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STABILITY OF CLOSED LIE SUBGROUPS IN COMPACT LIE GROUPS

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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Introduction. In [C-L-N], they dealt systematically the stability of totally geodesic submanifolds of a compact Riemannian symmetric space as minimal submanifolds. Using the method, Takeuchi [Tak] and Ohnita [O] studied the stability of some kinds of totally geodesic submanifolds. The class of closed subgroups in compact Lie groups with bi-invariant Riemannian metrics is one of the most typical totally geodesic submanifolds. On their stability, there are some results by Fomenko [F], Thi [Th] and Brothers [Br].

In [D], the index of a (complex) simple Lie subalgebra in a (complex) simple Lie algebra was defined and it played an important role. The results mentioned above and a result by the second named author (Theorem A) made us get interested in the problem to find some relationship between the index of a Lie subgroup and the stability of it as a totally geodesic submanifold.

Let U be a compact connected simple Lie group whose rank is greater than 1 and U_1 be an analytic subgroup of U associated with the highest root of U. It is known that U_1 is isomorphic to $SU(2)$ ([W]). The second named author [Tasl] proved the following: *U_1 is homologically volume minimizing (especially it is a stable minimal submanifold) with respect to a bi-invariant Riemaenian metric on U.* By the definition, U_1 is a subgroup of index 1. On the other hand, a 3-dimensional connected simple closed subgroup of index 1 in U is conjugate to U_1. Thus we can restate the above Theorem as follows:

THEOREM A. *Let U be a compact connected simple Lie group whose rank is greater than 1. A connected 3-dimensional simple closed subgroup of index 1 in U is a stable minimal submanifold with respect to a bi-invariant Riemannian metric on U.*

In this paper we generalize the above Theorem. Precisely speaking, we will

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prove the following:

THEOREM B. *Let U be a compact connected simple Lie group with a bi-invariant Riemannian metric. A connected simple closed Lie subgroup G of index 1 in U is a stable minimal submanifold.*

For the case that G is isomorphic to $SU(2)$ the converse to Theorem B is true. Namely we will prove the following:

THEOREM C. *Let G be a a simple Lie subgroup which is isomorphic to $SU(2)$ in a compact connected simple Lie group U. Then, G is stable if and only if G is of index* 1.

In general, the converse to Theorem B is not true. For the case that G is isomorphic to $SO(3)$ a necessary and sufficient condition that G is stable in U will be given in section 4 (Theorem D). Moreover we will determine all stable 3-dimensional connected simple Lie subgroups in each compact connected simple Lie group (Theorem E). And we get some examples which is stable but not of index 1.

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1. Stability of totally geodesic submanifolds.

In this section, we give a brief review on basic results on the stability of totally geodesic submanifolds in compact symmetric spaces after [O].

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and M be a homogeneous space of G. Let o be a point in M and K be the isotropy subgroup of G at o. Let E be a G-homogeneous complex vector bundle on M. Then the fiber E_o over o is a K-module. The space of smooth sections of E on M is denoted by $\Gamma(E)$. Let $C^\infty(G; E_0)$ be the space of smooth E_o-valued functions on G and $C^\infty(G; E_o)_K=\{f\in C^\infty(G; E_o): f(uk)=k^{-1}f(u),$ for $u\in G$ and $k\in K\}$. Then G acts on $\Gamma(E)$ and $C^\infty(G; E_o)_K$ in a natural manner. Define a mapping

$$s: C^\infty(G; E_o)_K \longrightarrow \Gamma(E); f \longrightarrow [g\cdot o \longrightarrow gf(g)].$$

Then s is a G-isomorphism. Each element of the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G acts on $C^\infty(G; E_o)$ as a left invariant (linear) differential operator. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then the action of $\mathfrak{g}$ on $C^\infty(G; E_o)$ is extended to that of $U(\mathfrak{g})$ in a natural manner. An element $L\otimes X$ of $\mathrm{Hom}(E_o, E_o)\otimes U(\mathfrak{g})$ acts, as a linear differential operator, on $C^\infty(G; E_o)$ by

$$(L\otimes X)(f)=L(Xf), \qquad f\in C^\infty(G; E_o).$$

Define an action of K on $\mathrm{Hom}(E_o, E_o)\otimes U(\mathfrak{g})$ by $k(L\otimes X)=(kLk^{-1})\otimes \mathrm{Ad}(k)X$ for $k\in K$. Then a K-invariant element D of $\mathrm{Hom}(E_o, E_o)\otimes U(\mathfrak{g})$ leaves the subspace

$C^\infty(G; E_o)_K$ invariant. Thus D induces a G-invariant linear differential operator of $\Gamma(E)$. Converesely every G-invariant linear differential operators of $\Gamma(E)$ can be obtained in the above manner.

Let P be a compact Riemannian symmetric space and U be the identity component of the group of isometries of P. We denote by R^P the curvature tensor of P. Let M be a compact totally geodesic submanifold of P. Take an analytic subgroup G of U which leaves M invariant and is locally isomorphic to the group of isometries of M. Then the normal bundle $N(M)$ of M in P is a G-homogeneous vector bundle. Let $\{M_t\}$ be a smooth variation of M in P and V be its variational vector field. We denote by V^N the normal component of V. Define a section S of $\mathrm{End}(N(M))$ by

$$\langle S(\xi), \eta\rangle = \sum_i \langle R^P(e_i, \xi)e_i, \eta\rangle \qquad \text{for} \quad \xi, \eta \in N_p(M),$$

taking an orthonormal basis $\{e_i\}$ of $T_p(M)$. We denote by $\Delta^{N(M)}$ the rough Laplacian of the normal connection on $N(M)$. Then the second variational formula is given by the following:

$$d^2 \mathrm{Vol}(M_t)/dt^2|_{t=0} = \int_M \langle \mathcal{J}(V^N), V^N\rangle d\, vol_M,$$

where $\mathcal{J}$ is defined by

$$\mathcal{J} = -\Delta^{N(M)} + S,$$

and is called the *Jacobi differential operator*. It is easily verified that $\mathcal{J}$ is a G-invariant linear differential operator of $\Gamma(N(M))$.

We denote by L the isotropy subgroup of U at $o \in M \subset P$ and put $K = G \cap L$. Let $\mathfrak{u}$, $\mathfrak{l}$ and $\mathfrak{k}$ be the Lie algebras of U, L and K respectively. Take an $\mathrm{Ad}(U)$-invariant inner product $\langle\,,\rangle$ on $\mathfrak{u}$ which induces the Riemannian metric on P. Take the orthogonal complement $\mathfrak{m}$ [resp. $\mathfrak{p}$] of $\mathfrak{k}$ [resp. $\mathfrak{l}$] in $\mathfrak{g}$ [resp. $\mathfrak{u}$]. Let $\mathfrak{m}^\perp$ [resp. $\mathfrak{k}^\perp$] be the orthogonal complement of $\mathfrak{m}$ [resp. $\mathfrak{k}$] in $\mathfrak{p}$ [resp. $\mathfrak{l}$]. Let $\mathfrak{g}^\perp$ be the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{u}$. Then $\mathfrak{g}^\perp = \mathfrak{k}^\perp \oplus \mathfrak{m}^\perp$. The action of $\mathfrak{g}$ on $\mathfrak{g}^\perp$ is extended to that of $U(\mathfrak{g})$ on $\mathfrak{g}^\perp$. Let C be the Casimir element of $U(\mathfrak{g})$ with respect to the inner product $\langle\,,\rangle|_{\mathfrak{g}\times\mathfrak{g}}$. Since $\mathrm{ad}_{\mathfrak{u}}(C)$ leaves $\mathfrak{m}$ invariant, it is considered as an element of $\mathrm{Hom}(\mathfrak{m}^\perp, \mathfrak{m}^\perp)$. Then the Jacobi differential operator $\mathcal{J}$ is identified with a linear differential operator on $C^\infty(G; N_o(M))_K$.

THEOREM 1.1.

$$\mathcal{J} = \mathrm{ad}_{\mathfrak{u}}(C) \otimes I - I \otimes C.$$

Proof. We refer to [0].

Since the Jacobi differential operator $\mathcal{J}$ is a strongly elliptic linear differential operator, it has discrete eigenvalues

$$\lambda_1 < \lambda_2 < \cdots \longrightarrow \infty$$

and all eigenspaces are of finite dimension. We put $E_\lambda = \{V \in \Gamma(N(M)) : \mathcal{J}(V) =$

$\lambda V\}$. We call the number $i(M)=\sum_{\lambda<0}\dim E_\lambda$ the *index* of M in P, and $n(M)=\mathrm{din}\, E_0$ the *nullity* of M in P. When the index $i(M)=0$, the submanifold M is said to be *stable* in P. We call the dimension of the subspace $\{X^N : X$ is a Killing vector field on $P\}$ of E_0 the *Killing nullity* of M in P and denote it by $n_K(M)$.

We denote by $\mathcal{D}(G)$ the set of all equivalence classes of the complex irreducible representations of G. Let $V(\lambda)$ be a representation space of an element λ of $\mathcal{D}(G)$. Then $\lambda(C)$ is a scalar operator $a_\lambda I$ on $V(\lambda)$. Let θ be the involutive automorphism of $\mathfrak{u}$ defining the symmetric structure of $P=U/L$. We can take a direct sum decomposition $\mathfrak{g}^\perp=\mathfrak{g}_1{}^\perp\oplus\cdots\oplus\mathfrak{g}_k{}^\perp$, where each $\mathfrak{g}_i{}^\perp$ is θ-stable G-invariant and has no nontrivial θ-stable, G-invariant subspace. By Schur's lemma and $\theta(C)=C$, we have $\mathrm{ad}_\mathfrak{u}(C)=a_i I$ on each $\mathfrak{g}_i{}^\perp$ for some scalar a_i. Put $\mathfrak{k}_i{}^\perp=\mathfrak{k}^\perp\cap\mathfrak{g}_i{}^\perp$ and $\mathfrak{m}_i{}^\perp=\mathfrak{m}^\perp\cap\mathfrak{g}_i{}^\perp$. Then we have $\mathfrak{g}_i{}^\perp=\mathfrak{k}_i{}^\perp\oplus\mathfrak{m}_i{}^\perp$ and each $\mathfrak{m}_i{}^\perp$ is K-invariant.

THEOREM 1.2. *The index, nullity and Killing nullity are given as follows:*

(i) $i(M)=\sum_{i=1}^{k}\sum_{\substack{a_\lambda>a_i\\ \lambda\in\mathcal{D}(G)}}\dim \mathrm{Hom}_K(V(\lambda), (\mathfrak{m}_i{}^\perp)^C)\dim V(\lambda)$

(ii) $n(M)=\sum_{i=1}^{k}\sum_{\substack{a_\lambda=a_i\\ \lambda\in\mathcal{D}(G)}}\dim \mathrm{Hom}_K(V(\lambda), (\mathfrak{m}_i{}^\perp)^C)\dim V(\lambda)$

(iii) $n_K(M)=\sum_{i=1,\,\mathfrak{m}_i{}^\perp\neq 0}^{k}\dim\mathfrak{g}_i{}^\perp$.

Proof. We refer to [0].

2. Lie subgroups.

In this section we consider the case that P is a compact connected semisimple Lie group U with a bi-invariant Riemannian metric $\langle\, ,\rangle$ and M is a connected closed semisimple subgroup G. We denote by $\mathfrak{u}$ and $\mathfrak{g}$ the Lie algebras of U and G respectively. The bi-invariant Riemannian metric $\langle\, ,\rangle$ on U induces an $\mathrm{Ad}(U)$-invariant inner product on $\mathfrak{u}$, which we also denote by $\langle\, ,\rangle$. Let $\mathfrak{g}^\perp$ be the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{u}$. Take the identity element as the point o. We use the following notation:

$$U^*=U\times U\,,$$
$$G^*=G\times G\,,$$
$$L=\{(u,\ u)\in U^* : u\in U\}\,,$$
$$K=\{(g,\ g)\in G^* : g\in G\}\,,$$
$$\mathfrak{p}=\{(X,\ -X) : X\in\mathfrak{u}\}\,,$$
$$\mathfrak{m}=\{(X,\ -X) : X\in\mathfrak{g}\}\,,$$
$$(\mathfrak{g}^*)^\perp=\{(X,\ Y) : X,\ Y\in\mathfrak{g}^\perp\}\,,$$

$$\mathfrak{k}^{\perp}=\{(X, X): X\in\mathfrak{g}^{\perp}\},$$
$$\mathfrak{m}^{\perp}=\{(X, -X): X\in\mathfrak{g}^{\perp}\}.$$

We take the direct sum of $\langle\ ,\ \rangle$ as an inner product on the Lie algebra $\mathfrak{u}\oplus\mathfrak{u}$ of U^*. Take a G-irreducible decomposition of $\mathfrak{g}^{\perp}$:

$$\mathfrak{g}^{\perp}=\mathfrak{g}_1{}^{\perp}\oplus\cdots\oplus\mathfrak{g}_k{}^{\perp}.$$

Then it induces a G^*-invariant decomposition:

$$(\mathfrak{g}^*)^{\perp}=\bigoplus_{i=1}^{k}(\mathfrak{g}_i^*)^{\perp},\qquad (\mathfrak{g}_i^*)^{\perp}=\{(X, Y): X, Y\in\mathfrak{g}_i{}^{\perp}\}$$

Each $(\mathfrak{g}_i^*)^{\perp}$ is θ-stable and G^*-invariant, where θ is the involutive automorphism $\theta:\mathfrak{u}\oplus\mathfrak{u}\to\mathfrak{u}\oplus\mathfrak{u}$; $(X, Y)\to(Y, X)$. We have a decomposition $(\mathfrak{g}_i^*)^{\perp}=\mathfrak{k}_i{}^{\perp}\oplus\mathfrak{m}_i{}^{\perp}$, where

$$\mathfrak{k}_i{}^{\perp}=\{(X, X): X\in\mathfrak{g}_i{}^{\perp}\}.$$
$$\mathfrak{m}_i{}^{\perp}=\{(X, -X): X\in\mathfrak{g}_i{}^{\perp}\}.$$

If $\dim\mathfrak{g}_i{}^{\perp}=1$, then both of $\mathfrak{k}_i{}^{\perp}$ and $\mathfrak{m}_i{}^{\perp}$ are θ-stable and G^*-irreducible. It is well-known that $\mathcal{D}(G^*)=\{(V(\lambda), \lambda)\boxtimes(V(\mu), \mu): \lambda, \mu\in\mathcal{D}(G)\}$, where $\boxtimes$ means the outer tensor product. Let C^* be the Casimir element of $U(\mathfrak{g}^*)$ and let C be the Casimir element of $U(\mathfrak{g})$. Let a_λ be the eigenvalue of $\lambda(C)$ on $V(\lambda)$ for each $\lambda\in\mathcal{D}(G)$. Since $(\lambda\boxtimes\mu)(C^*)=\lambda(C)\otimes I+I\otimes\mu(C)$, we have $(\lambda\boxtimes\mu)(C^*)=(a_\lambda+a_\mu)I$. We simply denote by a_i the eigenvalue of $\mathrm{ad}_{\mathfrak{u}}(C)$ on $\mathfrak{g}_i{}^{\perp}$, then $\mathrm{ad}_{\mathfrak{u}^*}(C^*)=a_iI$ on each $(\mathfrak{g}_i^*)^{\perp}$. Since $\mathfrak{m}_i{}^{\perp}$ is a K-irreducible module, we must decompose each G^*-irreducible module into a direct sum of K-irreducible modules. Since K is the diagonal subgroup of G^*, the problem is to decompose the (inner) tensor product $V(\lambda)\otimes V(\mu)$ into a direct sum of G-irreducible modules. We can reagard each K-module $\mathfrak{m}_i{}^{\perp}$ as a G-module $\mathfrak{g}_i{}^{\perp}$. Applying the Theorem 1.2 to our case, we have the following:

THEOREM 2.1. *The index, nullity and Killing nullity are given as follows*:

(i) $$i(G)=\sum_{i=1}^{k}\sum_{\substack{a_\lambda+a_\mu>a_i\\ \lambda,\mu\in\mathcal{D}(G)}}\dim\mathrm{Hom}_G(V(\lambda)\otimes V(\mu), (\mathfrak{g}_i{}^{\perp})^C)\dim(V(\lambda)\otimes V(\mu))$$

(ii) $$n(G)=\sum_{i=1}^{k}\sum_{\substack{a_\lambda+a_\mu=a_i\\ \lambda,\mu\in\mathcal{D}(G)}}\dim\mathrm{Hom}_G(V(\lambda)\otimes V(\mu), (\mathfrak{g}_i{}^{\perp})^C)\dim(V(\lambda)\otimes V(\mu))$$

(iii) $$n_K(G)=\#\{i:\dim\mathfrak{g}_i{}^{\perp}=1\}+2\sum_{\dim(\mathfrak{g}_i{}^{\perp})\neq1}\dim(\mathfrak{g}_i{}^{\perp})$$

In order to count $\dim\mathrm{Hom}_G(V(\lambda)\otimes V(\mu), (\mathfrak{g}_i{}^{\perp})^C)$ we must remember that there are two possibilities for $(\mathfrak{g}_i{}^{\perp})^C$:

(i) $(\mathfrak{g}_i{}^{\perp})^C$ is G-irreducible,

(ii) $(\mathfrak{g}_i{}^{\perp})^C$ is decomposed into a direct sum of G-irreducible modules V and $\overline{V}$, the conjugate module of V.

Let T be a maximal torus of G and $\mathfrak{t}$ be its Lie algebra. Let (V, ρ) be a complex representation of G. For each element λ in $\mathfrak{t}$, put

$$V_\lambda=\{X\in V : \rho(H)(X)=\sqrt{-1}\langle\lambda, H\rangle X, \text{ for any } H\in\mathfrak{t}\}.$$

If $V_\lambda\neq\{0\}$, then λ is called a weight and V_λ is called a weight space. Especially, if $(V, \rho)=(\mathfrak{g}^C, \mathrm{ad})$, then a weight is called a root of G and a weight space is called a root space. We denote by $\Sigma(G)$ the set of all non-zero roots of G. Fix a lexicographic ordering on $\mathfrak{t}$.

THEOREM 2.2 (Freudenthal). *Let (V, ρ) be a complex irreducible representation of G with highest weight λ. Then the eigenvalue a_λ of the Casimir operator $\rho(C)$ with respect to $\langle\,,\,\rangle|_{\mathfrak{g}\times\mathfrak{g}}$ is given by the following*

$$a_\lambda=-\langle\lambda+2\delta, \lambda\rangle \tag{2.1}$$

where δ is half the sum of positive roots of G.

If we assume that U is a compact simple Lie group, then an $Ad(U)$-invariant inner product on $\mathfrak{u}$ is unique up to a constant. As a normalizing condition, we assume that the square of the length of the longest root is equal to 2. We call such an inner product the *canonical inner product*. We assume that both of U and G are simple. Let $\langle\,,\,\rangle_{\mathfrak{u}}$ and $\langle\,,\,\rangle_{\mathfrak{g}}$ be the canonical inner products on $\mathfrak{u}$ and $\mathfrak{g}$ respectively. Since $\langle\,,\,\rangle_{\mathfrak{u}}$ is also an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, there exists a real number j such that

$$\langle X, Y\rangle_{\mathfrak{u}}=j\langle X, Y\rangle_{\mathfrak{g}} \qquad \text{for any } X, Y\in\mathfrak{g},$$

which we call the *index* of $\mathfrak{g}$ in $\mathfrak{u}$ or the *index* of G in U. In fact it is known that the index is a positive integer ([A-H-S], [D], [Y]).

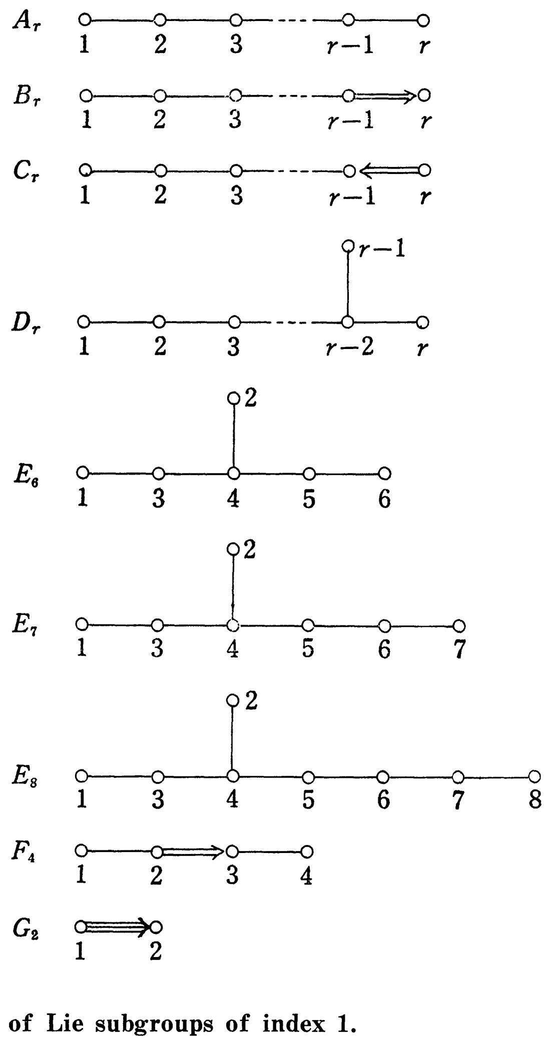
THEOREM 2.3 (Dynkin, [D, p. 133]). *Let U be a compact connected simple Lie group and G be a closed connected simple Lie subgroup. If the index of G in U is equal to* 1, *then roots of maximal length, and the corresponding root vectors in $\mathfrak{g}^C$ are roots and root vectors in $\mathfrak{u}^C$ respectively with respect to a maximal torus of U containing T.*

A complex subalgebra $\bar{\mathfrak{g}}$ of a complex semisimple Lie algebra $\bar{\mathfrak{u}}$ is said to be a *regular subalgebra*, if there exists a basis of $\bar{\mathfrak{g}}$ consisting of elements of some Cartan subalgebra $\bar{\mathfrak{h}}$ of $\bar{\mathfrak{u}}$ and root vectors of the Lie algebra $\bar{\mathfrak{u}}$ with respect to $\bar{\mathfrak{h}}$. In our compact case, if $\mathfrak{g}^C$ is a regular subalgebra of $\mathfrak{u}^C$, $\mathfrak{g}$ is said to be a *regular subalgebra* of $\mathfrak{u}$. Theorem 2.3 asserts that if every roots of G is of the same length and G is of index 1, then $\mathfrak{g}$ is a regular subalgebra of $\mathfrak{u}$.

We denote by $\{\alpha_1, \cdots, \alpha_r\}$ a fundamental root system of $\Sigma(G)$ and by α_0 the highest root of $\Sigma(G)$. Let $\alpha_0=\sum_{j=1}^{r} m_j\alpha_j$. Throughout this paper the fundamental roots are numbered as in the Table 1 at the end of this section. The fundamental weights $\bar{\omega}_j$ are numbered correspondingly.

Let T' be a maximal torus of U which contains T and $\Sigma(U)$ be the set of non-zero roots of U with respect to T'. We denote by $\{\beta_1, \cdots, \beta_{r'}\}$ a fundamental root system of $\Sigma(U)$ and by β_0 the highest root of $\Sigma(U)$.

Table 1. Numbering of the simple roots



3. Stability of Lie subgroups of index 1.

Let U be a compact connected simple Lie group with a bi-invariant Riemannian metric and G be a simple connected closed Lie subgroup. In this section we assume that G is a subgroup of index 1 and study the stability of G in U

as a totally geodesic submanifold. The purpose of this section is to prove the following:

THEOREM B. *Let U be a compact connected simple Lie group with a bi-invariant Riemannian metric. A simple connected closed Lie subgroup G of index 1 in U is a stable minimal submanifold.*

We will employ the same notation as in section 2. By our assumption, there are no need to distinguish the canonical inner product on $\mathfrak{g}$ and $\mathfrak{u}$. Thus we denote them by $\langle\, ,\rangle$, which will be used to define a bi-invariant Riemannian metric.

3.1 First we determine the structure of the normal space of $\mathfrak{g}$ in $\mathfrak{u}$. Since the index of G in U is 1, we may assume that $\alpha_0=\beta_0$, by Theorem 2.3. Let λ be the highest weight of an irreducible component V of the G-module $(\mathfrak{u}^C, \mathrm{ad})$. Let $\pi: \mathfrak{t}'\to\mathfrak{t}$ be the orthogonal projection. Take $\beta\in\Sigma(U)$ such that $\pi(\beta)=\lambda$. Then by Schwarz' inequality, we have $\langle\pi(\beta), \beta_0\rangle=\langle\beta, \beta_0\rangle\leqq 2$, where the equality holds if and only if $\beta=\beta_0$. If $\beta=\beta_0$, then the component V must coincide with $\mathfrak{g}^C$. Thus if V is an irreducible componenet of the G-module $((\mathfrak{g}^\perp)^C, \mathrm{ad})$, then we have $\langle\pi(\beta), \beta_0\rangle<2$. Since λ is a dominant integral weight, we have $\langle\lambda, \alpha_0\rangle= 2\langle\lambda, \alpha_0\rangle/\langle\alpha_0, \alpha_0\rangle=0, 1$. We put $\lambda=\sum_{j=1}^{r} n_j\bar{\omega}_j$. Then we have $\langle\lambda, \alpha_0\rangle=\sum_{j=1}^{r} n_j m_j\langle\alpha_j, \alpha_j\rangle/2 =0, 1$. Since $m_j\langle\alpha_j, \alpha_j\rangle/2=m_j\langle\alpha_j, \bar{\omega}_j\rangle=2\langle\alpha_0, \bar{\omega}_j\rangle/\langle\alpha_0, \alpha_0\rangle$ is a positive integer, $\langle\lambda, \alpha_0\rangle$ is equal to

(1) 0, if and only if all of the n_j's are 0,
(2) 1, if and only if there exists k such that

$$n_k=m_k\langle\alpha_k, \alpha_k\rangle/2=1,$$
$$n_j=0 \quad \text{if} \quad j\neq k.$$

For each simple Lie algebra, we can calculate the number $m_k\langle\alpha_k, \alpha_k\rangle/2$ (cf. [B]), and we can pick up all possible k's with the above property.

PROPOSITION 3.1. *Let λ be the highest weight of an irreducible component of the G-module* $((\mathfrak{g}^\perp)^C, \mathrm{ad})$. *Then λ is one in the following table* 2.

Now we inspect the possibility of λ more carefully.

Case 1. $\mathfrak{g}=\mathfrak{su}(r+1)$, $r\geqq 1$. Let V be an irreducible component of the G-module $((\mathfrak{g}^\perp)^C, ad)$ and λ be the highest weight of V. The highest weight vector Y is expressed as follows

$$Y=\sum_{\substack{\beta\in\Sigma(U)\\ \pi(\beta)=\lambda}} c_\beta X_\beta,$$

where X_β is a root vector of $\mathfrak{u}^C$ corresponding to β. Since λ is the highest

Table 2.

Type of $\mathfrak{g}^C$	λ
A_r $(r\geqq 1)$	$0, \bar{\omega}_1, \cdots, \bar{\omega}_r$
B_r $(r\geqq 2)$	$0, \bar{\omega}_1, \bar{\omega}_r$
C_r $(r\geqq 3)$	$0, \bar{\omega}_1, \cdots, \bar{\omega}_r$
D_r $(r\geqq 4)$	$0, \bar{\omega}_1, \bar{\omega}_{r-1}, \bar{\omega}_r$
E_6	$0, \bar{\omega}_1, \bar{\omega}_6$
E_7	$0, \bar{\omega}_7$
E_8	0
F_4	$0, \bar{\omega}_4$
G_2	$0, \bar{\omega}_1$

weight, we have

$$0=[X_{\alpha_i}, Y]=\sum_{\substack{\beta\in\Sigma(U)\\ \pi(\beta)=\lambda}} c_\beta[X_{\alpha_i}, X_\beta], \tag{3.1}$$

for each i. Take and fix β with $c_\beta\neq 0$. By Theorem 2.3, $\alpha_i\in\Sigma(U)$ and $[X_{\alpha_i}, X_\beta]\in\mathfrak{u}^C_{\alpha_i+\beta}$. Thus by (3.1), $[X_{\alpha_i}, X_\beta]=0$, $\alpha_i+\beta\notin\Sigma(U)$. Put

$$\Gamma=\{\alpha_1, \cdots, \alpha_r, -\beta\}.$$

Then Γ satisfies the following property:

$$\gamma-\delta\notin\Sigma(U) \text{ holds for any } \gamma, \delta\in\Gamma. \tag{C_1}$$

If a subset of $\Sigma(U)$ with the property (C_1) is linearly independent, then it corresponds uniquely to a Dynkin diagram [He, p. 470]. However even if a subset of $\Sigma(U)$ is linearly dependent, we associate with it a diagram in an analogous fashion to the construction of the Dynkin diagram. The subsets of $\Sigma(U)$ with the property (C_1) are classified in [He, p. 503]. In our case, the set Γ has two restrictive conditions:

(i) $\alpha_1, \cdots, \alpha_r$ forms a fundamental root system of $\mathfrak{su}(r+1)$,

(ii) $-\beta$ is joined to only one vertex in $\{\alpha_1, \cdots, \alpha_r\}$, if $\lambda\neq 0$ (by Proposition 3.1).

From the classification given in [He], we pick up diagrams which is possible for our Γ. And we get the following:

PROPOSITION 3.2. *If $\mathfrak{g}=\mathfrak{su}(r+1)$, $r\geqq 1$, then the highest weight λ of an irreducible component of the G-module $((\mathfrak{g}^\perp)^C, ad)$ is one of the following*:

(1) $0, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_{r-1}, \bar{\omega}_r$, *if* $r\geqq 9$,
(2) $0, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_6, \bar{\omega}_7, \bar{\omega}_8$, *if* $r=8$,
(3) $0, \bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_r$, *if* $1\leqq r\leqq 7$.

Case 2. $\mathfrak{g}=\mathfrak{so}(2r)$, $r\geqq 4$. Since each root is long, we can discuss similarly to the case 1. We get the following:

PROPOSITION 3.3. *If* $\mathfrak{g}=\mathfrak{so}(2r)$, $r\geqq 4$, *then the highest weight* λ *of an irreducible component of the G-module* $((\mathfrak{g}^{\perp})^C, ad)$ *is one of the following*:

(1) $0, \bar{\omega}_1$, *if* $r\geqq 9$,
(2) $0, \bar{\omega}_1, \bar{\omega}_{r-1}, \bar{\omega}_r$, *if* $4\leqq r\leqq 8$.

Case 3. $\mathfrak{g}=\mathfrak{so}(2r+1)$, $r\geqq 2$. A Lie algebra $\mathfrak{l}$ which is isomorphic to $\mathfrak{so}(2r)$ is canonically embedded in $\mathfrak{g}$. If $r\geqq 3$, $\mathfrak{l}$ is also a simple subalgebra of $\mathfrak{u}$ of index 1. We denote by $V(\bar{\omega})$ the complex irreducible $\mathfrak{g}$-module with highest weight $\bar{\omega}$ and by $W(\rho)$ the complex irreducible $\mathfrak{l}$-module with highest weight ρ. Let $\rho_1, \cdots, \rho_r$ penote the fundamental weights of $\mathfrak{l}$. It is easily verified that

$$V(\bar{\omega}_r)=W(\rho_{r-1})\oplus W(\rho_r).$$

If $((\mathfrak{g}^{\perp})^C, \mathrm{ad})$ contains a $\mathfrak{g}$-irreducible component $V(\bar{\omega}_r)$, then $((\mathfrak{l}^{\perp})^C, \mathrm{ad})$ contains an $\mathfrak{l}$-irreducible component $W(\rho_r)$. Thus by Propositions 3.2 and 3.3, r must be smaller than or equal to 8 and we get the following:

PROPOSITION 3.4. *If* $\mathfrak{g}=\mathfrak{so}(2r+1)$, $r\geqq 2$, *then the highest weight* λ *of an irreducible component of the G-module* $((\mathfrak{g}^{\perp})^C, ad)$ *is one of the following*:

(1) $0, \bar{\omega}_1$, *if* $r\geqq 9$,
(2) $0, \bar{\omega}_1, \bar{\omega}_r$, *if* $2\leqq r\leqq 8$.

Case 4. $\mathfrak{g}=\mathfrak{sp}(r)$, $r\geqq 3$. In this case we have the following:

PROPOSITION 3.5. *If* $\mathfrak{g}=\mathfrak{sp}(r), r\geqq 3$, *then the highest weight* λ *of an irreducible component of the G-module* $((\mathfrak{g}^{\perp})^C, ad)$ *is one of the following*:

(1) $0, \bar{\omega}_1, \bar{\omega}_2$, *if* $r\geqq 5$,
(2) $0, \bar{\omega}_1, \cdots, \bar{\omega}_r$, *if* $r=3, 4$.

Proof. If $\mathfrak{u}$ is of exceptional type, then $\mathfrak{sp}(r)$, $r\geqq 5$, cannot be realized as a subalgebra of index 1 ([D]). So we assume that $\mathfrak{u}$ is of classical type. If $\mathfrak{g}$ is a regular subalgebra of $\mathfrak{u}$, then we can argue similarly to the case 1. And we have one possibility that $\lambda=\bar{\omega}_1$. Tasaki classified complex simple Lie subalgebra of index 1 in classical complex simple Lie algebras (see Remark 3.9(1)). By his classification, if $\mathfrak{g}$ is not a regular subalgebra, then there exists a Lie subalgebra $\mathfrak{l}$ of $\mathfrak{u}$ which satisfies

(i) $\mathfrak{l}$ is isomorphic to $\mathfrak{su}(2r)$,
(ii) $\mathfrak{l}$ is a regular subalgebra of $\mathfrak{u}$ of index 1,
(iii) $\mathfrak{g}$ is a canonically embedded Lie subalgebra of $\mathfrak{l}$.

The orthogonal complement $\mathfrak{g}^{\perp}$ is decomposed as $\mathfrak{g}^{\perp}=\mathfrak{g}_0{}^{\perp}\oplus\mathfrak{l}^{\perp}$, where $\mathfrak{g}_0{}^{\perp}$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{l}$ and $\mathfrak{l}^{\perp}$ is the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{u}$.

We can easily see that $(\mathfrak{g}_0^{\perp})^C$ is $V(\bar{\omega}_2)$. Let $\rho_1, \cdots, \rho_{2r}$ denote the fundamental weights of $\mathfrak{l}$. We denote by $W(\rho)$ the complex irreducible $\mathfrak{l}$-module with highest weight ρ. Now we decompose $(\mathfrak{l}^{\perp})^C$ as a $\mathfrak{g}$-module. Take an irreducible component $W(\rho)$ of $(\mathfrak{l}^{\perp})^C$ as an $\mathfrak{l}$-module. We know the possibility of ρ by Proposition 3.2. For each possible ρ, we decompose $W(\rho)$ as a $\mathfrak{g}$-module. We have only to consider the case $\rho=\rho_1, \rho_2, \rho_{2r-2}, \rho_{2r-1}$. For the other cases r must be less than or equal to 4. We can easily see

$$W(\rho_1)=V(\bar{\omega}_1),$$
$$W(\rho_{2r-1})=V(\bar{\omega}_1),$$
$$W(\rho_2)=V(\bar{\omega}_2)\oplus V(0),$$
$$W(\rho_{2r-2})=V(\bar{\omega}_2)\oplus V(0).$$

Thus we have the Proposition. Q.E.D.

3.2 *Proof of Theorem B.*

Case 1. $\mathfrak{g}=\mathfrak{su}(r+1),\ r\geqq 1$.

By Theorem 2.2, we can calculate the eigenvalues of the Casimir operator with respect to the canonical inner product in $\mathfrak{g}$,

$$a_{\bar{\omega}_i}=-(r+2)i(r+i-1)/(r+1).$$

Remember that $a_{\bar{\omega}_1}=a_{\bar{\omega}_r}>a_{\bar{\omega}_2}=a_{\bar{\omega}_{r-1}}>\cdots$. By examining the eigenvalues, we determine the set of pairs $(\bar{\omega}, \bar{\omega}')$ such that

$$a_{\bar{\omega}}+a_{\bar{\omega}'}>a_{\bar{\omega}_j} \tag{3.2}$$

for each $\bar{\omega}_j$ given in Proposition 3.2. If $j=3$ ($r\leqq 8$), $r-2$ ($r\leqq 8$) or 4 ($r=7$), then the set of pairs $(\bar{\omega}, \bar{\omega}')$ are

$$(\bar{\omega}_1, \bar{\omega}_1),\ (\bar{\omega}_1, \bar{\omega}_r),\ (\bar{\omega}_r, \bar{\omega}_1),\ (\bar{\omega}_r, \bar{\omega}_r),$$

otherwise such a pair does not exist. On the other hand we have

$$V(\bar{\omega}_1)\otimes V(\bar{\omega}_1)=V(2\bar{\omega}_1)\oplus V(\bar{\omega}_2),$$
$$V(\bar{\omega}_1)\otimes V(\bar{\omega}_r)=V(\bar{\omega}_1+\bar{\omega}_r)\oplus V(0),$$
$$V(\bar{\omega}_r)\otimes V(\bar{\omega}_r)=V(2\bar{\omega}_r)\oplus V(\bar{\omega}_{r-1}).$$

Thus by Theorem 2.1, G is stable as a totally geodesic submanifold in U.

For the other cases we can argue in a similar fashion. So we list

(i) the eigenvalues of the Casimir operator,

(ii) the set of pairs with (3.2),

(iii) the decomposition of the tensor product $V(\bar{\omega})\otimes V(\bar{\omega}')$, for the pair $(\bar{\omega}, \bar{\omega}')$ given in (ii).

Case 2. $\mathfrak{g}=\mathfrak{so}(2r+1),\ r\geqq 2$.

(i) $a_{\bar\omega_i}=-i(2r+1-i),\quad 1\leqq i\leqq r-1,$
$a_{\bar\omega_r}=-r(2r+1)/4,$

(ii) if $j=r=8$, then the set of pairs $(\bar\omega, \bar\omega')$ are $(\bar\omega_1, \bar\omega_1)$, otherwise such a pair does not exist.

(iii) $V(\bar\omega_1)\otimes V(\bar\omega_1)=V(2\bar\omega_1)\oplus V(\bar\omega_2)\oplus V(0)$.

Thus by Theorem 2.1, G is stable as a totally geodesic submanifold in U.

Case 3. $\mathfrak{g}=\mathfrak{sp}(r),\ r\geqq 3$.

(i) $a_{\bar\omega_i}=-i(2r+2-i),\quad 1\leqq i\leqq r,$

(ii) if $(j,r)=(3,3)$, $(3,4)$ or $(4,4)$, then the set of pairs $(\bar\omega, \bar\omega')$ are $(\bar\omega_1, \bar\omega_1)$, otherwise such a pair does not exist.

(iii) $V(\bar\omega_1)\otimes V(\bar\omega_1)=V(2\bar\omega_1)\oplus V(\bar\omega_2)\oplus V(0)$.

Thus by Theorem 2.1, G is stable as a totally geodesic submanifold in U

Case 4. $\mathfrak{g}=\mathfrak{so}(2r),\ r\geqq 4$.

(i) $a_{\bar\omega_i}=-ir(2-i),\quad 1\leqq i\leqq r-2,$
$a_{\bar\omega_{r-1}}=a_{\bar\omega_r}=-r(2r+1)/4,$

(ii) such a pair does not exist.

Thus by Theorem 2.1, G is stable as a totally geodesic submanifold in U.

Case 5. $\mathfrak{g}$ *is of exceptional type.*

If $\mathfrak{g}$ is of exceptional type, then the eigenvalue of the Casimir operator for λ given in Proposition 3.1 is the largest except zero. Thus by Theorem 2.1, G is stable as a totally geodesic submanifold in U.

3.3 *Examples and remarks.*

Now we give some examples of simple connected closed Lie subgroups G of index 1 in compact connected simple Lie groups and give the decomposition of $(\mathfrak{g}^\perp)^C$.

Example 3.6. (1) Let U be the special unitary group $SU(r+s+1)$ and $G=\{\mathrm{Diagonal}(A, I_s): A\in SU(r+1)\}$. Then the index of G in U is equal to 1. If $r\geqq 2$, the G-irreducible decomposition of $(\mathfrak{g}^\perp)^C$ is as follows:

$$(\mathfrak{g}^\perp)^C=\underbrace{(V(\bar\omega_1)\oplus V(\bar\omega_r))\oplus\cdots\oplus(V(\bar\omega_1)\oplus V(\bar\omega_r))}_{s}\oplus\underbrace{V(0)\oplus\cdots\oplus V(0)}_{s^2}.$$

(2) Let U be the special orthogonal group $SO(2r+2)$ and embed $G=SU(r+1)$ as a subgroup in a standard way. Then the index of G in U is equal to 1. If

$r \geqq 4$, the G-irreducible decomposition of $(\mathfrak{g}^{\perp})^{C}$ is as follows:

$$(\mathfrak{g}^{\perp})^{C}=V(\bar{\omega}_2)\oplus V(\bar{\omega}_{r-1})\oplus V(0).$$

(3) Let U be the compact connected simple exceptional Lie group E_8. Then U has $G=SU(q)/\boldsymbol{Z}_3$ as a subgroup of index 1. The G-irreducible decomposition of $(\mathfrak{g}^{\perp})^{C}$ is as follows (see [M-P, p. 305]):

$$(\mathfrak{g}^{\perp})^{C}=V(\bar{\omega}_3)\oplus V(\bar{\omega}_6).$$

Remark 3.7. For each dominant integral weight λ appeared in Proposition 3.2, there exist a compact connected simple Lie group U and its closed connected subgroup G with the following:

(i) the index of G in U is equal to 1.
(ii) G is locally isomorphic to $SU(r+1)$,
(iii) $V(\lambda)$ is a G-irreducible component of $(\mathfrak{g}^{\perp})^{C}$.

We give further examples of pairs of compact connected simple Lie groups U and their closed connected subgroups G of index 1. We omit the G-irreducible decompositions of $(\mathfrak{g}^{\perp})^{C}$.

Example 3.8. (1) $SO(N)\supset SO(n)$.
(2) $Sp(N)\supset Sp(r)$.
(3) $SU(N)\supset(SU(2r)\supset)Sp(r)$.
(4) $SO(8)\supset Spin(7)$.
(5) $SO(7)\supset G_2$.
(6) $F_4\supset Sp(3)$.

Remark 3.9. (1) Complex simple Lie subalgebras of index 1 in classical complex simple Lie algebras were classified in [Tas2]. By the classification, such a subalgebra corresponds to one of the subgroups given in Example 3.6 (1), (2) and Example 3.8 (1)-(5).

(2) Let λ be a dominant integral weight appeared in Proposition 3.3, 3.4 or 3.5 except the cases that $\lambda=\bar{\omega}_8$ for $\mathfrak{g}=\mathfrak{so}(17)$ and $\lambda=\bar{\omega}_3$ for $\mathfrak{g}=\mathfrak{sp}(4)$. There exist a compact connected simple Lie group U and its closed connected subgroup G with the following:

(i) the index of G in U is equal to 1,
(ii) the Lie algebra of G is isomorphic to $\mathfrak{g}$,
(iii) $V(\lambda)$ is a G-irreducible component of $(\mathfrak{g}^{\perp})^{C}$.

It is easily seen that the assumption on G in Theorem B is weakened as follows:

THEOREM B′. *Let U be a compact connected simple Lie group with a bi-invariant Riemannian metric. A connected semisimple closed Lie subgroup G all of whose simple factors are of index* 1 *is a stable minimal submanifold.*

By Theorem B′, we conclude that the subgroup $G=\{\text{Diagonal}(A, B): A\in SO(p), B\in SO(q)\}$ of $SO(p+q)$, $p, q\geqq 4$, is a stable minimal submanifold.

4. Stability of 3-dimensional simple subgroups.

We shall give a necessary and sufficient condition that a connected 3-dimensional simple Lie subgroup in a compact connected simple Lie group is stable.

A compact connected 3-dimensional simple Lie group is isomorphic to one of $SU(2)$ and $SO(3)$ and its Lie algebra is always isomorphic to $\mathfrak{so}(3)$. We state our results separately for $SU(2)$ and $SO(3)$.

THEOREM C. *Let G be a simple Lie subgroup which is isomorphic to $SU(2)$ in a compact connected simple Lie group U with a bi-invariant Riemannian metric. Then, G is stable if and only if G is of index* 1. *If G is stable, then $n(G)=n_K(G)$.*

In order to state Theorem for G which is isomorphic to $SO(3)$, we fix some notation. We choose a basis $\{H, E, F\}$ for a 3-dimensional compact simple Lie algebra $\mathfrak{g}$ with

$$[H, E]=2F, \quad [H, F]=-2E, \quad [E, F]=H.$$

With respect to the canonical inner product $\langle\, ,\rangle_0$ on $\mathfrak{g}$, $\{H/\sqrt{2}, E, F\}$ is an orthonormal basis of $\mathfrak{g}$.

Let G be a 3-dimensional connected simple Lie subgroup in a compact connected simple Lie group U of rank r. Then the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{so}(3)$, hence we can take a basis $\{H, E, F\}$ for the Lie algebra $\mathfrak{g}$ of G as above. Let $\langle\, ,\rangle$ denote the canonical inner product on the Lie algebra $\mathfrak{u}$ of U. Let $\mathfrak{t}$ be a maximal Abelian subalgebra in $\mathfrak{u}$ such that $H\in\mathfrak{t}$. With respect to a suitable ordering, we may assume $\langle\beta_j, H\rangle\geqq 0$ for the fundamental root system $\{\beta_1, \cdots, \beta_r\}$ of $\Sigma(U)$. The Dynkin diagram of $\{\beta_1, \cdots, \beta_r\}$ marked with the non-negative integer $\langle\beta_j, H\rangle$ at the j-th vertex is called the *characteristic diagram* of G. The characteristic diagram determines the conjugacy class of G (see [D, Theorem 8.2] and [Tas2, Proposition 3.3]). Let $\beta_0=\sum_{j=1}^{r} n_j\beta_j$ be the highest root of $\Sigma(U)$.

THEOREM D. *Let G be a simple Lie subgroup which is isomorphic to $SO(3)$ in a compact connected simple Lie group U with a bi-invariant Riemannian metric. Then, G is stable if and only if there exists k, $1\leqq k\leqq r$, such that*

$$n_k=1, \quad \langle\beta_j, H\rangle=2\delta_{jk}, \quad 1\leqq j\leqq r.$$

If G is stable, then $n(G)=n_K(G)$.

Let λ be a weight of a (complex) G-module. Since $\mathfrak{g}$ is of rank 1, λ is determined by its (integral) value $\langle\lambda, H\rangle_0$. On the other hand an integer n

determines an integral weight $nH/2$ of G. For the sake of brevity, we simply denote the weight $nH/2$ by n. Let $V(n)$ be the irreducible (complex) G-module with the highest weight $n \geqq 0$. Then the weight space decomposition of $V(n)$ is as follows:

$$V(n)=\sum_{k=0}^{n} V(n)_{n-2k}, \qquad \dim V(n)_{n-2k}=1. \tag{4.1}$$

By (4.1) and counting the multiplicities of weights, we have the well-known theorem of Clebsh-Gordan.

$$V(n)\otimes V(m)=\sum_{j=0}^{\mathrm{Min}(n,m)} V(|n-m|+2j). \tag{4.2}$$

Let j be the index of G in U. By the definition of the index,

$$\langle X, Y\rangle=j\langle X, Y\rangle_0, \qquad \text{for} \quad X, Y\in\mathfrak{g}. \tag{4.3}$$

Let X_β be a root vector of $\mathfrak{u}^C$ corresponding to a root $\beta\in\Sigma(U)$. Then by its definition $[H, X_\beta]=\sqrt{-1}\langle\beta, H\rangle X_\beta$. Thus X_β is a weight vector of the G-module $\mathfrak{u}^C$ corresponding to the weight $\langle\beta, H\rangle=\langle j\beta, H\rangle_0$. Therefore the set of weights of G-module $\mathfrak{u}^C$ is given as follows:

$$W(\mathfrak{u}^C)=\{\langle\beta, H\rangle : \beta\in\Sigma(U)\cup\{0\}\}. \tag{4.4}$$

For an integer k, we put

$$\Gamma_k=\{\beta\in\Sigma(U)\cup\{0\} : \langle\beta, H\rangle=k\}.$$

Then the weight space of $\mathfrak{u}^C$ corresponding to the weight k is given by the following:

$$(\mathfrak{u}^C)_k=\sum_{\beta\in\Gamma_k}\mathfrak{u}^C_\beta.$$

Since $\mathfrak{g}^C=V(2)$ is an irreducible component of $\mathfrak{u}^C$, we have $2\in W(\mathfrak{u}^C)$ and $\langle\beta_0, H\rangle \geqq 2$, for $\langle\beta_0, H\rangle$ is the highest weight in $W(\mathfrak{u}^C)$. Define a basis $\{H, X_+, X_-\}$ of $\mathfrak{g}^C$ by

$$X_+=(E-\sqrt{-1}F)/2, \qquad X_-=(E+\sqrt{-1}F)/2.$$

Then $X_+\in(\mathfrak{u}^C)_2$, $X_-\in(\mathfrak{u}^C)_{-2}$ and we can put

$$X_+=\sum_{\beta\in\Gamma_2} X_\beta, \qquad X_\beta\in\mathfrak{u}^C_\beta,$$

$$X_-=\sum_{\beta\in\Gamma_2} X_{-\beta}, \qquad X_{-\beta}\in\mathfrak{u}^C_{-\beta}.$$

Since $H=[E, F]=-2\sqrt{-1}[X_+, X_-]$, we have

$$H\in\sum_{\beta\in\Gamma_2}\boldsymbol{R}\beta. \tag{4.5}$$

By (2.1), the eigenvalue a_n of the Casimir operator on $V(n)$ of $\mathfrak{g}$ with respect to $\langle\,,\rangle_0$ is given as follows:

$$a_n = -n(n+2)/2. \tag{4.6}$$

Since U is a simple Lie group, we have only to show the Theorem C and D with respect to the invariant Riemannian metric on U induced by $\langle\,,\rangle/j$. By (4.3), the induced Riemannian metric on G coincides with the invariant Riemannian metric induced by $\langle\,,\rangle_0$. We remember that

$$\mathcal{D}(SU(2)) = \{V(n) : n \in \boldsymbol{Z},\ n \geqq 0\}$$
$$\mathcal{D}(SO(3)) = \{V(2n) : n \in \boldsymbol{Z},\ n \geqq 0\}\,.$$

Proof of Theorem C. First we prove that if $\sum_{j=2}^{\infty} \#(\Gamma_j) \geqq 2$, then G is unstable. In fact, under the assumption there exists an n $(n \geqq 2)$, such that $V(n) \subset (\mathfrak{g}^{\perp})^C$. By (4.2) and (4.6),

$$a_1 + a_{n-1} > a_n\,,$$
$$V(1) \otimes V(n-1) = V(n) \oplus \cdots\,.$$

Thus by (i) of Theorem 2.1, we conclude that G is unstable. We can easily see that the converse is also true.

We consider the case that G is stable. As we remarked before, $\#(\Gamma_2) \geqq 1$. Thus if G is stable, then $\#(\Gamma_2)=1$, $\#(\Gamma_3)=\#(\Gamma_4)=\cdots=0$, and Γ_2 consists of β_0. By Schwarz' inequality and the definition of index, we have

$$2 = \langle \beta_0, H \rangle \leqq \sqrt{j \langle \beta_0, \beta_0 \rangle \langle H, H \rangle_0} = 2\sqrt{j}\,.$$

The equality holds, since β_0 and H are proportional by (4.5). Namely we have $j=1$. Thus, combined with Theorem A, the former half of Theorem C is proved.

Now we prove the latter half. As we have proved, G is stable if and only if each irreducible component of $(\mathfrak{g}^{\perp})^C$ is equivalent to $V(1)$ or $V(0)$. Let m [resp. n] be the multiplicity of $V(1)$ [resp. $V(0)$] in $(\mathfrak{g}^{\perp})^C$. Note that m is even: $m=2m'$. Then, by (ii) of Theorem 2.1, we have

$$\begin{aligned} n(G) &= m \sum_{\substack{a_\lambda + a_\mu = -3/2 \\ \lambda, \mu \in \mathcal{D}(G)}} \dim \mathrm{Hom}_G(V(\lambda) \otimes V(\mu), V(1)) \dim (V(\lambda) \otimes V(\mu)) \\ &\quad + n \sum_{\substack{a_\lambda + a_\mu = 0 \\ \lambda, \mu \in \mathcal{D}(G)}} \dim \mathrm{Hom}_G(V(\lambda) \otimes V(\mu), V(0)) \dim (V(\lambda) \otimes V(\mu)) \\ &= 4m + n\,. \end{aligned}$$

On the other hand, by (iii) of Theorem 2.1, we also have

$$n_K(G) = 4m + n\,.$$

Q. E. D.

The Proof of Theorem D. Remember that each weight of a G-module is an even integer. By a similar manner to that of the proof of Theorem C, we can prove that G is unstable if and only if $\sum_{j=4}^{\infty} \#(\Gamma_j) \geqq 1$.

We consider the case that G is stable. In this case we have $\langle \beta_0, H \rangle = 2$. Since a weight $\langle \beta_j, H \rangle$ is equal to 0 or 2, $2 = \langle \beta_0, H \rangle = \sum_{j=1}^{r} n_j, \langle \beta_j H \rangle$ implies that there exists an integer k such that $n_k = 1$, and $\langle \beta_j, H \rangle = 2\delta_{jk}$. Conversely, if the condition is satisfied we have $\langle \beta, H \rangle = 0$, 2 or -2 for any $\beta \in \Sigma(U)$. Thus the former half of Theorem D is proved.

The latter half is proved by a similar manner to the latter half of Theorem C. Q. E. D.

5. Classification of stable 3-dimensional simple subgroups.

Now we determine all stable 3-dimensional simple subgroups which satisfy the condition in Theorem D in each compact simple Lie group.

In the case that the ambient group U is of classical type we imbedd $\mathfrak{u}^C$ in $\mathfrak{sl}(N, \boldsymbol{C})$. We denote by ε_i the complex $N \times N$-matrix of which (i, i)-component is equal to $\sqrt{-1}$ and all of the other components are equal to 0. Put

$$\mathfrak{h} = \left\{ \sum_{i=1}^{N} t_i \varepsilon_i : t_i \in \boldsymbol{R},\ t_1 + \cdots + t_N = 0 \right\}.$$

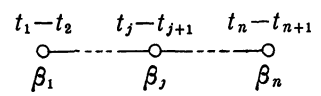
Case 1. $\mathfrak{u} = \mathfrak{su}(n+1)$, $n \geqq 1$. In this case case $\mathfrak{h}^C$ is a Cartan subalgebra of $\mathfrak{su}(n+1)^C = \mathfrak{sl}(n+1, \boldsymbol{C})$. Let $\mathfrak{g}$ be a 3-dimensional simple subalgebra in $\mathfrak{su}(n+1)$. We may assume

$$H = \sum_{i=1}^{N} t_i \varepsilon_i, \qquad t_1 \geqq t_2 \geqq \cdots \geqq t_{n+1}.$$

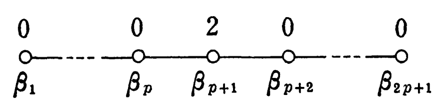
Note that $\{t_1, t_2, \cdots, t_{n+1}\}$ is the set of all weights of $\mathfrak{g}$ acting on $\boldsymbol{C}^{n+1}$. Since

$$\beta_j = \varepsilon_j - \varepsilon_{j+1}, \qquad 1 \leqq j \leqq n,$$

is a system of fundamental roots, the characteristic diagram of $\mathfrak{g}$ is as follows:



By (4.1) $t_i = -t_{n+2-i}$, hence the characteristic diagram of $\mathfrak{g}$ is symmetrical. Since $\beta_0 = \beta_1 + \cdots + \beta_n$, the diagram for $n = 2p+1$, $p \geqq 1$:



is a unique one satisfying the condition in Theorem D. Thus we get $t_1=\cdots=t_{p+1}=1$ and $t_{p+2}=\cdots=t_{2p+2}=-1$. The corresponding subgroup $\tilde{G}$ in $SU(2p+2)$ is

$$\tilde{G}=\{\text{Diagonal}(\underbrace{A, \cdots, A}_{p+1});\ A\in SU(2)\}.$$

Since $\tilde{G}$ is isomorphic to $SU(2)$ and its index is $p+1$, $\tilde{G}$ is unstable by Theorem C.

Each Lie group which is locally isomorphic to $SU(2p+2)$ is of the form $SU(2p+2)/D$ for some subgroup D of the center of $SU(2p+2)$. If a subgroup D of the center of $SU(2p+2)$ contains -1, $G=\tilde{G}/\{\pm 1\}\cong SO(3)$ is stable in $U=SU(2p+2)/D$.

Case 2. $\mathfrak{u}=\mathfrak{so}(2n+1)$, $n\geqq 2$. We imbedd $\mathfrak{so}(2n+1)^C=\mathfrak{so}(2n+1, \boldsymbol{C})$ in $\mathfrak{sl}(2n+1, \boldsymbol{C})$ as follows:

$$\mathfrak{so}(2n+1, \boldsymbol{C})=\left\{\begin{bmatrix} 0 & a & b \\ -{}^t b & X & Y \\ -{}^t a & Z & -{}^t X \end{bmatrix} : \begin{array}{l} {}^tY=-Y,\ {}^tZ=-Z, \\ X, Y, Z\in M_n(\boldsymbol{C}),\ a, b\in \boldsymbol{C}^n \end{array}\right\}$$

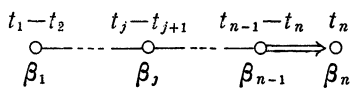
In this case $\mathfrak{h}^C\cap\mathfrak{so}(2n+1, \boldsymbol{C})$ is a Cartan subalgebra of $\mathfrak{so}(2n+1, \boldsymbol{C})$. Let $\mathfrak{g}$ be a 3-dimensional simple subalgebra in $\mathfrak{so}(2n+1)$. We may assume

$$H=\sum_{i=1}^{n} t_i(\varepsilon_{i+1}-\varepsilon_{i+n+1}), \qquad t_1\geqq t_2\geqq \cdots \geqq t_n\geqq 0.$$

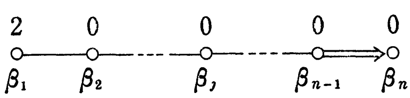
Then $\{0, t_1, \cdots, t_n, -t_1, \cdots, -t_n\}$ is the set of all weights of $\mathfrak{g}$ acting on $\boldsymbol{C}^{2n+1}$.

$$\beta_j=(\varepsilon_{j+1}-\varepsilon_{j+2}-\varepsilon_{j+n+1}+\varepsilon_{j+n+2})/\sqrt{2}, \qquad 1\leqq j\leqq n-1,$$
$$\beta_n=(\varepsilon_{n+1}-\varepsilon_{2n+1})/\sqrt{2}$$

is a system of fundamental roots, the characteristic diagram of $\mathfrak{g}$ is as follows:



Since $\beta_0=\beta_1+2\beta_2+\cdots+2\beta_n$, the diagram:



is a unique one satisfying the condition in Theorem D. Thus we get $t_1=2$ and $t_2=\cdots=t_n=0$. The corresponding subgroup G in $SO(2n+1)$ is

$$G=\{\text{Diagonal}(A, I_{2n-2});\ A\in SO(3)\}.$$

Since G is isomorphic to $SO(3)$, it is stable in $SO(2n+1)$. The index of G is equal to 2.

The corresponding subgroup in $Spin(2n+1)$ is isomorphic to $SU(2)$. Thus, by Theorem C, it is not stable.

Case 3. $\mathfrak{u}=\mathfrak{sp}(n)$, $n\geqq 3$. We imbedd $\mathfrak{sp}(n)^C=\mathfrak{sp}(n, \boldsymbol{C})$ in $\mathfrak{sl}(2n, \boldsymbol{C})$ as follows:

$$\mathfrak{sp}(n, \boldsymbol{C})=\left\{\begin{bmatrix} X & Y \\ Z & -{}^tX \end{bmatrix} : {}^tY=Y,\ {}^tZ=Z,\ X, Y, Z\in M_n(\boldsymbol{C})\right\}$$

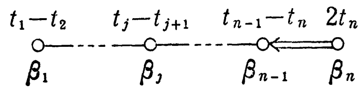
In this case $\mathfrak{h}^C\cap\mathfrak{sp}(n, \boldsymbol{C})$ is a Cartan subalgebra of $\mathfrak{sp}(n, \boldsymbol{C})$. Let $\mathfrak{g}$ be a 3-dimensional simple subalgebra in $\mathfrak{sp}(n)$. We may assume

$$H=\sum_{i=1}^{n} t_i(\varepsilon_i-\varepsilon_{i+n}), \qquad t_1\geqq t_2\geqq \cdots \geqq t_n\geqq 0.$$

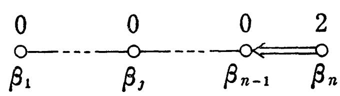
Note that $\{t_1, \cdots, t_n, -t_1, \cdots, -t_n\}$ is the set of all weights of $\mathfrak{g}$ acting on $\boldsymbol{C}^{2n}$. Since

$$\beta_j=(\varepsilon_j-\varepsilon_{j+1}-\varepsilon_{j+n}+\varepsilon_{j+n+1})/2, \qquad 1\leqq j\leqq n-1,$$
$$\beta_n=\varepsilon_n-\varepsilon_{2n}$$

is a system of fundamental roots, the characteristic diagram of $\mathfrak{g}$ is as follows:



Since $\beta_0=2\beta_1+\cdots+2\beta_{n-1}+\beta_n$, the diagram:



is a unique one satisfying the condition in Theorem D. Thus we get $t_1=\cdots=t_n=1$. The corresponding subgroup $\tilde{G}$ in $Sp(n)$ is

$$\tilde{G}=\{\text{Diagonal}(\underbrace{A, \cdots, A}_{n}) : A\in Sp(1)\}.$$

Since $\tilde{G}$ is isomorphic to $SU(2)$ and its index is n, $\tilde{G}$ is unstable by Theorem C.

The center of $Sp(n)$ is $\{\pm 1\}$. The corresponding subgroup $G=\tilde{G}/\{\pm 1\}$ in $U=Sp(n)/\{\pm 1\}$, which is isonorphic to $SO(3)$, is stable.

Case 4. $\mathfrak{u}=\mathfrak{so}(2n)$, $n\geqq 4$. We imbedd $\mathfrak{so}(2n)^C=\mathfrak{so}(2n, \boldsymbol{C})$ in $\mathfrak{sl}(2n, \boldsymbol{C})$ as follows:

$$\mathfrak{so}(2n, \boldsymbol{C})=\left\{\begin{bmatrix} X & Y \\ Z & -{}^tX \end{bmatrix} : {}^tY=-Y,\ {}^tZ=-Z,\ X, Y, Z\in M_n(\boldsymbol{C})\right\}$$

In this case $\mathfrak{h}^C\cap\mathfrak{so}(2n, \boldsymbol{C})$ is a Cartan subalgebra of $\mathfrak{so}(2n, \boldsymbol{C})$. Let $\mathfrak{g}$ be a 3-dimensional simple subalgebra in $\mathfrak{so}(2n)$. We may assume

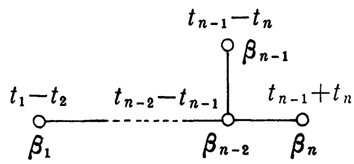
$$H=\sum_{i=1}^{n} t_i(\varepsilon_i-\varepsilon_{i+n}), \qquad t_1\geqq t_2\geqq \cdots \geqq t_{n-1}\geqq |t_n| .$$

Then $\{t_1, \cdots, t_n, -t_1, \cdots, -t_n\}$ is the set of all weights of $\mathfrak{g}$ acting on $\boldsymbol{C}^{2n}$. Since

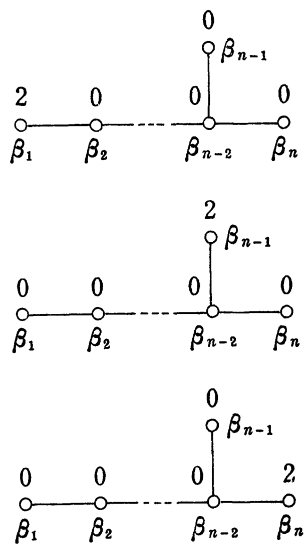
$$\beta_j=(\varepsilon_j-\varepsilon_{j+1}-\varepsilon_{j+n}+\varepsilon_{j+1+n})/\sqrt{2}, \qquad 1\leqq j\leqq n-1,$$

$$\beta_n=(\varepsilon_{n-1}+\varepsilon_n-\varepsilon_{2n-1}-\varepsilon_{2n})/\sqrt{2}.$$

is a system of fundamental roots, the characteristic diagram of $\mathfrak{g}$ is as follows:



Since $\beta_0=\beta_1+2\beta_2+\cdots+2\beta_{n-2}+\beta_{n-1}+\beta_n$, the diagrams satisfying the condition in Theorem D are



Thus $t_1=2$, $t_2=\cdots=t_n=0$ or $t_1=\cdots=t_{n-1}=1$, $t_n=\pm 1$.

(i) If $t_1=2,\ t_2=\cdots=t_n=0$, then the corresponding subgroup in $SO(2n)$ is

$$\{\mathrm{Diagonal}(A,\ I_{2n-3}):\ A\in SO(3)\}$$

and its index is 2. Since the corresponding subgroup $\tilde{G}$ in $Spin(2n)$ is isomorphic to $SU(2)$, $\tilde{G}$ is unstable. Let Z be the center of $Spin(2n)$. Then $\tilde{G}\cap Z=\{\pm 1\}$.

If n is odd, Z is isomorphic to $\boldsymbol{Z}_4$ and the groups which is locally isomorphic to $Spin(2n)$ are $Spin(2n)$, $SO(2n)$ and $Spin(2n)/Z$. Since the corresponding subgroups in $SO(2n)$ and $Spin(2n)/Z$ are isomorphic to $SO(3)$, they are stable.

If n is even: $n=2m$, Z is isomorphic to $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$. The subgroups of Z are $\{(0, 0)\}$, $\{(0, 0), (1, 0)\}$, $\{(0, 0), (0, 1)\}$, $\{(0, 0), (1, 1)\}$ and Z. The element $(1, 1)\in Z$ corresponds to $-1\in Spin(4m)$. Let U be a Lie group locally isomorphic to $Spin(4m)$ and D be the subgroup of Z such that U is isomorphic to $Spin(4m)/D$. If D is $\{(0, 0), (1, 1)\}$ or Z, then the subgroup corresponding to $\tilde{G}$ in U is isomorphic to $SO(3)$ and is stable. Otherwise, the subgroup corresponding to $\tilde{G}$ in U is isomorphic to $SU(2)$ and is unstable.

(ii) If $t_1=\cdots=t_{n-1}=1,\ t_n=\pm 1$ then n must be even: $n=2m$ and the corresponding subgroup in $SO(4m)$ is

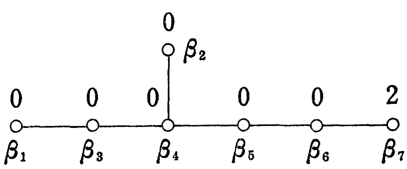
$$\{\mathrm{Diagonal}(\underbrace{A,\ \cdots,\ A}_{m}):\ A\in Sp(1)\},$$

where we regard $Sp(1)$ as a subgroup of $SO(4)$. The index of it is m. Let $\tilde{G}$ be the corresponding subgroup in $Spin(4m)$ and Z be the center of $Spin(4m)$. Let U be a Lie group which is locally isomorphic to $Spin(4m)$ and D be the subgroup of Z such that U is isomorphic to $Spin(4m)/D$. Since $\tilde{G}/(\tilde{G}\cap Z)$ is isomorphic to $SO(3)$, if D is $(\tilde{G}\cap Z)$ or Z, then the subgroup corresponding to $\tilde{G}$ in U is isomorphic to $SO(3)$ and is stable. Otherwise, the subgroup corresponding to $\tilde{G}$ in U is isomorphic to $SU(2)$ and is unstable.

Case 5. $\mathfrak{u}=\mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8$. Due to Table 18 in [D], there is no subgroup in E_6 which satisfies the condition in Theorem D.

Due to Table 19 in [D], there is a subgroup $\tilde{G}$ in E_7 corresponding to the following characteristic diagram.

(5.1)



It is isomorphic to $SU(2)$ and its index is 3. Thus $\tilde{G}$ is not stable in E_7. The center Z of E_7 is isomorphic to $\boldsymbol{Z}_2$. Therefore $G=\tilde{G}/Z$ is isomorphic to $SO(3)$ and stable in $E_7/Z=Ad(E_7)$.

There is no coefficient of the highest root of E_8 which is equal to 1.

Case 6. $\mathfrak{u}=\mathfrak{f}_4$. There is no coefficient of the highest root of F_4 which is equal to 1.

Case 7. $\mathfrak{u}=\mathfrak{g}_2$. There is no coefficient of the highest root of G_2 which is equal to 1.

Now we summarize the above argument.

THEOREM E. *All stable* 3*-dimensional simple subgroups G isomorphic to SO*(3) *in compact connected simple Lie groups with bi-invariant Riemannian metrics are as follows.*

(1) *Let* $\tilde{G}=\{\text{Diagonal}(\underbrace{A, \cdots, A}_{n}) : A\in SU(2)\}\subset SU(2n)$ *and D be a subgroups of the center of* $SU(2n)$ *containing* $\{\pm 1\}$. *Then* $G=\tilde{G}/D$ *is stable in* $SU(2n)/D$. *Its index is equal to n.*

(2) *Let* $\tilde{G}=\{\text{Diagonal}(\underbrace{A, \cdots, A}_{n}) : A\in Sp(1)\}\subset Sp(n)$. *Then* $G=\tilde{G}/\{\pm 1\}$ *is stable in* $Ad(Sp(n))=Sp(n)/\{\pm 1\}$. *Its index is equal to n.*

(3) $G=\{\text{Diagonal}(A, I_{n-3}) : A\in SO(3)\}$ *is stable in* $SO(n)$. *If n is even*: $n=2m$, *then* $Ad(G)$ *is also stable in* $Ad(SO(2m))=PSO(2m)$. *Their indices are equal to* 2.

(4) *Let Z be the center of* $Spin(4n)$ *and* $\tilde{G}$ *be the subgroup of* $Spin(4n)$ *obtained by pulling back* $\{\text{Diagonal}(\underbrace{A, \cdots, A}_{n}) : A\in Sp(1)\}$ *in* $SO(4n)$, *where we regard* $Sp(1)$ *as a subgroup of* $SO(4)$ *in a natural manner. Then* $G=\tilde{G}/\tilde{G}\cap Z$ *is stable in* $Spin(4n)/\tilde{G}\cap Z$ *and* $Spin(4n)/Z=PSO(4n)$. *Their indices are equal to n.*

(5) *Let G be a subgroup of* $Ad(E_7)$ *corresponding to the characteristic diagram* (5.1). *Then it is stable. Its index is equal to* 3.

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