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"MASS CONCENTRATION" PHENOMENON FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH THE CRITICAL POWER NONLINEARITY II

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§ 0. Introduction and main results.

This paper is a sequel to the previous one [20]. We continue the study of the blow-up problem for the nonlinear Schrödinger equation:

$$(\mathrm{Cp})\qquad \begin{cases} (\mathrm{NLS}) & 2i\dfrac{\partial u}{\partial t}+\Delta u+F(u)=0, \quad (t, x)\in \boldsymbol{R}^{+}\times\boldsymbol{R}^{N}, \\ (\mathrm{IV}) & u(0, x)=u_0(x), \quad x\in\boldsymbol{R}^{N}, \end{cases}$$

where $i=\sqrt{-1}$, $u_0\in H^1=H^1(\boldsymbol{R}^N)$, Δ is the Laplace operator on $\boldsymbol{R}^N$ and F is a complex valued function satisfying, at least, the following assumptions:

$$(\mathrm{F}.1)\qquad F(0)=0,$$

$$(\mathrm{F}.2)\qquad F\in C(\boldsymbol{C}\,;\boldsymbol{C}),$$

$$(\mathrm{F}.3)\qquad |F(z)-F(w)|\leqq M(1+|z|^{4/N}+|w|^{4/N})|z-w|, \quad z, w\in\boldsymbol{C},$$

for some positive constant M.

Typical examples of F are

$$(\mathrm{NF})\qquad F(u)=|u|^{p-1}u+\chi|u|^{q-1}u, \quad \chi\in\boldsymbol{R}, \quad 1\leqq q<p\leqq 1+4/N.$$

Here, we list several basic notations which will be used throughout this paper.

μ: Lebesgue measure on $\boldsymbol{R}^N$,

$B(y\,; R)=\{x\in\boldsymbol{R}^N\,; |x-y|\leqq R\}$,

$[f>\varepsilon]=\{x\in\boldsymbol{R}^N\,; f(x)>\varepsilon\}$ or the characteristic function of this set,

$\partial_t=\partial/\partial t$, $\nabla=(\partial_1, \partial_2, \cdots, \partial_N)$, $\partial_j=\partial/\partial x_j$,

$L^\alpha=L^\alpha(\boldsymbol{R}^N)$ denotes the space of α-summable functions on $\boldsymbol{R}^N$ with the norm $\|\cdot\|_q$,

$\|\cdot\|=\|\cdot\|_2$,

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$W^{s,\alpha}=W^{s,\alpha}(\boldsymbol{R}^N)$ represents the standard Sobolev space of order s and exponent α on $\boldsymbol{R}^N$,

$H^s=W^{s,2}$,

$\Sigma=\{v\in H^1 ; \|v\|^2+\|\nabla v\|^2+\|xv\|^2<+\infty\}$,

$\langle\cdot, \cdot\rangle=L^2$-inner product,

$\mathcal{S}=\mathcal{S}(\boldsymbol{R}^N)$: the Schwartz space of rapidly decreasing C^∞-functions,

$\mathcal{S}'=\mathcal{S}'(\boldsymbol{R}^N)$: the dual of $\mathcal{S}$,

$\sigma=2+4/N$,

$2^*=2N/(N-2)$ if $N\geqq 3$, $2^*=\infty$ if $N=1$ and $N=2$,

$E(v)=\|\nabla v\|^2-\dfrac{2}{\sigma}\|v\|_\sigma^\sigma$.

We regard (NLS) as an abstract evolution equation in $H^{-1}=H^{1\prime}$, and we say that u is a solution to (Cp) on $[0, T)$ if and only if u satisfies the integral equation

$$u(t)=U(t)u_0+\frac{i}{2}S(t ; F(u)) \qquad \text{in } L^2 \tag{0.1}$$

for any $t\in[0, T)$, where $U(t)$, $S(t ; \cdot)$ are linear operators given by

$$U(t)=\exp\left(\frac{i}{2}t\Delta\right) \qquad \text{(free propagator)}, \tag{0.2}$$

$$S(t ; v)=\int_0^t U(t-\tau)v(\tau)d\tau\,, \tag{0.3}$$

respectively. For the precise definitions and properties of these operators, see Kato [10] and Yajima [30]. The integral in (0.3) is understood to be the Bochner integral in H^{-1}.

In the particular case of $F(z)=|z|^{p-1}z$ with $1<p<2^*-1$ (where $2^*=2N/(N-2)$ if $N>2$, otherwise $2^*=+\infty$), it is well known that for $p\geqq 1+4/N$ there are singular solutions of (Cp) for certain initial data (see Glassy [9] and M. Tsutsumi [25]). That is, there are some solutions $u(t)$ of (Cp) such that

$$u(\cdot)\in C([0, T) ; H^1) \quad \text{and} \quad \lim_{t\to T}\|\nabla u(t)\|=+\infty\,.$$

However, the formation of singularities in blow-up solutions for the critical case $p=1+4/N$ seems to be quite different from that of blow-up solutions for the supercritical case $1+4/N<p<2^*-1$. In the critical case there are blow-up solutions which lose their L^2-continuity because of the so-called "mass concentration" phenomenon (Weinstein [28], Nawa and M. Tsutsumi [21] and Merle and Y. Tsutsumi [17]), while Merle [15] suggests that in the supercritical case every blow-up solution has a strong limit in L^2 at the blow-up time.

In the case of $F(u)\sim|u|^{4/N}u$ as $|u|\to+\infty$, Merle and Y. Tsutsumi [17] show that no blow-up solution to (Cp) has a strong limit in L^2 as $t\to T$ (T is the blow-up time), and that L^2-concentration occurs at the origin for all the radially symmetric blow-up solutions to (Cp), when $N\geqq 2$. In [20] we also investigate the "mass concentration" phenomenon, and proved the following theorems.

THEOREM A. *Assume that F satisfies* (F.1)-(F.3). *Then for any $u_0\in H^1$ there exist a positive number T (maximal existence time, i.e, blow-up time) and a unique H^1-solution $u(t)$ to (Cp) such that*

$$u,\ \nabla u\in C([0,T);L^2)\cap L^{2+4/N}_{\mathrm{loc}}(0,T;L^{2+4/N})$$

and $u(t)$ satisfies (0.1). *Assume further that $T<+\infty$, so that $\lim_{t\to T}\|\nabla u(t)\|=+\infty$ (Blow-up). Then $u(t)$ does not have a strong limit in L^2 as $t\to T$.*

THEOREM B. *Let F be (NF) with $p=\sigma-1=1+4/N$. Suppose that the solution $u(t)$ to (Cp) blows up at $t=T\in(0,\infty]$, i.e.,* $\lim_{t\to T}\|\nabla u(t)\|=\lim_{t\to T}\|u(t)\|_\sigma=\infty$. *Set*

(B.1) $$\lambda(t)=1/\|u(t)\|_\sigma^{\sigma/2}.$$

(B.2) $$S_\lambda u(t,x)=\lambda^{N/2}u(t,\lambda x).$$

(B.3) $$A\equiv\sup_{R>0}\liminf_{t\uparrow T}\left\{\sup_{y\in\boldsymbol{R}^N}\int_{B(y;R)}|S_{\lambda(t)}u(t,x)|^2dx\right\}.$$

(1) *If $\|u_0\|^2=A$, then for any $\varepsilon>0$, there are a constant $K>0$ and a function $y(t)\in C([0,T);\boldsymbol{R}^N)$ such that*

(B.4) $$\liminf_{t\uparrow T}\int_{B(K)}|S_{\lambda(t)}u(t,x+y(t))|^2dx>(1-\varepsilon)A.$$

If we impose the condition $u_0\in\Sigma$, $T<\infty$ and $\chi\leqq 0$, then we have

(B.5) $$\sup_{t\in[0,T)}|y(t)\lambda(t)|<+\infty,$$

(B.6) $$\liminf_{t\uparrow T}\int_{B_t}|u(t,x)|^2dx>(1-\varepsilon)A,$$

where $B_t=B(y(t)\lambda(t);K\lambda(t))$.

(2) *If u_0 is radially symmetric and $N\geqq 2$, we have* (B.4) *and* (B.6) *with $y(t)\equiv 0$ and $A=\|Q\|^2$, where Q is a ground state (non trivial minimal L^2 norm solution of*

(B.7) $$\Delta Q-Q+|Q|^{4/N}Q=0,\qquad Q\in H^1.$$

(*For the equation* (B.7), *see e.g.* [1], [3], [22] *and* [27].)

(3) *If $\|u_0\|=\|Q\|$, we have the results of* (1) *with $A=\|Q\|^2$ in* (B.4) *and* (B.6).

Here we note that:

(1) $\lambda(t)\to 0$ as $t\to T$.

(2) We do not assume the radial symmetricity of solutions to (Cp).

Recently, Y. Tsutsumi [26] has investigated the rate of L^2-concentration for radially symmetric blow-up solutions to (Cp) with $N \geqq 2$ and $F(u) \sim |u|^{4/N}u$ as $|u| \to +\infty$, and showed that for any $\varepsilon > 0$, there exists a $K > 0$ such that

$$\liminf_{t \to T} \|u(t)\,;\, L^2(|x| < K(T-t)^{1/2})\| \geqq (1-\varepsilon)\|Q\|, \tag{0.4}$$

where Q is a ground state (minimal L^2 norm) solution of (B.7).

In this paper we have the following theorem, which is an improvement of Theorem B and (0.4).

THEOREM C. *Let F be (NF) with $p=\sigma-1=1+4/N$. Suppose that the solution $u(t)$ to (Cp) blows up at $t=T \in (0, \infty]$, i.e., $\lim_{t \to T}\|\nabla u(t)\| = \lim_{t \to T}\|u(t)\|_\sigma = +\infty$. Let $(t_n)_n$ be any sequence such that $t_n \to T$ as $n \to \infty$. Set*

$$\lambda_n \equiv \lambda(t_n) = 1/\|u(t_n)\|_\sigma^{\sigma/2}, \tag{C.1}$$

$$S_\lambda u(t, x) = \lambda^{N/2}u(t, \lambda x). \tag{C.2}$$

Then there exists a subsequence of $(t_n)_n$ (we still denote it by $(t_n)_n$) which satisfies the following properties: one can find a sequence $(y_n)_n$ in $\boldsymbol{R}^N$ such that, for any $\varepsilon > 0$, there is a positive constant K;

$$\liminf_{n \to \infty} \int_{B(K)} |S_{\lambda_n} u(t_n, x+y_n)|^2 dx \geqq (1-\varepsilon)\|Q\|^2. \tag{C.3}$$

If we impose the condition $u_0 \in \Sigma$, $\chi \leqq 0$ and $T < \infty$, then we have

$$\sup_{n \in \boldsymbol{N}} |y_n \lambda_n| < +\infty, \tag{C.4}$$

$$\liminf_{n \to \infty} \int_{B_n} |u(t_n, x)|^2 dx \geqq (1-\varepsilon)\|Q\|^2, \tag{C.5}$$

where $B_n = B(y_n\lambda_n\,;\, K\lambda_n)$.

Our proof of Theorem C depends heartily on the following

PROPOSITION D. *Let F be (NF) with $p=\sigma-1=1+4/N$. Suppose that the solution $u(t)$ to (Cp) blows up at $t=T \in (0, \infty]$, i.e., $\lim_{t \to T}\|\nabla u(t)\| = \lim_{t \to T}\|u(t)\|_\sigma = +\infty$. Let $(t_n)_n$ be any sequence such that $t_n \to T$ as $n \to \infty$. Set*

$$\lambda_n \equiv \lambda(t_n) = 1/\|u(t_n)\|_\sigma^{\sigma/2}, \tag{D.1}$$

$$u_n(t, x) \equiv S_{\lambda_n} u(t, x) = \lambda_n^{N/2} u(t, \lambda_n x). \tag{D.2}$$

Then there exists a subsequence of $(t_n)_n$ (we still denote it by $(t_n)_n$) which satisfies the following properties· one can find $L \in \boldsymbol{N} \cup \{\infty\}$ and sequences $(y^j{}_n)_n$ in $\boldsymbol{R}^N$ for $1 \leqq j \leqq L$ such that

(D.3) $\lim_{n\to\infty}|y_n^j-y_k^n|=\infty \quad (j\neq k)$,

(D.4) $f_n^1\equiv u_n(t_n,\ x+y_n^1)\longrightarrow f^1$ *weakly in* H^1,

(D.5) $f_n^j\equiv(f_n^{j-1}-f^{j-1})(t_n,\ \cdot+y_n^j)\longrightarrow f^j \quad (j\geqq 2)$ *weakly in* H^1

(D.6) $\lim_{n\to\infty}\{E(f_n^j)-E(f_n^j-f^j)\}=E(f^j)$,

(D.7) $\lim_{j\to L}\lim_{n\to\infty}\|f_n^j-f^j\|_\sigma=0 \quad (L=+\infty)$,

(D.7)′ $\lim_{n\to\infty}\|f_n^L-f^L\|_\sigma=0 \quad (L<+\infty)$,

(D.8) $\lim_{j\to L}\lim_{n\to\infty}\left\{\sup_{y\in\boldsymbol{R}^N}\int_{B(y;R)}|(f_n^j-f^j)(x)|^2dx\right\}=0$ *if* $L=+\infty$,

(D.8)′ $\lim_{n\to\infty}\left\{\sup_{y\in\boldsymbol{R}^N}\int_{B(y;R)}|(f_n^L-f^L)(x)|^2dx\right\}=0$ *if* $L<+\infty$,

where R *is any positive constant.*

In view of this proposition, we can understand the assertion of Theorem B (2) and (3).

COROLLARY E. (1) *If* u_0 *is radially symmetric and* $N\geqq 2$, *we have* $L=1$ *with* $y^1{}_n\equiv 0$ *in Proposition* D, *so that we have Theorem* B (2).

(2) *If* $\|u_0\|=\|Q\|$, *we have* $L=1$ *in Proposition* D, *so that we have Theorem* B (3).

Remarks. 1. Roughly speaking, Proposition D suggests that u behaves like;

$$|u(t,\ x)|^2\longrightarrow\sum_{j=1}^{L}\|f^j\|^2\delta(x-a_j)+r(x) \quad \text{in } \mathcal{S}' \tag{0.5}$$

as $t\to T$, where $a_j=\lim_{n\to\infty}\lambda_n y_n^j$, $\delta(x-a)$ is a Dirac mass at $a\in\boldsymbol{R}^N$ and $r(x)$ is a remainder term (it may be a function). We note that L may be infinite. It could happen that $a_j=a_k$ $(j\neq k)$. However we see from this theorem that "mass concentration" occurs at some points.

In [21], the author and M. Tsutsumi characterized the initial datum (in Σ) leading to the solution which develops the singularity like Dirac mass δ:

$$\lim_{t\to T}\|(x-a)u(t)\|=0 \quad \text{for some } a\in\boldsymbol{R}^N, \tag{0.6}$$

so that

$$|u(t,\ x)|^2\longrightarrow\|u_0\|^2\delta(x-a) \quad \text{in } \mathcal{S}' \tag{0.7}$$

as $t\to T$ ([21; Theorem 1]). Thus we know there are many blow-up solutions satisfying

$$\lim_{t\to T}\|(x-a)u(t)\|>0 \tag{0.8}$$

for each $a\in\boldsymbol{R}^N$ ([21; Corollary 1.1]). So $L\geqq 2$ or $r\neq 0$ occurs in (0.5), which also explains how the blow-up solution u performs (0.8).

Recently Merle [16] has constructed blow-up solutions which concentrates their "L^2 mass" at exactly m points $\{a_1, a_2, \cdots a_m\}$ such that

$$\lim_{t\to T}\|u(t, x);\ L^2_{\mathrm{loc}}(\boldsymbol{R}^N\setminus\{a_1, a_2, \cdots, a_m\})\|=0. \tag{0.9}$$

This example corresponds to (0.5) with $L=m$ and $r=0$.

On the other hand, some numerical computations suggest that there are blow-up solutions which behave like (0.5) with $L=1$ and r being some function (see e.g. [24] and [27]).

2. The spatial dilation operator S_λ was introduced by Weinstein for the first time in [28]. We note that our scaling function λ is different from the one in [28]. Our choice of scaling function λ simplifies our calculations in § 3, and we can treat more general nonlinearities than those in [28].

3. The proof of Theorem C is inspired by the method of concentration-compactness due to Lions [12, 13] and the argument performed in Weinstein [28]. We, however, repeatedly use the same compactness device as in Lieb [11] and Brézis and Lieb [3] to decompose $(u_n)_n$ iteratively into several parts (possibly infinite parts) with the help of Brezis and Lieb's lemma [4]. It is worth while to note that the case $L=1$ in Theorem C (1) does not always correspond to the terminology "tightness" in the method of concentration-compactness for the concentration function of $|u_n|^2$: If $L=1$, it could happen that

$$\lim_{n\to\infty}\|u_n(t_n, \cdot+y_n^1)-f^1\|\neq 0, \tag{0.10}$$

although it holds that

$$\lim_{n\to\infty}\|u_n(t_n, \cdot+y_n^1)-f^1\|_\sigma=0. \tag{0.11}$$

Thus $L=1$ is not equivalent to (0.7).

4. Weinstein [32] proved the similar result to Theorem C. However he treated only the single power case and in his paper there is only a proof for the radially symmetric case.

§ 1. Preliminaries.

In this section we collect several well-known facts about solutions to (Cp) and those to (0.5), and recall the week compactness result due to Lieb [11], a related lemma from [6] (see also [3]) and Bézis and Lieb's lemma [4], which will be crucial for the proof of Theorem C.

We use the notation $\sigma=2+4/N$.

LEMMA 1.1. *Let*

$$I\equiv\inf\{\|\nabla v\|^2\|v\|^{4/N}/\|v\|_\sigma^\sigma;\ v\in H^1 \text{ and } v\neq 0\}. \tag{1.1}$$

The infimum I is attained at a function Q with the following properties:

(1) *Q is positive and radially symmetric.*

(2) *$Q\in H^1(\mathbf{R}^N)\cap C^2(\mathbf{R}^N)$ and satisfies*

$$E(Q)\equiv\|\nabla Q\|^2-(2/\sigma)\|Q\|_\sigma^\sigma=0\,. \tag{1.2}$$

(3) *Q is a solution to* (0.5) *of minimal L^2 norm (the ground state). In addition,*

$$I=2\|Q\|^{\sigma-2}/\sigma\,. \tag{1.3}$$

(4) *Q is a solution to the following variational problem.*

$$\text{minimize}\ \|v\|\,;\ E(v)\leqq 0\ \textit{and}\ v\in H^1/\{0\}\}. \tag{1.4}$$

(5) *Let S' be the best constant for the interpolation estimate:*

$$\|v\|_\sigma^\sigma\leqq S'\|\nabla v\|^2\|v\|^{\sigma-2}\,,\qquad N\geqq 1\,. \tag{1.5}$$

Then $S'=1/I$.

For Lemma 1.1, see Weinstein [28], Beretycki and Lions [1] and Strauss [22] (see also [19] for part (4)).

LEMMA 1.2. (1) *Assume that F satisfies* (F.1)-(F.3). *For any $u_0\in H^1$, there exist a positive number T and a unique solution*

$$u(\cdot)\in C([0,\,T)\,;\,H^1)\cap L^\sigma_{\mathrm{loc}}(0,\,T\,;\,L^\sigma) \tag{1.6}$$

to (Cp) *satisfying* (0.1) *with the alternatives; either $T=+\infty$ or $T<+\infty$ and*

$$\lim_{t\to T}\|\nabla u(t)\|=\lim_{t\to T}\|u(t)\|_\sigma=+\infty\,. \tag{1.7}$$

(2) *In addition to* (F.1)-(F.3), *assume that F satisfies*

$$\operatorname{Im} F(z)\bar{z}=0\,,\qquad z\in\mathbf{C}\,, \tag{F.4}$$

$$\textit{there exists}\ G\in C^2(\mathbf{C}\,;\,\mathbf{R})\ \textit{such that}\ F=\frac{\partial G}{\partial\bar{z}}\,. \tag{F.5}$$

Then the above solution u satisfies:

$$\|u(t)\|=\|u_0\|\,, \tag{1.8}$$

$$\begin{aligned}H(u(t))&\equiv\|u(t)\|^2-\langle G(u(t)),\,1\rangle \\ &=H(u_0)\,,\end{aligned} \tag{1.9}$$

for $t\in[0,\,T)$. If $u_0\in\Sigma$, then $u\in C([0,\,T)\,;\,\Sigma)$ and satisfies·

$$\|xu(t)\|^2=\|xu_0\|^2+2t\,\mathrm{Im}\,\langle x\cdot\nabla u_0,\ u_0\rangle+t^2H(u_0)\tag{1.10}$$
$$-N\int_0^t(t-\tau)\langle F(u),\ u\rangle(\tau)d\tau$$
$$+(N+2)\int_0^t(t-\tau)\langle G(u),\ 1\rangle(\tau)d\tau\,.$$

(3) *If $u_0\in H^1$ and $\|u_0\|<\|Q\|$, then the solution u exists globally in time.*

We can find the proof of part (1) and (2) in Kato [10] and Ginibre and Velo [7]. See also the proof of Lemma 2.4 in the previous paper [20]. For the identity (1.10), see e.g., Glassy [9] and M. Tsutsumi [25]. One can find the proof of part (3) in Weinstein [27].

LEMMA 1.3 (Frölich, Lieb and Loss [6]). *Let $1<\alpha<\beta<\gamma$ and let g be a measurable function on $\boldsymbol{R}^N$ such that, for some positive constants C_α, C_β, C_γ,*

(i) $\|g\|_\alpha\leqq C_\alpha$,

(ii) $\|g\|_\beta\geqq C_\beta>0$,

(iii) $\|g\|_\gamma\leqq C_\gamma$.

Then $\mu([|g|>\eta])>C$ for some η, $C>0$ depending on α, β, γ, C_α, C_β, C_γ, but not on g.

LEMMA 1.4 (Lieb [11]). (1) *Let $1\leqq\alpha<\infty$ and let v be a function such that $v\in L^\alpha_{\mathrm{loc}}$, $\nabla v\in L^\alpha$, $\|\nabla v\|_\alpha\leqq\Lambda$ and $\mu([|v|>\eta])\geqq C$ for some positive constants Λ, η, C. Then, there exists a shift $T_yv(x)=v(x+y)$ such that, for some constant $\delta=\delta(\Lambda,\ C,\ \eta)$, $\mu(B\cap[|T_yv|>\eta/2])>\delta$, where $B=B(1)$.*

(2) *Let $1<\alpha<\infty$ and let $(f_n)_n$ be a uniformly bounded sequence of functions in $W^{1,\alpha}(\boldsymbol{R}^N)$ with the property that $\mu([|f_n|>\eta])\geqq C$ for some η, $C>0$. Then there exists a seauence $(y_n)_n$ in $\boldsymbol{R}^N$, $\phi_n(x)\equiv f_n(x+y_n)$, such that, for some subsequence $\{n_j\}$, $\phi_{n_j}\to\phi$ weakly in $W^{1,\alpha}(\boldsymbol{R}^N)$ and $\phi\neq0$.*

We note that part (2) is a direct consequence of the Banach-Alaoglu theorem and part (1).

LEMMA 1.5 (Brézis aud Lieb [4]). *Let $0<\alpha<\infty$ and let $(f_n)_n$ be a uniformly bounded sequence in L^α. Suppose that $f_n\to f$ a.e. in $\boldsymbol{R}^N$. (By Fatou's Lemma $f\in L^\alpha$.) Then,*

$$\lim_{n\to\infty}\int_{\boldsymbol{R}^N}||f_n(x)|^\alpha-|f_n(x)-f(x)|^\alpha-|f(x)|^\alpha|dx=0.\tag{1.11}$$

Our proof of Theorem C depends heartily on the above two lemmas, which enable us to derive Proposition D.

§ 2. Proofs of main results.

The main purpose of this section is to prove Theorem C. For simplicity, we suppose $N \geqq 3$ (in the case of $N \leqq 2$, we need a slight modification in (2.7) below).

Let the nonlinearity F be (NF), which satisfies (F.1)-(F.5). Suppose that the solution $u(t)$ to (Cp) blows up at time $t=T \in (0, \infty]$, i.e., $\lim_{t\to T}\|\nabla u(t)\|=\infty$. From Lemma 1.2, $u(t)$ satisfies the mass conservation law (1.8) and the energy conservation law (1.9) for $t \in [0, T)$. We note that, in this case,

$$G(u)=\frac{2\chi}{q+1}|u|^{q+1}+\frac{2}{\sigma}|u|^{\sigma}. \tag{2.1}$$

Proof of Theorem C.

We recall that

$$\lambda \equiv \lambda(t)=1/\|u(t)\|_{\sigma}^{\sigma/2}, \tag{2.2}$$

$$u_{\lambda} \equiv S_{\lambda}u(t, x)=\lambda^{N/2}u(t, \lambda x). \tag{2.3}$$

One can see that

$$\|u_{\lambda}\|=\|u\|=\|u_0\|, \tag{2.4}$$

$$\|u_{\lambda}\|_{\sigma}=1. \tag{2.5}$$

Moreover we have that

$$\begin{aligned} E(u_{\lambda})&=\lambda^2 E(u(t)) \\ &=\lambda^2\left\{H(u_0)+\frac{2\chi}{q+1}\|u(t)\|_{q+1}^{q+1}\right\} \longrightarrow 0 \quad (t \to T). \end{aligned} \tag{2.6}$$

since, by Hölder's inequality, it holds that

$$\begin{aligned} \lambda^2\|u(t)\|_{q+1}^{q+1} &\leqq \lambda^2\|u(t)\|^{1-a}\|u(t)\|_{\sigma}^{a} \\ &=\|u_0\|^{1-a}(\lambda^2\|u(t)\|_{\sigma}^{a}), \end{aligned}$$

where $a=(N/4)(q-1)\sigma<\sigma$ (because $q-1<4/N$). From (2.5), (2.6) and Sobolev's inequality one has

$$\|u_{\lambda}\|_{2*} \leqq S\|\nabla u_{\lambda}\| \leqq S \tag{2.7}$$

for sufficiently small λ, where S is the Sobolev best constant.

By (2.4), (2.5) and (2.7) we have, for some constants η, $C>0$,

$$\mu([|u_{\lambda}(t)|>\eta])>C \tag{2.8}$$

with the help of Lemma 1.3.

For any sequence $(t_n)_n$ such that $t_n \to T$ $(n \to \infty)$, we use the notations:

$$\lambda_n \equiv \lambda(t_n)\,, \tag{2.9}$$

$$u_n(t_n,\, x) \equiv \lambda_n^{N/2} u(t_n,\, \lambda_n x)\,. \tag{2.10}$$

We shall prove Proposition D.

Proof of Proposition D. By (2.8) and Lemma 1.4 (2), we can shift each u_n so that

$$f_n^1 \equiv u_n(t_n,\, x+y_n^1) \longrightarrow f^1 \neq 0 \qquad \text{weakly in } H^1\,. \tag{2.11}$$

This is valid only for a subsequence. We shall however often extract subsequence without explicitly mentioning this fact.

From (2.1), one has

$$f_n^1 \longrightarrow f^1 \qquad \text{in } L^2_{\text{loc}}\,, \tag{2.12}$$

so that

$$f_n^1 \longrightarrow f^1 \qquad \text{a.e. in } \boldsymbol{R}^N\,. \tag{2.13}$$

Hence we have, by Lemma 1.5,

$$\lim_{n\to\infty}(\|f_n^1\|_\sigma^\sigma - \|f^1{}_n - f^1\|_\sigma^\sigma) = \|f^1\|_\sigma^\sigma\,, \tag{2.14}$$

and by the weak convergence of $\nabla f^1{}_n$ and the uniqueness of the limit,

$$\lim_{n\to\infty}(\|\nabla f_n^1\|^2 - \|\nabla f_n^1 - \nabla f^1\|^2) = \|\nabla f^1\|^2\,. \tag{2.15}$$

Combining (2.14) and (2.15), we get

$$\lim_{n\to\infty}\{E(f_n^1) - E(f_n^1 - f^1)\} = E(f^1)\,. \tag{2.16}$$

We deduce from (2.5), (2.10) and (2.14) that following two limit exist and equalities hold;

$$\lim_{n\to\infty}\|f_n^1 - f^1\|_\sigma^\sigma = 1 - \|f^1\|_\sigma^\sigma\,, \tag{2.17}$$

$$\lim_{n\to\infty} E(f_n^1 - f^1) = -E(f^1)\,. \tag{2.18}$$

Suppose $\lim_{n\to\infty}\|f_n^1 - f^1\|_\sigma \neq 0$. Then one can verify that $f_n^1 - f^1$ satisfies the assumptions in Lemma 1.3 with $\alpha=2$, $\beta=\sigma$ and $\gamma=2^*$. So at this stage, we consider $f_n^1 - f^1$, and repeat the above argument. There exists a sequence $(y_n^2)_n$ in $\boldsymbol{R}^N$ such that

$$f_n^2 \equiv (f_n^1 - f^1)(t_n,\, \cdot + y_n^2) \longrightarrow f^2 \neq 0 \qquad \text{weakly in } H^1\,, \tag{2.19}$$

$$\lim_{n\to\infty}\{E(f_n^2) - E(f_n^2 - f^2)\} = E(f^2)\,. \tag{2.20}$$

We note that $\lim_{n\to\infty}|y_n^1-y_n^2|=\infty$, since $f_n^1-f^1\to 0$ weakly in H^1. By (2.17)-(2.20), one can also verify that

$$\lim_{n\to\infty}\|f_n^2-f^2\|_\sigma^\sigma=1-\|f^2\|_\sigma^\sigma-\|f^1\|_\sigma^\sigma\,, \tag{2.21}$$

$$\lim_{n\to\infty}E(f_n^2-f^2)=-E(f^1)-E(f^2)\,, \tag{2.22}$$

since we have that

$$\lim_{n\to\infty}\|f_n^1-f^1\|_\sigma=\lim_{n\to\infty}\|f_n^2\|_\sigma\,,$$

$$\lim_{n\to\infty}E(f_n^1-f^1)=\lim_{n\to\infty}E(f_n^2)\,,$$

by the translation invariance of the norm $\|\cdot\|_\sigma$ and the functional $E(\cdot)$.

Repeating this procedure, we obtain sequences $(y_n^j)_n$ in $\boldsymbol{R}^N$ for $1\leqq j$ such that

$$\lim_{n\to\infty}|y_n^j-y_n^k|=\infty \qquad (j\neq k)$$

and corresponding functions

$$f_n^j\equiv(f_n^{j-1}-f^{j-1})(t_n,\ \cdot+y_n^j)\longrightarrow f^j \qquad \text{weakly in } H^1\,,$$

where f_n^j satisfies

$$\lim_{n\to\infty}(\|f_n^j\|_\sigma^\sigma-\|f_n^j-f^j\|_\sigma^\sigma)=\|f_j\|_\sigma^\sigma\,, \tag{2.23}$$

$$\lim_{n\to\infty}(\|\nabla f_n^j\|^2-\|\nabla f_n^j-\nabla f^j\|^2)=\|\nabla f^j\|^2\,, \tag{2.24}$$

so that we have

$$\lim_{n\to\infty}\{E(f_n^j)-E(f_n^j-f^j)\}=E(f^j)\,. \tag{2.25}$$

We also obtain by induction that

$$\lim_{n\to\infty}\|f_n^j-f^j\|_\sigma^\sigma=1-\sum_{k=1}^{j}\|f^k\|_\sigma^\sigma\,, \tag{2.26}$$

$$\lim_{n\to\infty}E(f_n^j-f^j)=-\sum_{k=1}^{j}E(f^k)\,, \tag{2.27}$$

(D.8) and (D.8)′ immediately follow from (D.7) and (D.7)′. (D.7)′ is obvious by the construction of f^j's. Therefore it remains to prove (D.7). Suppose the contrary that there exists a constant $C_0>0$ and $J\in\boldsymbol{N}$ such that

$$\lim_{n\to\infty}\|f_n^j-f^j\|_\sigma^\sigma>C_0 \tag{2.28}$$

for any $j\geqq J$. Thus there is a constant $C_1>0$ such that

$$\|f^{j+1}\|_\sigma^\sigma>C_1 \tag{2.29}$$

for any $j\geqq J$, since the size of $\|f^{j+1}\|$, essentially depends on the lower bound of $\|f_n^j-f^j\|_\sigma$ by Lemma 1.3, Lemma 1.4 and the construction of f^j. We choose and $k\in\boldsymbol{N}$ (sepecified later). Using the formula (2.26) for $j\in\{J, J+1, \cdots ; J+k\}$, we have by (2.28) and (2.29) that

$$\begin{aligned}1-C_0 &> \lim_{n\to\infty}\{\|f_n^j-f^j\|_\sigma^\sigma-\|f_n^{j+k}-f^{j+k}\|_\sigma^\sigma\}\\ &=\sum_{j=1}^{k}\|f^{J+j}\|_\sigma^\sigma\\ &>kC_1\,.\end{aligned}$$

Thus we reach a contradiction, if we take k as $kC_1\geqq 1-C_0$.

Remark 2.1. Proposition D asserts that u_n behaves like a superposition of several parts u_n^1, u_n^2, u_n^3, $\cdots$, u_n^L (L may be infinite) as $n\to\infty$. The above argument is somewhat related to those used in Lions [14], Brézis and Coron [2] and Struwe [23].

We now distinguish two cases:

Case I $L=\infty$ and $E(f^j)>0$ for any $j\in\boldsymbol{N}$,

Case II $L<\infty$ or $E(f^k)\leqq 0$ for some $k\in\boldsymbol{N}$.

We shall establish that Case I cannot occur.

Suppose that we are in Case I. We recall (2.27) and define the sequence $(E_j)_j$ by

$$E_j\equiv\lim_{n\to\infty}\{-E(f_n^j-f^j)\}=\sum_{k=1}^{j}E(f^k)\,. \tag{2.30}$$

Hence the sequence $(E_j)_j$ is positive and increasing, since $E(f^k)>0$ for any $k\in\boldsymbol{N}$. On the other hand we have by the definition of E that

$$\begin{aligned}(0<)E_j &= \lim_{n\to\infty}\{-E(f_n^j-f^j)\}\\ &\leqq (2/\sigma)\lim_{n\to\infty}\|f_n^j-f^j\|_\sigma^\sigma\,.\end{aligned} \tag{2.31}$$

Thus $(E_j)_j$ has a subsequence decreasing to 0 by (D.7) in Proposition D. Therefore we reach a contradiction and Case I is excluded.

Hence the only case which occurs is Case II. Since $L<\infty$ implies that

$$\sum_{j=1}^{L}E(f^j)\leqq 0\,,$$

we have $E(f^k)\leqq 0$ for some $k\in\boldsymbol{N}$ in this case. Thus we get, by Lemma 1.1(4),

$$\|f^k\|\geqq\|Q\|\,. \tag{2.32}$$

We also have (2.11) and (2.13) for all $j\in\{1, 2, \cdots, L\}$, so it follows from Lemma 1.5 that

$$\lim_{n\to\infty}\int_{R^N}||f_n^j(x)|^2-|f_n^j(x)-f^j(x)|^2-|f^j(x)|^2|dx=0. \tag{2.33}$$

Hence, for any sequence $(y_n)_n$ in $\boldsymbol{R}^N$, it holds (2.33) with $f^j{}_n$ and f^j replaced by $f_n^j(t_n, x+y_n)$ and $f^j(t_n, x+y_n)$, respectively. So we have that, for any domain $\Omega\subset\boldsymbol{R}^N$,

$$\lim_{n\to\infty}\int_{x\in y_n+\Omega}||f_n^j(x)|^2-|f_n^j(x)-f^j(x)|^2-|f^j(x)|^2|dx=0. \tag{2.34}$$

Therefore, for f^k and any $K>0$, we have by (2.34) with $j\in\{1, 2, \cdots, k\}$ and $\Omega=B(K)$ that

$$\int_{B(K)}|f^k|^2dx\leqq\int_{B(K)}|\phi_n|^2dx-\sum_{j=1}^{k-1}\int_{B(K)}|\phi^j|^2dx+o(1), \tag{2.35}$$

where $o(1)$ is a quantity converging to 0 as $n\to\infty$ and

$$\phi_n=u_n\Big(t_n, x+\sum_{m=1}^{k}y_n^m\Big), \tag{2.36}$$

$$\phi^j=f^j\Big(x+\sum_{m=j+1}^{k}y_n^m\Big). \tag{2.37}$$

The main conclusion of Theorem C thus follows from (2.35), since one can see that, for any $\varepsilon>0$,

$$\|\Omega\|^2-\varepsilon\leqq\int_{B(K)}|f^k|^2dx \tag{2.38}$$

for sufficiently large $K>0$. In the case of $u_0\in\sum$, $\chi\leqq0$ and $T<+\infty$ (1.10), implies the boundedness of $\|xu(t)\|$. Therefore we obtain (C.4) and (C.5), since we have by Chebychev's inequality;

$$\int_{|x|>R}|u(t, x)|^2dx\leqq\frac{1}{R^2}\|xu(t)\|^2. \tag{2.39}$$

Therefore we conclude the proof of Theorem C.

Proof of Corollary E.

We recall (2.11). In view of (D.7)′ and (D.8)′ in Proposition D, it is enough to prove that we have

$$f_n^1\equiv u_n(t_n, x+y_n^1)\longrightarrow f^1\neq0 \qquad (n\to\infty). \tag{2.40}$$

strongly in L^σ or L^2 for some $f\in H^1$.

Proof of (1). We will show that (2.40) with $y_n^1\equiv0$ holds true in the strong topology of L^σ. If the initial datum u_0 is radially symmetric, so is the solution to (Cp) in $C([0, T); H^1)$. Thus each u_n is also radially symmetric. Since $(u_n)_n$ is a bounded sequence in H^1 by (2.4), (2.7) and (2.10), we have (2.40) with $y_n^1 \equiv 0$ in the strong topology of L for a subsequence (we still denote it by the same letter) by a radial compactness lemma due to Strauss [22] (see also [1]). We note that $f^1 \neq 0$ by (2.5) and (2.10).

Proof of (2). We will show that (2.40) holds true in the strong topology in L^2. Now we suppose $\|f^1\| < \|Q\|$ where Q is the ground state solution to (0.5). Then one has $E(f) > 0$ by (1.4) in Lemma 1.1. This together with (2.18) implies that

$$\lim_{n\to\infty} \|f_n^1 - f^1\| \geqq \|Q\| \tag{2.41}$$

by (1.4) again. We note that the limit in the left hand side of (2.41) exists, since the weak convergence of $f_n^1 \to f$ in L^2 together with the fact $\|f_n^1\| = \|u_0\| = \|Q\|$ implies that

$$\lim_{n\to\infty} \|f_n^1 - f^1\|^2 = \|Q\|^2 - \|f^1\|^2 . \tag{2.42}$$

Here we reach a contradiction, since (2.41) and (2.42) yield $\|Q\| < \|Q\|$. Therefore we have $\|f^1\| = \|Q\|$, so that we obtain

$$\lim_{n\to\infty} \|f_n^1 - f^1\| = 0 . \tag{2.43}$$

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Note added in proof. After completing this work, we found that we could improve the proof of Theorem C to refine Theorem B (1) with the result that we have $A \geqq \|Q\|^2$ for any blow-up solution. This will appear elsewhere.

References

[1] Berestycki, H. and Lions, P. L., Nonlinear scalar field equations, I, II, Archi. Rat. Mech. Anal. **82**, 313-376 (1983).

[2] Brézis, H. and Coron, J. M., Convergence of H-system or how to blow bubbles. Archi. Rat. Mech. Anal. **89**, 21-56 (1985).

[3] Brézis, H. and Lieb, E. H., Minimum action solutions of some vector field equations. Commun. Math. Phys. **96**, 97-113 (1983).

[4] Brézis, H. and Lieb, E. H., A relation between pointwise convergence of functions and convergence of functionals. Proc. Am. Math. Soc. **88**, 486-490 (1983).

[5] CAZENAVE, T. AND WEISSLER, F.B., The Cauchy problem for the critical nonlinear Schrôdinger equation in H^s, preprint.

[6] FRÖHLICH, J., LIEB, E.H. AND LOSS, M., Stability of coulomb systems with magnetic fields, I, The one-electron atom, Commun. Math. Phys. **104**, 251-270 (1986).

[7] GINIBRE, J. AND VELO, G., On a class of nonlinear Schrödinger equations, I, II, J. Funct Anal. **32**, 1-71 (1979).

[8] GIGA, Y., Solution for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, J. Diff. Eqs. **62**, 186-212 (1986).

[9] GLASSY, R.T., On the blowing up solutions to the Cauchy problem for nonlinear Schrödinger equations, J. Math. Phys. **18**, 1794-1797 (1979).

[10] KATO, T., On nonlinear Schrödinger equations. Ann. Inst. Henri Poincaré, Physique Theorique, **46**, 113-129 (1987).

[11] LIEB, E.H., On the lowest eigenvalue of the Laplacian for the intersection of two domains. Invent. Math. **74**, 441-448 (1983).

[12] LIONS, P.L., The concentration-compactness principle in the calculus of variations. The locally compact case, part I. Ann. I.H.P., Analyse Nonlinéaire, **1**, 109-145 (1984) and part II, **1**, 223-283 (1984).

[13] LIONS, P.L., The concentration-compactness principle in the calculus of variations. The limit case, Part I, Riv. Mat. Iberoamericana. **1**, 45-121 (1985) and Part II, **1**, 145-201 (1985).

[14] LIONS, P.L., Solutions of Hartree-Fock equations for coulomb systems, Commun. Math. Phys. **109**, 33-97 (1987).

[15] MERLE, F., Limit of solution of the nonlinear Schrödinger equation at the blow-up time. J. Funct. Anal. to appear.

[16] MERLE, F., Construction of solutions with exactly k blow-up points for the Schrödinger equation with the critical nonlinearity. (preprint 1988).

[17] MERLE, F. AND TSUTSUMI, Y., L^2-concentration of blow-up solutions for the nonlinear Schrödinger equation with the critical power nonlinearity. (preprint 1988).

[18] NAGASAKI, K. AND SUZUKI, T., Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially-dominated nonlinearities, (preprint 1989).

[19] NAWA, H., Blow-up solutions for the nonlinear Schrödinger equations. (unpublished).

[20] NAWA, H., "Mass concentration" phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity. (preprint 1989).

[21] NAWA, H. AND TSUTSUMI, M., On Blow-up for the pseudo-conformally invariant nonlinear Schrödinger equation, Funk. Ekva. to appear.

[22] STRAUSS, W.A., Existence of solitary waves in higher dimensions, Commun. Math. Phys. **55**, 149-162 (1977).

[23] STRUWE, M., A global compactness result for elliptic boundary value problem involving limiting nonlinearities, Math. Z. **187**, 511-517 (1984).

[24] SULUM, P.L., SULUM, C. AND PATERA, A., Numerical simulation of singular-solutions to the two dimensional cubic Schrödinger equation, Commun. pure Appl. Math. **37**, 755-778 (1984).

[25] TSUTSUMI, M., Nonexistence and instability of solutions of non-linear Schrödinger equations. (unpublished).

[26] TSUTSUMI, Y., Rate of L^2 concentration of blow-up solutions for the nonlinear Schrödinger equation with the critical power nonlinearity. (preprint 1989).

[27] WEINSTEIN, M. I., Nonlinear Schrödinger equations and Sharp interpolation estimates, Commun. Math. Phys. **87**, 567-576 (1983).

[28] WEINSTEIN, M. I., On the structure and formation of singulatrities in solutions to nonlinear dispersive evolution equations, Comm. in Partial Differential Equations **11**, 545-565 (1986).

[29] WEISSLER, F. B., Existence and non-existence of global solutions for semilinear heat equation, Israel J. Math. **38**, 29-49 (1981).

[30] YAJIMA, K., Existence of solutions for Schrödinger equations. Commun. Math. Phys. **110**, 415-426 (1987).

[31] ZAKHAROV, V. E. AND SYNAKH, V. E., The nature of the self-forcusing singularity, Sov. Phys. JETP **41**, 441-448 (1976).

[32] WEINSTEIN, M. I., The nonlinear Schrödinger equation—Singularity formation, stability and dispertion, Contemporary Mathematics **99** (1989), to appear.

[33] OGAWA. T. AND TSUTSUMI, Y., Blow-up H^1-solution for the nonlinear Schrödinger equation. (preprint 1989).

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