n, k\sigma_{\phi_n}+\varphi_n^0\cdot\mu_n) \longrightarrow F_r(g\cdot\sigma_0, k\sigma_{\phi_0}+\varphi_0^0\cdot\mu_0) \qquad (n\to\infty)$

Proof. By [19.1], it is easily shown that

$$T_0(g\cdot\sigma_{\phi_n}, \sigma_n, \mu_n) \longrightarrow T_0(g\cdot\sigma_{\phi_0}, \sigma_0, \mu_0), \tag{19.9}$$

$$F_r(g\cdot\sigma_{\phi_n}, \sigma_n) \longrightarrow F_r(g\cdot\sigma_{\phi_0}, \sigma_0) \qquad (n\to\infty). \tag{19.10}$$

On the other hand by [19.3]

(19.11) $$T_0(g\cdot\sigma_{\phi_n}, \sigma_n, \mu_n)+T_0(g\cdot\sigma_n, \sigma_{\phi_n}, \mu_n)$$

$$\longrightarrow T_0(g\cdot\sigma_{\phi_0}, \sigma_0, \mu_0)+T_0(g\cdot\sigma_0, \sigma_{\phi_0}, \mu_0)\,,$$

(19.12) $$F_r(g\cdot\sigma_{\phi_n}, \sigma_n)+F_r(g\cdot\sigma_n, \sigma_{\phi_n})$$

$$\longrightarrow F_r(g\cdot\sigma_{\phi_0}, \sigma_0)+F_r(g\cdot\sigma_0, \sigma_{\phi_0})$$

and

(19.13) $$F_r(g\cdot\sigma_n, \psi_n^0\cdot\mu_n)\longrightarrow F_r(g\cdot\sigma_0, \psi_0^0\cdot\mu_0)\qquad(n\to\infty)\,.$$

Now (1) is proved by (19.9) and (19.11). (2) is proved by (19.10), (19.12) and (19.13).

[19.8] REMARK. Let f be in $C_p(R)$ and a be positive. Assume that a function ϕ_λ defined on D^a satisfies (1) and (2) in definition [4.16]. Then noting [5.19], [5.20] and lemma [6.1], we can see that ϕ_λ is *B^λ-solution* for f in D^a if and only if

(19.14) $$U_0^\lambda(\phi)(x)\sigma_\lambda(dx)=U_0^\lambda(x)\sigma_{\phi_\lambda}(dx)\,,$$

where U_0^λ is given by (19.8), and $U_0(\phi_\lambda)$ is represented by

(19.15) $$U_0^\lambda(\phi_\lambda)=-T_0(x, \sigma_{\phi_\lambda}, \mu_\lambda)-F_r(x, k\sigma_{\phi_\lambda}+\psi_\lambda^0\cdot\mu_\lambda)+C(\phi_\lambda)$$

with some constant $C(\phi_\lambda)$.

[19.9] Under the same assumption as in [19.6], ϕ_0 defined by [19.4] is a B^0-solution for f in D^a.

Proof. By (19.14) and (19.15)

$$\int_{I(r)}U_0^n(x)\sigma_{\phi_n}(dx)=\int_{I(r)}U_0^n(\phi_n)(x)\sigma_n(dx)$$

$$=-T_0(\sigma_n, \sigma_{\phi_n}, \mu_n)-F_r(\sigma_n, k\sigma_{\phi_n}+\psi_n^0\cdot\mu_n)+2\pi C(\phi_n)\,.$$

By (3) in [19.1] and [19.7], $\{C(\phi_n)\}$ converges. Set $C=C(\phi_0)=\lim_{n\to\infty} C(\phi_n)$. By (19.14) and (19.15), it also holds that for g in $C_p(R)$

$$\int_{I(r)}g(x)U_0^n(x)\sigma_{\phi_n}(dx)=\int_{I(r)}g(x)U_0^n(\phi_n)(x)\sigma_n(dx)$$

$$=-T_0(g\cdot\sigma_n, \sigma_{\phi_n}, \mu_n)-F_r(g\cdot\sigma_n, k\sigma_{\phi_n}+\psi_n^0\cdot\mu_n)+C(\phi_n)\int_{I(r)}g(x)\sigma_n(dx).$$

Using (3) in [19.1] and [19.7] again, we can show that

$$\int_{[r,r+2\pi)} g(x)U_0^0(x)\sigma_{\phi_0}(dx)$$

$$=-T_0(g\cdot\sigma_0, \sigma_{\phi_0}, \mu_0)-F_r(g\cdot\sigma_0, k\sigma_{\phi_0}+\phi_0^0\cdot\mu_0)+C\int_{I(r)} g(x)\sigma_0(dx).$$

Noting [19.8] again, we obtain [19.9].

[19.10] Let $\{\lambda_n\}$, f in $C_{p,N}(R)$ and $a>0$ satisfy the assumption [19.4]. Then ϕ_0 in (2) of [19.4] is a B_N^0-solution for f in D^a.

Proof. Define $\sigma_{i,N}$ and $\mu_{j,N}$ $(i, j=0, 1)$ by (7.2). Then by [7.4] $B_{i,j}^*=\{\sigma_{i,N}, \mu_{j,N}, k, p_{i,j}/N\}$ is in $\mathcal{B}$. As in (19.3) set $B^{*\lambda}=B(\lambda, B_{i,j}^*)$, then

$$B^{*\lambda}=\left\{\sigma_{\lambda,N}, \mu_{\lambda,N}, k, \frac{p_\lambda}{N}\right\}.$$

Since ϕ_n is a $B_N^n=B_N^{\lambda_n}$-solution for f in D^a, $\phi_{n,N}(z)=(1/N)\phi_n(Nz)$ is a $B^{*n}=B^{*\lambda_n}$-solution for $f_N(x)=(1/N)f(Nx)$ in $D^{a/N}$ by [7.5]. Since $\{\lambda_n\}$, f_N in $C_p(R)$ and a/N satisfy [19.4], $\phi_{0,N}=\lim_{n\to\infty}\phi_{n,N}$ is a $B^{*,0}$-solution by [19.9]. Using [7.5] again, we can see that ϕ_0 is a B_N^0-solution for f in D^a.

[19.11] PROPOSITION. *Let* $B^\lambda=B(\lambda, B_{i,j})$ $(0\leqq\lambda\leqq 1)$ *be given by* (19.3). *If* P^λ $(0<\lambda\leqq 1)$ *in* $\mathcal{P}_c$ *is* B^λ*-process with* $B^\lambda=B_{P^\lambda}$ *and satisfies* $[M]$ *and* $[V]$. *Then* $P^\lambda\to P$ $(\lambda\to 0)$ *in* $\mathcal{P}$, *where* P *is a* B^0*-process with* $B^0=B_P$ *and satisfies* $[M]$ *and* $[V]$.

Proof. Since $\sigma_\lambda\to\sigma_0$, $\mu_\lambda\to\mu_0$ and $U^\lambda\to U^0$ $(\lambda\to 0)$ by definition, it holds that $B^\lambda\to B^0$ $(\lambda\to 0)$ and $\sup_\lambda p_{B^\lambda}(a)\leqq k(a)<\infty$ for any $a>0$. Therefore, by [18.2] for any sequence $\{\lambda_n\}$ which converges to 0, we can choose a subsequence $\{\lambda_m\}$ such that $\lambda_m\to 0$ and $P^{\lambda_m}\to P$ $(m\to\infty)$ in $\mathcal{P}$. Set $P^m=P^{\lambda_m}$. By [18.3] and [18.10] P satisfies $[M]$ and $[V]$ and $B^0=B_P$. Let any function f be in $C_{p,N}(R)$ and $a>0$ be given. Set $\phi_m=H_{P^m}^a f$. Then by definition

$$\phi_m(z)\longrightarrow\phi_0(z)=H_P^a f(z).$$

Since ϕ_m is harmonic in D^a with $\|\phi_m\|\leqq\|f\|$,

$$\left\|(\phi_m)_x\left(\cdot, \frac{a}{2}\right)\right\|\leqq K(a)\|f\|$$

and

$$\frac{\left\|(\phi_m)_x\left(\cdot, \frac{a}{2}\right)\right\|}{\left\|s_m\left(\cdot, \frac{a}{2}\right)\right\|}\leqq K(a)\|f\|\coth\frac{a}{2}=K(a, f)<\infty.$$

Therefore, by [9.8], $|\sigma_{\phi_m}|(dx)\leqq K(a, f)d\sigma_{\lambda_m}$, and $\{\lambda_m\}$, f and a satisfy the assumption [19.4]. Therefore by [19.10] $\phi_0=H_P^a f$ is a B_N^0-solution for f in D^a. Thus P is a B^0-process. By the uniqueness of B^0-process (cf. [7.6]) P is independent of choice of subsequence $\{\lambda_n\}$. Therefore $P_\lambda\to P$ ($\lambda\to 0$) also holds.

Let σ be $M_p(R)$ with $\int_{[0,2\pi)}\sigma(dx)=2\pi$ and k be any constant. Set

$$s(z)=\int_{[0,2\pi)}\tilde{h}_\xi(z)\sigma(d\xi),\quad t(z)=\int_{[0,2\pi)}\tilde{k}_\xi(z)\sigma(d\xi)+k \tag{19.16}$$

and

$$\bar{m}(z)=(1+k^2)\frac{s}{s^2+t^2},\quad \bar{l}(z)=-(1+k^2)\frac{t}{s^2+t^2}. \tag{19.17}$$

[19.12] Let s, t, $\bar{m}$ and $\bar{l}$ be defined by (19.16) and (19.17). Then it holds that:

(1) $\bar{m}$ is positive, periodic and harmonic in D with $\lim_{y\to\infty}\bar{m}(z)=1$. $\bar{l}$ is a harmonic conjugate of $\bar{m}$ with $\lim_{y\to\infty}\bar{l}(z)=-k$.

(2) Let $\bar{\mu}$ be the boundary measure of $\bar{m}$ on ∂_0, that is,

$$\bar{m}(z)=\int_{[0,2\pi)}\tilde{h}_\xi(z)\bar{\mu}(d\xi).$$

Then $\{\sigma, \bar{\mu}\}$ satisfies condition $[P]$ in [5.11].

Proof. Since $\lim_{y\to\infty}s(z)=1$, $\lim_{y\to\infty}t(z)=k$ and

$$\bar{m}(z)+i\bar{l}(z)=\frac{1+k^2}{s(z)+it(z)},$$

(1) is obvious. Set $U=(1+k^2)y$. Then U is a nonnegative solution of

$$\begin{cases} U_x=\bar{m}t+\bar{l}s=0\,, \\ U_y=\bar{m}s-\bar{l}t=1+k^2\,. \end{cases} \tag{19.18}$$

By [5.11] and [4.6] $\{\sigma, \bar{\mu}\}$ satisfies $[P]$.

[19.13] DEFINITION. For σ in $M_p(R)$ with $\int_{[0,2\pi)}\sigma(dx)=2\pi$ and a constant k, set $\bar{\mu}=F_k\sigma$, where $\bar{\mu}$ is defined by (19.17) and [19.12] (2).

[19.14] *Remark.* (1) $F_{-k}\cdot F_k=$Identity.

(2) Since $U=(1+k^2)y$ is a solution of (19.18), $\{\sigma, F_k\sigma, k, 0\}$ is in $\mathcal{B}$.

[19.15] Let $\bar{\mu}=F_k\sigma$.

(1) If $\sigma([a, b])=0$ for $a<b$, then $\bar{\mu}$ has at most one point mass in (a, b).

(2) If σ is in $M_i(R)$, then $\bar{\mu}$ is in $M_i(R)$.

Proof. Since $\sigma\neq 0$, we can assume $[a, b]\subset(c, c+2\pi)$. Set $I=[c, c+2\pi)$. Then for $\xi\in(a, b)$

$$s_0(\xi)=\lim_{z\to\xi} s(z)=\lim_{z\to\xi}\frac{1}{2\pi}\int_{I-[a,b]}\frac{\sinh y}{\cosh y-\cos(\eta-x)}\sigma(d\eta)=0$$

and

$$t_0(\xi)=\lim_{z\to\xi} t(z)=\lim_{z\to\xi}\frac{1}{2\pi}\int_{I-[a,b]}\frac{\sin(\eta-x)}{\cosh y-\cos(\eta-x)}\sigma(d\eta)+k$$

$$=\frac{1}{2\pi}\int_{I-[a,b]}\cot\left(\frac{\eta-\xi}{2}\right)\sigma(d\eta)+k\,,$$

Therefore

$$\frac{d}{d\xi}t_0(\xi)=-\frac{1}{2\pi}\int_{I-[a,b]}\frac{1}{2\sin^2\left(\frac{\eta-\xi}{2}\right)}\sigma(d\xi)<0\,,$$

and $t_0(\xi)\neq 0$ for $\xi\in(a, b)$ except at most one point. For $\xi\in(a, b)$ with $t_0(\xi)\neq 0$,

$$\lim_{z\to\xi}\bar{m}(z)=\lim_{z\to\xi}\frac{(1+k^2)s}{s^2+t^2}=0\,,$$

which shows that $\bar{\mu}(d\xi)$ has no mass in (a, b) except at most one point. Hence (1) is proved. To prove (2), assume $\bar{\mu}([a, b])=0$ for some $a<b$. Then by (1) $\sigma=F_{-k}\bar{\mu}$ can not belong to $M_i(R)$. Thus (2) is proved.

[19.16] THEOREM. *For any $B=\{\sigma, \mu, k, p\}$ in $\mathcal{B}$, there exists a unique B-process P with $B=B_P$ and P satisfies $[M]$ and $[V]$.*

Proof. Set $\rho(dx)=dx$ (Lebesque measure on R) and $\sigma^*=(1/2)(\sigma+\rho)$, $\bar{\mu}=F_k\sigma^*$ and $\bar{\sigma}=F_{-k}((1/2)(\mu+\bar{\mu}))$. Then by (2) in [19.15], $\bar{\mu}$ and $\bar{\sigma}$ are in $M_i(R)$, since σ^* is in $M_i(R)$. By (2) in [19.12], $\{(1/2)(\sigma+\rho), \bar{\mu}\}$, and $\{\bar{\sigma}, (1/2)(\mu+\bar{\mu})\}$ satisfy condition $[P]$. Therefore, $\{\sigma, \bar{\mu}\}$, $\{\bar{\sigma}, \mu\}$ and $\{\bar{\sigma}, \bar{\mu}\}$ satisfy condition $[P]$. Therefore, $B_{01}=\{\sigma, \bar{\mu}, k, p_{01}\}$, $B_{10}=\{\bar{\sigma}, \mu, k, p_{10}\}$ and $B_{11}=\{\bar{\sigma}, \bar{\mu}, k, p_{11}\}$, are in $\mathcal{B}$ for sufficiently large p_{01}, p_{10} and p_{11}. Set $B_{00}=B=\{\sigma, \mu, k, p\}$ and $B^\lambda=B(\lambda, B_{ij})$ $(0\leqq\lambda\leqq 1)$ as in (19.3). Since $\sigma_\lambda=(1-\lambda)\sigma+\lambda\bar{\sigma}$ and $\mu_\lambda=(1-\lambda)\mu+\lambda\bar{\mu}$ are in $M_i(R)$ for $\lambda>0$, by theorem [18.12] there exists a B^λ-process P^λ with $B_{P^\lambda}=B^\lambda$, and P^λ is in $\mathcal{P}_c$ and satisfies $[M]$ and $[V]$. Therefore by proposition [19.11], $P^\lambda\to P$ $(\lambda\to 0)$ and P is $B=B^0$-process with $B_P=B$ which also satisfies $[M]$ and $[V]$. The uniqueness is proved in theorem [7.7].

[19.17] DEFINITION. For B in $\mathcal{B}$, let P_B be the unique B-process. Set $\mathcal{P}_B=\{P_B: B\in\mathcal{B}\}$.

If P is B-process, then by theorem [19.16] $B=B_P$ therefore B is uniquely determined by P. So we have:

[19.18] COROLLARY. *The mapping* $B \rightarrow P_B$ *is a bijection between* $\mathscr{B}$ *and* $\mathscr{P}_B$.

Combining theorem [19.16] with theorems [3.12], [15.10] and [18.12], we can characterize Feller type processes in $\bar{D}$ with continuods path functions in the class of B-processes $\mathscr{P}_B$.

[19.19] THEOREM. *There exists one-to-one correspondence between* P *in* $\mathscr{P}_c$ *with condition* C *and* $B=\{\sigma, \mu, k, p\}$ *such that* σ *and* μ *are in* $M_i(R)$ *and* σ *has no discrete mass. The correspondence is given by* $P=P_B$.

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