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INTEGRALS OF SOME TRIGONOMETRIC FUNCTIONS

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1. Introduction

Let a and b be positive numbers, and u, v and m be positive integers such that $u+v=m$, $2\leq m$. We define $I(m)$ and $I(u;v)$ by

$$I(m)=\int_0^\infty \frac{\sin^m at}{t^m}dt, \qquad I(u;v)=\int_0^\infty \frac{\sin^u at \sin^v bt}{t^m}dt.$$

Tables of integrals give values of $I(m)$ and $I(u;v)$ only for some special cases. For example, for $3\leq m\leq 6$, Gradshteyn and Ryzhik [4] (p. 449-p. 450) gives

$$I(3)=3a^2\pi/8, \qquad I(4)=a^3\pi/3,$$
$$I(5)=115a^4\pi/384, \qquad I(6)=11a^5\pi/40.$$

As for $I(u;v)$ with $a<b$, [4] (p. 451-p. 452) gives

$$\begin{aligned} I(2;2)&=(3b-a)a^2\pi/6 && (a<b),\\ I(3;1)&=a^3\pi/2 && (0<3a\leq b)\\ &=[24a^3-(3a-b)^3]\pi/48 && (0<a<b\leq 3a),\\ I(1;3)&=(9b^2-a^2)a\pi/24 && (a<b). \end{aligned}$$

In this note we give the general expressions of $I(m)$ and $I(u;v)$. These are special cases of Theorem A below. To state our Theorem A we need the following definition. Let $a_1, a_2, \cdots, a_m$ be positive numbers such that $0<a_1\leq a_2\leq \cdots \leq a_{m-1}\leq a_m$. For a subset $\lambda=\{k_1, k_2, \cdots, k_{m-r}\}$ of $\{1, 2, \cdots, m-1\}$, a polynomial

$$P_r(\lambda)=a_{k_1}+a_{k_2}+\cdots+a_{k_{m-r}}-a_{k_{m-r+1}}-\cdots-a_{k_{m-1}}-a_m$$

is said to be of *r-type*, if $\{a_{k_1}, a_{k_2}, \cdots, a_{k_{m-1}}, a_m\}=\{a_1, a_2, \cdots, a_m\}$ as sets and

$$P_r(\lambda)>0, \quad k_1<k_2<\cdots<k_{m-r}, \quad k_{m-r+1}<\cdots<k_{m-1}.$$

Note that a_m appears with negative sign and r is the number of negative signs contained in a polynomial of r-type. A polynomial of 1-type is unique if it exists.

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THEOREM A. *For constants $1<a_1\leq a_2\leq \cdots \leq a_{m-1}\leq a_m$, $2\leq m$, the following holds:*

$$\frac{2}{\pi}\int_0^\infty \frac{1}{t^m}\Big(\prod_{k=1}^m \sin a_k t\Big)dt=\prod_{k=1}^{m-1} a_k-\frac{1}{c'}\sum_{r=1}^{m-2}(-1)^{r-1}\sum_{r\text{-type}} P_r(\lambda)^{m-1}, \tag{1.1}$$

where $c'=(m-1)!\,2^{m-2}$ and $P_r(\lambda)$ denotes a polynomial of r-type.

If $a_m>a_1+a_2+\cdots+a_{m-1}$, then there is no polynomial of r-type ($r\geq 1$), and so the above integral does not depend on a_m.

For proof of Theorem A we use the volume expression of cube-slicing by Hensley [5] and Ball [1], and actually we give the volume of cube-slicing in terms of numbers a_i (after normalization) using polynomials of r-type by elementary geometric method.

A special case of Theorem A for $m=3$ is given, for example, at p. 79 of Erdélyi [3] and at p. 422 of Gradshteyn and Ryzhik [4]. A special case where there is no polynomial of r-type ($r\geq 1$) is given at p. 417 of [4].

Three special cases of Theorem A are given as Corollaries C, D and E in § 3. From these one can deduce many analogous formulas. Only several examples are given in Corollary F.

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2. The volume of cube-slicing

By $\{e_j, 1\leq j\leq m\}$ we denote the standard base of the Euclidean m-space E^m at the origen O, and by $\{x^j\}$ the standard coordinate system of E^m. Let K^m be the unit cube in E^m, which is expressed by $\{x; 0\leq x^j\leq 1, 1\leq j\leq m\}$. Let $b_1, b_2, \cdots, b_m$ be positive numbers such that

$$b_1\geq b_2\geq \cdots \geq b_m>0.$$

Let T be an m-simplex determined by $\{O, p_1, p_2, \cdots, p_m\}$, where p_j is the end point of the vector b_je_j, $1\leq j\leq m$.

LEMMA. *The volume $V(K^m\cap T)$ of the intersection of K^m and T is given by*

$$V(K^m\cap T)=\frac{b_1b_2\cdots b_m}{m!}\sum_{r=0}^m(-1)^r\sum^*_{k_1<k_2<\cdots<k_r}\Big(1-\frac{1}{b_{k_1}}-\cdots-\frac{1}{b_{k_r}}\Big)^m,$$

where $\sum^*$ denotes the sum over all positive terms $(1-1/b_{k_1}-\cdots-1/b_{k_r}>0)$.

Proof. (i) If $b_1\leq 1$, then the m-simplex T is contained in K^m and $V(K^m\cap T)=V(T)=b_1b_2\cdots b_m/m!$.

(ii) If $b_1>1\geq b_2$, then only one vertex p_1 of T lies outside K^m. The intersection $T\cap(K^m)^C$ of T and the complement $(K^m)^C$ of K^m in E^m defines an m-

simplex T_1, which we call the outer simplex at p_1. $V(K^m \cap T)$ is given by the difference of the volume $V(T)$ and the volume $V(T_1)$ of the outer simplex at p_1, and

$$V(K^m \cap T) = (b_1 b_2 \cdots b_m / m!)[1-(1-1/b_1)^m].$$

(iii) If $b_s > 1 \geq b_{s+1}$, for $2 \leq s \leq m$ (putting $b_{m+1}=0$), then vertices $p_1, p_2, \cdots, p_s$ of T lie outside K^m. In this case the outer simplex T_h at p_h, $1 \leq h \leq s$, is defined by $T_h = T \cap \{x; x^h \geq 1\}$.

(iii-1) If the outer simplexes at $p_1, p_2, \cdots, p_s$ are disjoint, then $V(K^m \cap T) = V(T) - V(T_1) - V(T_2) - \cdots - V(T_s)$, where $V(T_h) = (b_1 b_2 \cdots b_m/m!)(1-1/b_h)^m$, $1 \leq h \leq s$.

(iii-2) Two outer simplexes at p_1 and p_2 have a non trivial intersection T_{12}, if and only if $1-1/b_1-1/b_2>0$, which is equivalent to the fact that the vertex $(1, 1, 0, \cdots, 0)$ of K^m lies below the affine hyperplane determined by the face of T opposite to O. The volume $V(T_{12})$ of T_{12} is equal to $(b_1 b_2 \cdots b_m/m!)(1-1/b_1-1/b_2)^m$. Let $\{T_{12}, T_{13}, T_{23}, \cdots, T_{jh}\}$ be the set of all non trivial intersections of two outer simplexes. If three outer simplexes at p_1, p_2, and p_3 do not have non trivial intersecton T_{123} (or equivalently, $1-1/b_1-1/b_2-1/b_3 \leq 0$), then $V(K^m \cap T)$ is given by $V(T)-V(T_1)-V(T_2)-\cdots-V(T_s)+V(T_{12})+\cdots+V(T_{jh})$.

(iii-3) If T_{123} is non trivial, then we need the term $-V(T_{123}) = -(b_1 b_2 \cdots b_m/m!)(1-1/b_1-1/b_2-1/b_3)^m$ in the expression of $V(K^m \cap T)$.

(iii-4) Generally, the intersection $T_{t_1 t_2 \cdots t_u}$ of $T_{t_1}, T_{t_2}, \cdots, T_{t_u}$ is non trivial, if and only if $1-1/b_{t_1}-1/b_{t_2}-\cdots-1/b_{t_u}>0$, and its volume has sign $(-1)^u$ in the expression of $V(K^m \cap T)$. q. e. d.

Let B^m be the unit cube in E^m centered at the origin: $B^m = \{x; -1/2 \leq x^j \leq 1/2, 1 \leq j \leq m\}$. Let $a=(a_1, a_2, \cdots, a_m)$ be a unit vector in E^m and $H(a)$ be the hyperplane passing through the origin and orthogonal to a. Since the case where $a_1=0$ reduces to the lower dimensional case, we assume that the components of a satisfy

$$0 < a_1 \leq a_2 \leq \cdots \leq a_{m-1} \leq a_m .$$

Concerning the volume $V_m(a)$ of the slice $B^m \cap H(a)$ corresponding to a, Hensley [5] and Ball [1] gave the best possible inequality; $1 \leq V_m(a) \leq \sqrt{2}$, which was verified by using the following expression of $V_m(a)$:

$$V_m(a) = \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{k=1}^{m} \frac{\sin a_k t}{a_k t} dt . \tag{2.1}$$

Since (1.1) is homothetically invariant with respect to (a_i), to prove Theorem A it suffices to give the value of the left hand side of (2.1). Namely we prove the following.

PROPOSITION B. *The volume $V_m(a)$ of the slice corresponding to a is given by*

$$V_m(a) = \frac{1}{a_m} - \frac{1}{C} \sum_{r=1}^{m-2} (-1)^{r-1} \sum_{r\text{-}type} P_r(\lambda)^{m-1} , \tag{2.2}$$

where $c=(m-1)!\,2^{m-2}a_1a_2\cdots a_m$ *and* $P_r(\lambda)$ *denotes a polynomial of r-type.*

Proof. We define θ by $\cos\theta=\langle e_m, a\rangle=a_m$. By ρ we denote the orthogonal projection of E^m onto E^{m-1} defined by $x_m=0$. First we study the case where $a_m>a_1+a_2+\cdots+a_{m-1}$. The condition $a_m>a_1+a_2+\cdots+a_{m-1}$ is equivalent to the fact that $H(a)$ does not meet the upper face F^{m-1} of B^m defined by $x_m=1/2$. Therefore, $\rho(B^m\cap H(a))=B^{m-1}$. Hence, $V_m(a)=1/\cos\theta=1/a_m$.

Next we assume that $a_m<a_1+a_2+\cdots+a_{m-1}$ holds. Then $H(a)$ meets the upper face F^{m-1}. We denote the part of F^{m-1} which lies below $H(a)$ by $K(a)$. Then the volume $V(K(a))$ of $K(a)$ is given by the preceding Lemma. The relation between (b_k) and (a_k) is given by $b_k=(a_1+a_2+\cdots+a_{m-1}-a_m)/2a_k$, $1\leq k\leq m-1$. Consequently, we obtain the following:

$$1-1/b_k=(a_1+a_2+\cdots\check{a}_k\cdots+a_{m-1}-a_k-a_m)/A,$$

$$1-1/b_k-1/b_l=(a_1+a_2+\cdots\check{a}_k\cdots\check{a}_l\cdots+a_{m-1}-a_k-a_l-a_m)/A,$$

etc., where $A=a_1+a_2+\cdots+a_{m-1}-a_m$ and $\check{a}_k$ means that a_k is removed. Since $V_m(a)=[1-2V(K(a))]/\cos\theta$, we obtain (2.2). q. e. d.

3. Corollaries

First we give three special cases of Theorem A.

COROLLARY C. *For a positive number a and integer* $m\geq 2$, *the following holds.*

$$\int_0^\infty\frac{\sin^m at}{t^m}dt=\frac{a^{m-1}\pi}{(m-1)!\,2^{m-1}}\Big[(m-1)!\,2^{m-2}-\sum_{r=1}^{[(m-1)/2]}(-1)^{r-1}{}_{m-1}C_{r-1}(m-2r)^{m-1}\Big]. \tag{3.1}$$

COROLLARY D. *For positive numbers* $a<b$ *and positive integers* u, v $(u+v=m)$, *the following holds*·

$$\int_0^\infty\frac{\sin^u at\sin^v bt}{t^m}dt=\frac{\pi}{(m-1)!\,2^{m-1}}\Big[(m-1)!\,2^{m-2}a^u b^{v-1}-\sum_{r=1}^{m-2}(-1)^{r-1}\sum_{p=0}^{*}{}_uC_p\cdot{}_{v-1}C_{r-p-1}\{(u-2p)a+(v-2r+2p)b\}^{m-1}\Big], \tag{3.2}$$

where $\sum^*$ *denotes the sum over all polynomials* $(u-2p)a+(v-2r+2p)b>0$ *and* p *runs from* $\max\{r-v, 0\}$ *to* $\min\{r-1, u\}$.

COROLLARY E. *For positive numbers* $a<b<c$ *and positive integers* $u, v, w(u+v+w=m)$, *the following holds:*

$$(3.3)\quad \int_0^\infty \frac{\sin^u at \sin^v bt \sin^w ct}{t^m}dt=\frac{\pi}{(m-1)!\,2^{m-1}}\Big[(m-1)!\,2^{m-2}a^u b^v c^{w-1}$$

$$-\sum_{r=1}^{m-2}(-1)^{r-1}\sum_{p+q+s=r-1}^{*}{}_uC_p\cdot{}_vC_q\cdot{}_{w-1}C_s\cdot P(p,q,s)^{m-1}\Big],$$

where $P(p, q, s)=(u-2p)a+(v-2q)b+(w-2s-2)c$, *and* $\sum^*$ *denotes the sum over all polynomials* $P(p, q, s)>0$ *and* $0\leq p\leq \min\{r-1, u\}$, $0\leq q\leq \min\{r-1, v\}$ *and* $0\leq s=r-1-p-q\leq w-1$.

Proof of Corollary C. The number of polynomials $a+a+\cdots+a-a$ of 1-type is one for $m\geq 3$. The number of polynomials $a+a+\cdots+a-a-a$ of 2-type is $m-1$ for $m\geq 5$. Similarly, the number of polynomials $a+a+\cdots+a-a-\cdots-a-a$ of r-type is ${}_{m-1}C_{r-1}$. The range of r is from 1 to $[(m-1)/2]$. Therefore, Corollary C follows from Theorem A.

Proof of Corollary D. The number of polynomials of 1-type is at most one; $a+a+\cdots+a+b+\cdots+b-b$ for $m\geq 3$. Each of polynomials of 2-type is one of the following;

$$a+a+\cdots+a+a+b+\cdots+b-b-b \qquad (ua+(v-4)b>0),$$

$$a+a+\cdots+a+b+\cdots+b+b-a-b \qquad ((u-2)a+(v-2)b>0).$$

The numbers of such polynomials are ${}_uC_0\cdot{}_{v-1}C_1$ and ${}_uC_1\cdot{}_{v-1}C_0$. By p we denote the number of a with negative sign in the polynomial of r-type. Polynomials of r-type for general r and the number of such polynomials are similarly studied.

Proof of Corollary E is similar.

Corollary C enables us to calculate $I(m)$ for any m, for example, we obtain

$$I(7)=5887a^6\pi/23040, \qquad I(8)=151a^7\pi/630,$$

$$I(9)=259723a^8\pi/1146880, \qquad I(10)=15619a^9\pi/72576.$$

Also Corollary D enables us to calculate $I(u; v)$ for any u, v, for example, we obtain

$$I(3;4)=\frac{\pi}{6!\,2^6}[6!\,2^5a^3b^3-(3a+2b)^6+3(3a)^6+3(a+2b)^6$$

$$-3(3a-2b)^6-9a^6-3(-a+2b)^6] \qquad (2b\leq 3a<3b)$$

$$=\frac{\pi}{6!\,2^6}[6!\,2^5a^3b^3-(3a+2b)^6+3(3a)^6+3(a+2b)^6$$

$$-9a^6-3(-a+2b)^6+(-3a+2b)^6] \qquad (3a<2b).$$

The formulas in Corollaries produce many analogous formulas. Here we give some examples. For convenience sake we use the following notations:

$$I_m(u\,;\,a)=\int_0^\infty \frac{\sin^u at}{t^m}dt\,,$$

$$I_m(u,\,v\,;\,a,\,b)=\int_0^\infty \frac{\sin^u at\sin^v bt}{t^m}dt\,,$$

$$I_m(u,\,v,\,w\,;\,a,\,b,\,c)=\int_0^\infty \frac{\sin^u at\sin^v bt\sin^w ct}{t^m}dt\,.$$

Then, for $m\geq 2$, we obtain the following:

COROLLARY F.

$$\int_0^\infty \frac{\sin^{m+2} at}{t^m}dt=I_m(m\,;\,a)-\frac{1}{4}I_m(m-2,\,2\,;\,a,\,2a),\tag{3.4}$$

$$\begin{aligned}\int_0^\infty \frac{\sin^m at\sin^2 bt}{t^m}dt&=\frac{1}{2}I_m(m\,;\,a)-\frac{1}{2}I_m(m-2,\,2\,;\,a,\,a+b)\\&+\frac{1}{2}I_m(m-2,\,2\,;\,a,\,b)+\frac{1}{4}I_m(m-2,\,1,\,1\,;\,a,\,2a,\,2b),\end{aligned}\tag{3.5}$$

$$\int_0^\infty \frac{\sin^{m+4} at}{t^m}dt=I_m(m+2\,;\,a)-\frac{1}{4}I_m(m,\,2\,;\,a,\,2a),\tag{3.6}$$

$$\int_0^\infty \frac{\sin^m at\cos^2 bt}{t^m}dt=I_m(m\,;\,a)-I_m(m,\,2\,;\,a,\,b),\tag{3.7}$$

$$\int_0^\infty \frac{\sin^m at\cos bt}{t^m}dt=I_m(m-1,\,1\,;\,a,\,a+b)-\frac{1}{2}I_m(m-2,\,1,\,1\,;\,a,\,2a,\,b),\tag{3.8}$$

etc.

In the above, (3.5) is somewhat complicated. If $m\geq 4$, then it has a simpler expression: $I_m(m-2,\ ;\,a,\,b)-(1/4)I_m(m-4,\,2,\,2\,;\,a,\,2a,\,b)$.

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