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SOME PROPERTIES OF THE ENTIRE FUNCTIONS EXTREMAL FOR DENJOY'S CONJECTURE

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1. Introduction

In this paper we shall prove the following

THEOREM 1. *Let $F(Z)$ be an entire function extremal for Denjoy's Conjecture (that is, F is entire of finite order λ and has $k=2\lambda$ distinct finite asymptotic values) and satisfy the condition* $\varliminf_{r\to\infty} \log M(r, F)/r^{k/2}<\infty$, *then $F(Z)$ is right-prime.*

THEOREM 2. *Let $F(Z)$ be an entire function extremal for Denjoy's Conjecture and $P(Z)$ a nonconstant polynomial whose zeros are distinct from zeros of $F(Z)$, then $F(Z)/P(Z)$ is right-prime.*

THEOREM 3. *Let $A(Z)$ be an entire function extremal for Denjoy's Conjecture and f_1, f_2 two linear independent solutions of $f''+Af=0$, then at least one of f_1, f_2 has the property that the exponent of convergence of its zero-sequence is ∞.*

In 1907, A. Denjoy [1] posed the following famous conjecture:

Let $F(Z)$ be an entire function of finite order λ, if it has K distinct finite asymptotic values, then $K\leqq 2\lambda$.

L. Ahlfors [2] confirmed the conjecture in 1930.

An entire function $F(Z)$ is called to be extremal for Denjoy's conjecture $K\leqq 2\lambda$ if it is of finite order λ and has $K=2\lambda$ distinct finite asymptotic values. Since then, this kind of functions extremal for Denjoy's Conjecture was investigated by many mathematicians such as L. Ahlfors [2] P. Kennedy [3] D. Drasin [4] and Guang-hou Zhang [5]. Here we consider some other properties of this kind of functions.

2. Preliminary and lemmas

First, we introduce the notion of right-prime.

Let F be a meromorphic function on $|Z|<\infty$, if $F(Z)$ can be written as

$$F(Z)=f(g(Z)) \tag{1}$$

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where g is entire and f meromorphic, then (1) is called a factorization of F. If every factorization $F(Z)=f(g(Z))$ implies that g is linear whenever f is transcendental, then F is called right-prime.

In order to prove our results, we need some known results:

LEMMA 1 [2]. *Let F be an entire function of finite order, if F has K distinct finite asymptotic values, then* $\varliminf_{r\to\infty} \log M(r, F)/r^{k/2}>0$, *where* $M(r, F)=\max_{|z|=r} |F(Z)|$.

LEMMA 2 [5]. *Let F be extremal for Denjoy's Conjecture, $a_1, a_2, \cdots, a_k$ its distinct finite asymptotic values, $L_1, L_2, \cdots, l_k$ its asymptotic paths corresponding with $a_1, a_2, \cdots, a_k$. D_i, $(i=1, 2, \cdots, k)$ is the simply connected domain bounded by L_i and L_{i+1} $(i=1, 2, \cdots, k)$, $L_{k+1}=L_1$), then*

(i) *$F(Z)$ has no finite deficient values;*

(ii) *There exists an unbounded domain $\Omega_i \subset D_i$ such that if we denote $\theta_{it}=\{Z ; |Z|=t\}\cap D_i$ and $t\theta_i(t)$ its linear measure, then there exists a constant $r_0>0$, such that*

$$\int_{r_0}^{r} \frac{\left(\frac{2\pi}{k}-\theta_i(t)\right)^2}{\theta_i(t)} \frac{dt}{t}=o(\log r) \qquad (r\to\infty)$$

(ii) *can be obtained from the proof of the Lemma 1 in* [5].

LEMMA 3 [6]. *Let f and g be both nonconstant entire functions, then there exists a constant c $(0<c<1)$, which is independent of r, such that for sufficiently large r, we have*

$$M(r, f(g))>M\left(cM\left(\frac{r}{2}, g\right), f\right).$$

LEMMA 4 [7, p. 119]. *Suppose that f is a meromorphic function of order ρ, where $0\leqq\rho<\frac{1}{2}$, and that $\delta(a, f)>1-\cos\pi\rho$. Then there exists a sequence $r_n\to\infty$ $(n\to\infty)$, such that*

$$f(r_n e^{i\theta}) \longrightarrow a \text{ as } n\to\infty \text{ uniformly for } 0\leqq\theta\leqq 2\pi.$$

3. Proof of theorem 1

Let L_i $(i=1, 2, \cdots, k)$ and a_i $(i=1, 2, \cdots, k)$ be as in Lemma 2. Suppose that $F(Z)=f(g(Z))$ and we discuss three cases.

(i) f, g are both transcendental entire functions.

By Polya's theorem [6] we see that f is of order zero. From Lemma 4 we can deduce that f is unbounded on any unbounded paths. So g is bounded on L_i $(i=1, 2, \cdots, k)$. Suppose that R is sufficiently large, such that $g(L_i)\subset$

$\{\xi;\,|\xi|<R\}$ and there is no zero of $f(\xi)-a_i$ on $|\xi|=R$. Since $f(\xi)-a_i$ has only finitely many zeros in $|\xi|<R$ and $\lim\limits_{\substack{z\to\infty\\L_i}} F(Z)=a_i$ and $g(L_i)$ is connected, we see that g must tend to one of the zeros of $f(\xi)-a_i$ in $|\xi|<R$ as $z\to\infty$ along L_i, that is

$$\lim_{\substack{z\to\infty\\L_i}} g(Z)=b_i\,,$$

where b_i is a zero of $f(\xi)-a_i$.

Therefore, g has also k distinct finite asymptotic values b_i $(i=1, 2, \cdots, k)$. Since the order of g can not be greater than that of F, from Lemma 1 we deduce that $g(z)$ is of order λ.

Since $f(\xi)$ is transcendental, we have

$$\varliminf_{r\to\infty}\frac{\log M(r,\,f)}{\log r}=\infty\,.$$

Using Lemma 3, we obtain

$$\varliminf_{r\to\infty}\frac{\log M(r,\,F)}{\log cM\left(\frac{r}{2},\,g\right)}\geqq\varliminf_{r\to\infty}\frac{\log M\left(cM\left(\frac{r}{2},\,g\right),\,f\right)}{\log cM\left(\frac{r}{2},\,g\right)}$$

$$=\varliminf_{R\to\infty}\frac{\log M(R,\,f)}{\log R}=\infty\,.$$

Since $\varliminf\limits_{r\to\infty}\log M(r,\,F)/r^{k/2}<\infty$, there exists a sequence $r_n\to\infty$ $(n\to\infty)$ such that $\lim\limits_{n\to\infty}\log M(r_n,\,F)/{r_n}^{k/2}=M<\infty$.

From Lemma 1, we have

$$0<\varliminf_{r\to\infty}\log M(r,\,g)/r^{k/2}=\varliminf_{r\to\infty}\log cM\left(\frac{r}{2},\,g\right)\Big/\left(\frac{r}{2}\right)^{k/2}$$

$$<\varliminf_{n\to\infty}\log cM\left(\frac{r_n}{2},\,g\right)\Big/\left(\frac{r_n}{2}\right)^{k/2}$$

$$=\lim_{n\to\infty}\frac{\log cM(r_n/2,\,g)}{\log M(r_n,\,F)}\cdot\frac{\log M(r_n,\,F)}{\left(\frac{1}{2}\right)^{k/2}{r_n}^{k/2}}=0\,.$$

This indicates that (i) is impossible.

(ii) f is a transcendental entire function and g a polynomial.

Now suppose that $L_i'=g(L_i)$, then L_i' is a continuous curve tending to ∞. We can easily see that

$$\lim_{\substack{\xi\to\infty\\L_i'}} f(\xi)=\lim_{\substack{z\to\infty\\L_i}} F(Z)=a_i \qquad (i=1, 2, \cdots, k)$$

So f has k distinct finite asymptotic values and f is of order λ, therefore

$$\lambda=\overline{\lim_{r\to\infty}}\log T(r, f(g))/\log r=\overline{\lim_{r\to\infty}}\frac{\log T(r, f(g))}{\log|g|}\frac{\log|g|}{\log r}$$

$$=\lambda\deg g$$

So $\deg g=1$, that is, $g(z)$ is linear.

(iii) f is a transcendental meromorphic function having at least one pole and g transcendental entire. (it is obvious that g cannot be a polynomial)

In this case, we see that $f(\xi)=(\xi-\xi_0)^{-n}f_1(\xi)$, where n is a positive integer, f_1 a transcendental entire function such that $f_1(\xi_0)\neq 0$, and that $g(z)=\xi_0+e^{P(z)}$, where $P(z)$ is a polynomial.

By a theorem in [8], we know that f is of order zero. And from Lemma 4, f is unbounded on any unbounded paths. As in (i), we can prove that g has k distinct finite asymptotic values. But it is obvious that $\delta(\xi_0, g)=1$. This contradicts Lemma 2 and so (iii) is impossible. The proof of theorem 1 is complete.

COROLLARY 1. *Let F be extremal for Denjoy's conjecture, if F is not right-prime, then*

$$\underline{\lim_{r\to\infty}}\log M(r, F)/r^{k/2}=\infty\,.$$

It is worth noting that from Lemma 1 we only know that

$$\underline{\lim_{r\to\infty}}\log M(r, F)/r^{k/2}>0\,.$$

4. Proof of theorem 2.

Suppose that a_i, L_i, D_i, Ω_i, θ_{it} and $\theta_i(t)$ $(i=1, 2, \cdots, k)$ are defined as in Lemma 2 and set $F(z)/P(z)=f(g(z))$, we need only discuss two cases:

(i) f is transcendental meromorphic and g transcendental entire.

In this case, noting that $F(z)/P(z)$ has only finitely many poles, we have

$$f(\xi)=(\xi-\xi_0)^{-n}f_1(\xi)$$

where n is a positive integer, f_1 an entire function of order zero and $f_1(\xi_0)\neq 0$. We also have

$$g(z)=\xi_0+p_1(z)e^{p_2(z)}$$

where p_1 and p_2 are both nonconstant polynomials. For the convenience of the proof, we may assume $\xi_0=0$.

Using Lemma 4, as in the proof of theorem 1 (i), for each i we have $\lim_{\substack{z\to\infty\\ L_i}} g(z)=b_i$, where b_i is a zero of f_1.

Now we prove that g is unbounded in D_i. If this is not true, from Lindelof's theorem, g is uniformly bounded in D_i and g is uniformly convergent to b_i $(=b_{i+1})$ as z tend to ∞ in $\bar{D}_i$.

From Lemma 2, noting that $0<\theta_i(t)<2\pi$, we can find a sequence $r_n\to\infty$ $(n\to\infty)$ such that $\lim_{n\to\infty}\theta_i(r_n)=2\pi/k$.

Let $z=re^{i\theta}$, we have

$$|g(z)|=|p_1(z)e^{p_2(z)}|=|p_1(z)e^{(a_m+ib_m)z^m}+p_3(z)|$$
$$<|p_1(z)|\exp\left\{\frac{r^m}{\sqrt{a_m^2+b_m^2}}\cos m(\theta+\alpha)+o(r^{m-1})\right\}.$$

where p_3 is a polynomial whose degree is at most $m-1$ $(=\deg p_2-1)$. So the plane $|z|<+\infty$ is divided into $2m$ distinct angular domains $\Omega(\varphi_j, \varphi_{j+1})=\{re^{i\theta}; \varphi_j\leqq\theta<\varphi_{j+1}\}$, $(j=1, 2, \cdots, 2m, \varphi_{2m+1}=\varphi_1+2\pi)$, such that for sufficiently small $\varepsilon>0$, $g(z)$ is uniformly convergent to ∞ (or 0) as z tends to ∞ in $\bar{\Omega}(\varphi_j+\varepsilon, \varphi_{j+1}-\varepsilon)$. Since $\lim_{n\to\infty}\theta_i(r_n)=\dfrac{2\pi}{k}$, if we set $\varepsilon_0=\dfrac{2\pi}{8km}$, then for sufficiently large n, there must exist $z_n=r_ne^{i\theta_n}\in\theta_i r_n$, such that $\varphi_j+\varepsilon_0<\theta_n<\varphi_{j+1}-\varepsilon_0$ for some j $(1\leqq j\leqq 2m)$. Since there are only $2m$ distinct angular domains $\Omega(\varphi_j+\varepsilon_0, \varphi_{j+1}-\varepsilon_0)$, we can choose a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\{z_{n_k}\}\subset\bar{\Omega}(\varphi_{j_0}+\varepsilon_0, \varphi_{j_0+1}-\varepsilon_0)$, where j_0 $(1\leqq j_0\leqq 2m)$ is fixed. Since $\lim_{\substack{z\to\infty\\ \bar{D}_i}} g(z)=b_i\neq\infty$, g cannot be convergent to ∞ in $\bar{\Omega}(\varphi_{j_0}+\varepsilon_0, \varphi_{j_0+1}-\varepsilon_0)$, so we have $b_i=0$ and $f_1(0)=0$. This contradicts the fact $f_1(0)\neq 0$. So g is unbounded in D_i.

Therefore, g has k distinct asymptotic paths L_i and is unbounded in D_i. Using the same method as in the proof of Lemma 1 and Lemma 2, we can show that for g the conclusions of Lemma 1 and Lemma 2 remain valid. But we also have $\delta(0, g)=1$, so (i) is impossible.

(ii) f is transcendental meromorphic and g a polynomial.

In this case, we have $f(\xi)=f_1(\xi)/p_1(\xi)$, where f_1 is transcendental entire and p_1 a polynomial such that f_1 and p_1 have no common zero. So we have

$$\frac{F(z)}{P(z)}=\frac{f_1(g(z))}{p_1(g(z))}.$$

We see that $F(z)=cf_1(g(z))$. As in the proof of theorem 1 (ii), we can also deduce that g is linear. The proof of theorem 2 is complete.

5. Discussion of theorem 1

First we give an example to show that a nonprime entire function $F(Z)$ can satisfy the condition of theorem 1.

Example 1. Let

$$F(Z)=\int_0^z\frac{\sin t^2}{t^2}dt$$

and $G(Z)=(F(Z))^3$. Then $G(Z)$ is of order 2 and has 4 asymptotic values:

$e^{3\nu\pi i/2}\left(\int_0^\infty \frac{\sin r^2}{r^2}dr\right)^3$ ($\nu=1, 2, 3, 4$). We can easily show $\varliminf_{r\to\infty}\frac{\log M(r, G)}{r^2}<\infty$. So $G(Z)$ is only right-prime and not prime.

Now we give another example to show that the condition $\varliminf_{r\to\infty}\frac{\log M(r, F)}{r^{k/2}}<\infty$ in theorem 1 is necessary.

Example 2. Let

$$f(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\exp(\exp k)}\right).$$

Since

$$n(r, o, f)=O(\log\log r) \qquad (r\to\infty)$$

and

$$\log M(r, f)=O((\log r)(\log\log r)), \qquad (r\to\infty)$$

we see that $f(z)$ is of order zero, and for any entire function g, we haxe

$$\log M(r, f(g))<\log M(M(r, g), f)=O(\log M(r, g)\log\log M(r, g)).$$

So $f(g)$ has the same order as g. Now we put $g(z)=\int_0^z\frac{\sin t}{t}dt$, then $f(g)$ has order 1.

Since $f(z)$ is transcendental, we have

$$\varliminf_{r\to\infty}\log M(r, f)/\log r=\infty$$

and for any $k>0$,

$$\log M(r, f)>k\log r \qquad (r\to\infty)$$

so

$$\varliminf_{r\to\infty}\frac{\log M(r, f(g))}{r}\geqq\varliminf_{r\to\infty}\frac{k\log cM\left(\frac{r}{2}, g\right)}{r}$$

$$\geqq\frac{k}{2}\varliminf_{r\to\infty}\frac{\log M(r, g)}{r}.$$

Since k can be arbitrarily large, we deduce

$$\varliminf_{r\to\infty}\frac{\log M(r, f(g))}{r}=\infty.$$

On the other hand, we have

$$\lim_{r\to\infty}f(g(r))=f\left(\frac{\pi}{2}\right),$$

$$\lim_{r\to-\infty}f(g(r))=f\left(-\frac{\pi}{2}\right)$$

and

$$f\left(\frac{\pi}{2}\right)=\prod_{k=1}^{\infty}\left(1-\frac{\pi}{2\exp(\exp k)}\right)\neq\prod_{k=1}^{\infty}\left(1+\frac{\pi}{2\exp(\exp k)}\right)=f\left(-\frac{\pi}{2}\right).$$

So function $f(g)$ is extremal for Denjoy's Conjecture but is not pseudo-prime.

6. Proo of theorem 3.

In this part, we shall prove theorem 3. First, we shall say something about the linear differential equation $f''+Af'=0$.

In recent years, there are many papers on the properties of the solutions of differential equation $f''+Af=0$ where $A(Z)$ is entire or meromorphic, a major asspect of which is what conditions on $A(Z)$ will guarantee that every solution or one of the two linear independent ones of $f''+Af=0$ has the property that the exponent of convergence of its zero-sequence is ∞. We consider here that $A(Z)$ is an entire function extremal for Denjoy's Conjecture and obtain the interesting result which is stated in theorem 3.

In order to prove theorem 3, we need two lemmas.

LEMMA 5. *Let $A(Z)$ be an entire function extremal for Denjoy's conjecture $k=2\lambda$ and $b_1, b_2, \cdots, b_N$ its zeros. Suppose $H(z)=A(z)/\prod_{i=1}^{N}(z-b_i)$, then we have the following conclusions:*

(a) $\varliminf_{r\to\infty}\log M(r, H)/r^{k/2}>0$; (6.1)

(b) *$H(z)$ has the same order as $A(z)$;*

(c) *Zero is an asymptotic value of $H(z)$, and $H(z)$ has k distinct asymptotic paths L_i $(i=1, 2, \cdots, k)$ which divide the plane $|Z|<\infty$ into k disjoint simply-connected domains D_i $(i=1, 2, \cdots, k)$ (By suitable choice of the subscripts, we may assume that D_i is bounded by L_i and L_{i+1}, $(1\leqq i\leqq k, L_{k+1}=L_1)$)*

(d) *For each i $(1\leqq i\leqq k)$, there exists a curve contained in D_i tending to ∞ such that*

$$\lim_{\substack{z\to\infty\\ \Gamma_i}}\frac{\log\log|H(z)|}{\log|z|}=\lambda. \tag{6.2}$$

Proof. From Lemma 1 and Lemma 2, (a), (b) and (c) are obvious.

From theorem 1 in [5(I)], we know that for each i, there exists a curve Γ_i contained in D_i tending to ∞ such that

$$\lim_{\substack{z\to\infty\\ \Gamma_i}}\log\log|A(z)/\log|z|=\lambda \qquad (i=1, 2, \cdots, k)$$

From this we can easily deduce (d).

LEMMA 6 [9]. *Let $f(z)$ be an entire function and $N>1$ a given constant. Put*

$$D=\{z;\ |f(z)|>N\}.$$

If we define $A_k(t)$ $(k=1, 2, \cdots, n(t))$ the arcs of $|z|=t$ contained in D and $t\theta_k(t)$ their lengths, and

$$\theta_f(t)=\begin{cases} \infty, & |z|=t \text{ contained in } D \\ \max\limits_{1\leq k\leq n(t)} \theta_k(t), & \text{otherwise,} \end{cases} \tag{6.3}$$

then for any $0<\alpha<1$, we have

$$\log\log M(r, f)>\pi\int_{r_0}^{\alpha r}\frac{dt}{t\theta_f(t)}+C(\alpha, r_0)$$

where $0<r_0<\alpha r$ and $c(\alpha, r_0)$ is a constant independent of r.

Proof of the theorem

Let f_1, f_2 be two linear independent solutions of $f''+Af=0$. Set $F=f_1f_2$. Bank and Laine [10, p. 354] deduced that the function satisfies the equation

$$-4A=\frac{c^2}{F^2}+2\left(\frac{F''}{F}\right)-\left(\frac{F'}{F}\right)^2, \tag{6.4}$$

where c is the constant Wronskian of f_1 and f_2. Thus by applying the Nevanlinna theory to (6.4), they obtained

$$T(r, F)=O\left(N\left(r, \frac{1}{F}\right)+T(r, A)+\log r\right) \tag{6.5}$$

as $r\to\infty$ outside a set of finite logarithmic measure.

If the order ρ of F is finite. From a Lemma [12], there exists a set $E\subset(0, \infty)$ having finite logarithmic measure such that for $|z|\notin E$

$$|F''(z)/F(z)|+|F'(z)/F(z)|<|z|^{[4\rho+1]}=|z|^q \tag{6.6}$$

From lemma 5(a), we know that $A(z)$ must have infinitely many zeros, $\{b_i\}$ say. Put

$$H(z)=4A(z)\Big/\prod_{i=1}^{q+1}(z-b_i)=A(z)/P(z) \text{ (say)}$$

Now we have a contradition as follows, by the similar arguments to those in the proof of theorem 1 in [5 (I)].

Let $N>\max\{1, \sup\limits_{z\in L}|H(z)|\}$ be a constant, where $L=\bigcup\limits_{i=1}^{k}L_i$. To $H(z)$ applying lemma 5(d), we know that there exist $z_i\in D_i$ $(i=1, 2, \cdots, k)$ such that

$$|H(z_i)|\geqq 2N. \tag{6.7}$$

Write

$$r_0=\max\{1, |z_1|, |z_2|, \cdots, |z_k|\}.$$

Since there exists $N'\in[N, 2N]$ such that there is no zero of $H'(z)$ on the curves defined by $|H(z)|=N'$ and there is no zero of $F'(z)$ on the curve defined by $|F(z)|=N'$, the curves are analytic. We put

$$\tilde{D}_1=\{z;\ |F(z)|>N'\} \tag{6.8}$$

$$\tilde{D}_2=\{z;\ |H(z)|>N'\}, \tag{6.9}$$

$$E^*=\{z;\ z=re^{i\theta},\ 0\leqq\theta<2\pi,\ r\in E\}$$

From (6.4) and (6.5), we deduce that if $z\in\tilde{D}_1-E^*$, then

$$4|A(z)|<|c|^2+|z|^q<\frac{1}{2}|P(z)|. \qquad (|z|=r\geqq r_0) \tag{6.10}$$

But for $z\in\tilde{D}_2-E^*$, we have

$$4|A(z)|>|P(z)|. \tag{6.11}$$

From (6.10) and (6.11), we see that $(\tilde{D}_1-E^*)\cap\{z;\ |z|\geqq r_0\}\cap(\tilde{D}_2-E^*)=\emptyset$.

Let $\Omega_i\subset D_i$ be the connected component of $\tilde{D}_2$ containing z_i, $(i=1, 2, \cdots, k)$. From the maximum modulus principle, we deduce that each Ω_i $(i=1, 2, \cdots, k)$ is an unbounded domain. Let θ_{it} $(i=1, 2, \cdots, k,\ r_0\leqq t<\infty)$ be the arc $|z|=t$ contained in Ω_i, and $t\theta_i(t)$ its linear measure, then we have

$$\sum_{i=1}^{k}\theta_i(t)+\theta_F(t)\leqq 2\pi, \qquad t\notin E$$

From Lemma 5(a), (b), we deduce that

$$\log\log M(r, H)=\frac{k}{2}\log r+o(\log r) \tag{6.12}$$

By a theorem in [11, p. 116], we have

$$\log|H(z_i)|<\log N'+9\sqrt{2}\exp\left(-\pi\int_{2|z_i|}^{r/2}\frac{dt}{t\theta_i(t)}\right)\log M(r, H). \qquad (i=1, 2, \cdots, k)$$

Then we obtain

$$\sum_{i=1}^{k}\int_{2r_0}^{r/2}\frac{\pi dt}{t\theta_i(t)}\leqq K\log\log M(r, H)+O(1),$$

and we have

$$\int_{2r_0}^{r/2}\sum_{i=1}^{k}\left(\frac{\pi}{\theta_i(t)}-\frac{k}{2}\right)\frac{dt}{t}\leqq K\left[\log\log M(r, H)-\frac{k}{2}\log r\right]+O(1).$$

Since

$$K^2=\left(\sum_{i=1}^{k}\sqrt{\theta_i(t)}/\sqrt{\theta_i(t)}\right)^2\leqq\left(\sum_{i=1}^{k}\theta_i(t)\right)\left(\sum_{i=1}^{k}\frac{1}{\theta_i(t)}\right),$$

we have

$$\sum_{i=1}^{k}\left(\frac{\pi}{\theta_i(t)}-\frac{k}{2}\right)\geqq 0,$$

Hence we deduce from (6.12)

$$\int_{2r_0}^{r/2}\sum_{i=1}^{k}\left(\frac{\pi}{\theta_i(t)}-\frac{k}{2}\right)\frac{dt}{t}=o(\log r). \tag{6.13}$$

For any $\delta>0$, put

$$E(\delta)=\left\{t\,;\ \sum_{i=1}^{k}\theta_i(t)\leqq 2\pi-2\delta\right\}$$

$$E(\delta,\, r)=E(\delta)\cap\left[2r_0,\ \frac{1}{2}r\right].$$

$$E^c(\delta,\, r)=\left[2r_0,\ \frac{1}{2}r\right]-E(\delta).$$

We see that if $t\in E(\delta)$, then $\sum_{i=1}^{k}\left(\frac{\pi-\delta}{\theta_i(t)}-\frac{k}{2}\right)\geqq 0$. From (6.13), we deduce that for any $\varepsilon>0$, there exsits $R>0$ such that for $r\geqq R$,

$$\begin{aligned}\varepsilon\log r &> \int_{2r_0}^{r/2}\sum_{i=1}^{k}\left(\frac{\pi}{\theta_i(t)}-\frac{k}{2}\right)\frac{dt}{t}\\ &=\left(\int_{E(\delta,r)}+\int_{E^c(\delta,r)}\right)\sum_{i=1}^{k}\left(\frac{\pi}{\theta_i(t)}-\frac{k}{2}\right)\frac{dt}{t}\\ &\geqq \int_{E(\delta,r)}\sum_{i=1}^{k}\left(\frac{\pi}{\theta_i(t)}-\frac{\pi-\delta}{\theta_i(t)}\right)\frac{dt}{t}\\ &\geqq \frac{k\delta}{2\pi}\int_{E(\delta,r)}\frac{dt}{t}.\end{aligned}$$

Therefore

$$\lim_{r\to\infty}\frac{1}{\log r}\int_{E(\delta,r)}\frac{dt}{t}=0,$$

that is, the logarithmic dense of $E(\delta)$ is zero. If $t\notin E(\delta)\cup E$ and $t\geqq r_0$, then

$$\theta_F(t)\leqq 2\delta.$$

Let $J_r^*=\left[2r_0,\ \frac{1}{2}r\right]-(E(\delta)\cup E))$. From Lemma 6, we have

$$\begin{aligned}\log\log M(r,\, F) &\geqq \pi\int_{2r_0}^{r/2}\frac{dt}{t\theta_F(t)}+C\\ &\geqq \frac{\pi}{2\delta}\int_{J_r^*}\frac{dt}{t}+C\end{aligned}$$

Since

$$\lim_{r\to\infty}\frac{1}{\log r}\int_{J_r^*}\frac{dt}{t}=1, \tag{6.14}$$

we have

$$\underline{\lim}_{r\to\infty}\frac{\log\log M(r, F)}{\log r}\geqq\frac{\pi}{2\delta}. \tag{6.15}$$

As δ can be arbitrarily small, (6.15) contradicts the assumption that F is of finite order. So F must be of infinite order.

Noting that $A(z)$ is of finite order, then from (6.5) we deduce that $\overline{\lim}_{r\to\infty}\log N\left(r, \frac{1}{F}\right)\Big/\log r=\infty$, and this completes the proof.

7. Application

In this section, we define $\lambda(g)$ the exponent convergence of zero-sequence of g.

Example 1. If f_1, f_2 are two linear independent solutions of

$$f''+\left(\int_0^z\frac{\sin t^{m/2}}{t^{m/2}}dt\right)f=0, \tag{7.1}$$

where m is a positive integer, then $\max\{\lambda(f_1), \lambda(f_2)\}=\infty$.

Example 2. In [13], Bank and Laine considered the differential equations:

$$f''+z^q\sin^p(z^m)f=0, \tag{7.2}$$

$$f''+z^q\cos^p(z^m)f=0. \tag{7.3}$$

where q, p, m are positive integers. Under the additional condition $q>2(m-1)$, they obtained that if f_1, f_2 are two linear independent solutions of (7.2) or (7.3), then $\max(\lambda(f_1), \lambda(f_2))>1+m/2$.

From our method used to prove the theorem, we can get some strong results in more general cases. Now we consider the differential equation:

$$f''+p_1(z)P(\sin z^{m/2}/z^{m/2})f=0 \tag{7.4}$$

where $p_1(z)$, $P(w)$ ($\neq$const) are polynomials, and m is a positive integer. We prove that if f_1, f_2 are two linear independent solutions of (7.4), then $\max(\lambda(f_1), \lambda(f_2))=\infty$.

First we note that $A(z)=p_1P(\sin z^{m/2}/z^{m/2})$ is of finite order and m distinct paths L_ι from the oringin to ∞: $\arg z=\frac{2\nu\pi}{m}$ ($\nu=1, 2, \cdots, m$) are the asymptotic paths of $A(z)/p_1(z)$. In each angular domain D_ι bounded by L_ι and $L_{\iota+\iota}$

$(1\leqq i\leqq m,\ L_{m+1}=L_1)$, $A(z)/p_1(z)$ is unbounded.

It is worth noting that Lemma 5 is still true under the following conditions:

(1) $A(z)$ is of finite order $m/2$.

(2) There are m distinct paths L_i from the oringin to ∞ $(i=1, 2, \cdots, m)$ and m distinct simply-connected domains D_i bounded by L_i and L_{i+i} $(1\leqq i\leqq m,$ $L_{m+1}=L_1)$ such that $A(z)$ is bounded on $L=\bigcup_{i=1}^{m} L_i$ and unbounded in each D_i $(i=1, 2, \cdots, m)$.

Therefore, $A(z)/p_1(z)$ has infinitely many zeros. If we choose an integer q such that $q\geqq \deg p_1$, and (6.6) is also satisfied, and let N in Lemma 5 equal to $1+q$. We deduce that $A(z)\Big/\prod_{i=1}^{N}(z-a_i)=H(z)$ has the properties of Lemma 5(a), (b), (c), (d). Thus, using the same method in the proof of the theorem, we can prove the result mentioned above.

Applying the same method, we deduce that if f_1, f_2 are two linear independent solutions of

$$f''+p_1(z)P(\cos z^{m/2})f=0, \tag{7.5}$$

where $p_1(z)$, $P(w)$ $(\neq\text{const})$ are polynomials, and m is a positive integer, then $\max(\lambda(f_1), \lambda(f_2))=\infty$.

Especially, if f_1, f_2 are two linear independent solutions of (7.2) (or (7.3)), where p, m are positive integers, and q is a nonnegetive integer, then $\max(\lambda(f_1), \lambda(f_2))=\infty$.

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