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TORSION AND CRITICAL METRICS ON CONTACT THREE-MANIFOLDS

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1. Introduction. Let M be a compact orientable manifold of class C^∞. It is well known that a Riemannian metric g on M is a critical point of the functional "integral of the scalar curvature $\int_M r dv$" defined on the set of all Riemannian metrics of the same total volume on M, if and only if g is an Einstein metric.

Now let (M, ω) be a compact contact three-manifold. Then there exists an unique vector field X_0 on M such that $\omega(X_0)=1$ and $d\omega(X_0, \cdot)=0$. Consider the following functional

$$\mathcal{F}(g)=\int_M r dv \qquad g\in\mathcal{M}(\omega)$$

where $\mathcal{M}(\omega)$ denotes the space of all associated Riemannian metrics to the contact form ω. This functional was studied by Blair and Ledger [2] in general dimension. However the three-dimensional case has many special features to merit a separate study. Chern and Hamilton [7] introduced the torsion $\tau=L_{X_0}g$, namely the Lie derivative of g with respect to X_0, in their study of compact contact three-manifolds, and studied the Dirichlet energy

$$\mathcal{E}_C(g)=\int_M c^2 dv \qquad g\in\mathcal{M}(\omega),\ c^2=\frac{1}{2}|\tau|^2$$

over the set of "CR-structures" on M (see also Tanno [15]). Goldberg, the present author and Toth [10] studied the geometry of a compact contact Riemannian three-manifold (M, ω, g) with g critical metric of $\mathcal{E}_C$.

The main purpose of this paper is to study compact Riemannian three-manifolds (M, ω, g) with g critical metric of the functional $\mathcal{F}$.

In § 3, we show that a point g of $\mathcal{M}(\omega)$ is a critical point of $\mathcal{F}$, if and only if

$$(1.1)\qquad \nabla_{X_0}\tau=0.$$

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This condition is related to some interesting geometric properties, for example it is equivalent to the condition that the sectional curvatures of all planes, at a given point, perpendicular to $B=\ker\omega$ are equal to $(1-c^2/4)$. The (1.1) was incorrectly obtained in [7] as the condition for the critical metrics of the functional $\mathcal{E}_C$. Note that X_0 is a Killing vector field with respect to g, if and only if g is a critical point of $\mathcal{E}_C$ and $\mathcal{F}$. Besides if g is ω-Einstein or locally symmetric, then it is a critical point of $\mathcal{F}$.

In §4 we show that a metric g of $\mathcal{M}(\omega)$ is ω-Einstein if, and only if, the following hold: (1) g is a critical point of $\mathcal{F}$, and (2) the ϕ-torsion $\psi=-\tau\cdot\phi$ is perpendicular to the orbit of g under the group of diffeomorphisms of M. Moreover we give some properties of a tensor S_1 which measures the deviation from the ω-Einstein structure, for example $S_1=-\frac{1}{2}\nabla_{X_0}\tau$ if and only if the above condition (2) holds.

In §5, we extend some results of [8] and [10]. Precisely we show that the metric of a compact contact Riemannian three-manifold (M, ω, g, X_0) whose characteristic vector field X_0 is of Killing, may be deformed to a contact metric of positive sectional curvature if either the Ricci curvature is greater than $-2g$ or the ϕ-sectional curvature is greater than -3. Hence if, in addition, M is simply connected, then by [11] it is diffeomorphic with the three-sphere.

2. Contact manifolds. A $(2n+1)$-dimensional manifold M is said to be a *contact manifold* if it carries a global 1-form $\omega\neq 0$ with the property that $\omega\wedge(d\omega)^n\neq 0$ everywhere. It has an underlying almost contact structure (X_0, ω, ϕ), where $\omega(X_0)=1$, $\phi X_0=0$ and $\phi^2=-I+\omega\otimes X_0$. A metric g, called an *associated metric*, can then be found such that $\omega=g(X_0, \cdot)$, $d\omega(X, Y)=g(\phi X, Y)$ and hence $g(\phi X, Y)=-g(X, \phi Y)$. These metrics are constructed by the polarization of $d\omega$ evaluated on a local orthonormal basis of an arbitrary metric on the sub-bundle B of TM defined by $\ker\omega$. We refer to (ω, g) or (ω, g, X_0, ϕ) as a *contact Riemannian structure*. All metrics g of $\mathcal{M}(\omega)$, namely associated to the contact form ω, have the same volume element $(1/2^n n!)\omega\wedge(d\omega)^n$, and hence we will write dv instead of dv_g. Given a contact metric structure (ω, g, X_0, ϕ), the torsion $\tau=L_{X_0}g$ satisfies (cf. [9]):

$$\tau(X_0, \cdot)=0, \qquad \tau(X, Y)=\tau(Y, X) \tag{2.1}$$

$$\tau(\phi X, Y)=\tau(X, \phi Y), \qquad \tau(\phi X, \phi Y)=-\tau(X, Y). \tag{2.2}$$

Moreover (see for example formula (3.1)) $\tau(X, Y)=2g(\phi X, hY)$ where $h=\frac{1}{2}L_{X_0}\phi$. So h is a symmetric operator which anticommutes with ϕ. If X_0 is a Killing vector field with respect to g, the contact metric structure is said to be *K-contact*. It is easy to see that a contact metric structure is K-contact if and only if $\tau=0$ (or equivalently $h=0$). The reader is referred to [3] for details and other properties of contact manifolds. In the sequel we denote by R, S, r and K, respectively, the curvature tensor, the Ricci tensor, the scalar curvature and the sectional curvature of a given contact Riemannian manifold; moreover for tensor fields U and V of the same type, we put

$$\langle U, V\rangle = U^{ij\cdots}V_{ij\cdots} \quad \text{and} \quad |U|^2 = \langle U, U\rangle.$$

3. Torsion and critical metrics. Let $M(\omega, g, X_0, \phi)$ be a $(2n+1)$-dimensional contact Riemannian manifold and ∇ the Riemannian connection with respect to g. First we give the following.

PROPOSITION 3.1. *The tensor field $\nabla_{X_0}\tau$ satisfies the following properties:*

(i) $(\nabla_{X_0}\tau)(X, Y) = (\nabla_{X_0}\tau)(Y, X)$,

(ii) $(\nabla_{X_0}\tau)(X_0, \cdot) = 0$,

(iii) $(\nabla_{X_0}\tau)(\phi X, \phi Y) = -(\nabla_{X_0}\tau)(X, Y)$,

(iv) *for E in B, $|E|=1$, the sectional curvature $K(X_0, E)$ is given by*

$$K(X_0, E) = -\frac{1}{2}(\nabla_{X_0}\tau)(E, E) + 1 - |h(E)|^2,$$

(v) $\nabla_{X_0}\tau = 0$ *if, and only if,* $K(X_0, E') - K(X_0, E) = |h(E)|^2 - |h(E')|^2$ *for every E, E' in B, $|E| = |E'| = 1$,*

(vi) *if $n=1$, then*

$$(\nabla_{X_0}\tau)(X, Y) = S(\phi X, \phi Y) - S(X, Y) + \omega(X)S(X_0, Y) + \omega(Y)S(X_0, X) - S(X_0, X_0)\omega(X)\omega(Y).$$

Proof. (i) $\nabla_{X_0}\tau$ is symmetric because τ is symmetric.

(ii) $(\nabla_{X_0}\tau)(X_0, \cdot) = X_0\tau(X_0, \cdot) - \tau(\nabla_{X_0}X_0, \cdot) - \tau(X_0, \nabla_{X_0}\cdot) = 0$ because $\nabla_{X_0}X_0 = 0$ (cf. [3]) and $\tau(X_0, \cdot) = 0$.

(iii) follows from $\nabla_{X_0}\phi = 0$ (cf. [3]), i.e. $\nabla_{X_0}\phi(X) = \phi(\nabla_{X_0}X)$, and (2.2).

(iv) Since $L_{X_0}d\omega = 0$ (cf. [3]), we have

$$\begin{aligned} 0 &= X_0 d\omega(X, Y) - d\omega([X_0, X], Y) - d\omega(X, [X_0, Y]) \\ &= X_0 g(X, \phi Y) - g([X_0, X], \phi Y) - g(X, \phi[X_0, Y]) \\ &= \tau(X, \phi Y) + g(X, [X_0, \phi Y] - \phi[X_0, Y]) = \tau(X, \phi Y) + 2g(X, hY) \end{aligned}$$

and hence

$$\tau(X, Y) = 2g(\phi X, hY). \tag{3.1}$$

Since $\nabla_{X_0}\phi = 0$, from (3.1) it follows that

$$(\nabla_{X_0}\tau)(X, Y) = 2g(\phi X, (\nabla_{X_0}h)Y). \tag{3.2}$$

Moreover we have the following formula (cf. (3.3) of [4])

$$\nabla_{X_0}h = \phi - \phi h^2 - \phi R(\cdot, X_0)X_0. \tag{3.3}$$

From (3.3) and (3.2), since h is symmetric and anticommutes with ϕ, we get

$$K(X_0, E)=g(R(E, X_0)X_0, E)=g(-(\nabla_{X_0}h)E+\phi E-\phi h^2E, \phi E)$$
$$=-\frac{1}{2}(\nabla_{X_0}\tau)(E, E)+1-g(hE, hE).$$

(v) If $\nabla_{X_0}\tau=0$, from (iv) we have

$$K(X_0, E')-K(X_0, E)=|h(E)|^2-|h(E')|^2$$

for E, E' in B, $|E|=|E'|=1$. Conversely if this formula holds, then (iv) implies $(\nabla_{X_0}\tau)(E, E)=(\nabla_{X_0}\tau)(E', E')$. Choosing $E'=\phi E$, by (iii), we obtain

$$(\nabla_{X_0}\tau)(E, E)=0 \qquad \text{for } E \text{ in } B,\ |E|=1 .$$

So, by (ii), $\nabla_{X_0}\tau=0$.

(vi) For E in B, $|E|=1$, since $h\phi=-\phi h$, we have $|hE|=|h\phi E|$ and hence (iv) implies

$$K(X_0, E)-K(X_0, \phi E)=-(\nabla_{X_0}\tau)(E, E) \tag{3.4}$$

(see also Lemma 7.1 of [15]). Since $\dim M=3$, from (3.4) it follows that

$$S(\phi E, \phi E)-S(E, E)=(\nabla_{X_0}\tau)(E, E).$$

Consequently for E, E' in B,

$$S(\phi(E+E'), \phi(E+E'))-S(E+E', E+E')=(\nabla_{X_0}\tau)(E+E', E+E')$$

implies

$$S(\phi E, \phi E')-S(E, E')=(\nabla_{X_0}\tau)(E, E'). \tag{3.5}$$

Finally for X and Y in TM, ϕX and ϕY are in B and $\phi^2X=-X+\omega(X)X_0$, $\phi^2Y=-Y+\omega(Y)X_0$, therefore by (iii) and (3.5) we get the property (vi).

THEOREM 3.2. *Let (M, ω) be a compact contact three-manifold. Then a metric g in $\mathcal{M}(\omega)$ is a critical point of the functional $\mathcal{I}$ if and only if*

$$\nabla_{X_0}\tau=0.$$

Proof. Let $g(t)$ be a smooth curve in $\mathcal{M}(\omega)$ such that $g(o)=g$. We calculate $d\mathcal{I}/dt$ at $t=0$, where

$$\mathcal{I}(t)=\mathcal{I}(g(t))=\int_M r(t)dv .$$

We put

$$g(t)=g+tk+[t^2]$$

where $[t^2]$ denotes a set of terms of higher order ($\geqq 2$) in t, and k is a second order symmetric tensor that satisfies (see [6] p. 304)

$$k(X_0, \cdot)=0 \quad \text{and} \quad k(\phi X, \phi Y)=-k(X, Y).$$

Moreover the scalar curvature $r(t)$ is given (see [15] § 13) by

$$r(t)=r+t\{div-\langle k, S\rangle\}+[t^2]$$

where div denotes a term which is a divergence. So, by Green's Theorem, we get

$$\frac{d\mathcal{F}}{dt}(0)=\left\{\int_M \frac{dr(t)}{dt}dv\right\}(0)=-\int_M \langle k, S\rangle dv.$$

Since $k(X_0, \cdot)=0$, we can write $\langle k, S\rangle=\langle k, T\rangle$ where

$$T=S-S(X_0, \cdot)\otimes\omega-\omega\otimes S(X_0, \cdot)+S(X_0, X_0)\omega\otimes\omega$$

Moreover, since $k(\phi E, \phi E)=-k(E, E)$ for E in B, $\langle k, T\rangle=\langle k, V\rangle$ where $V=\frac{1}{2}T-\frac{1}{2}S(\phi\cdot, \phi\cdot)$. On the other hand by Proposition 3.1 property (vi),

$$\frac{1}{2}\nabla_{X_0}\tau=\frac{1}{2}\{S(\phi\cdot, \phi\cdot)-T\}=-V.$$

Therefore

$$\frac{d\mathcal{F}}{dt}(0)=\frac{1}{2}\int_M \langle k, \nabla_{X_0}\tau\rangle dv. \tag{3.6}$$

So if $\nabla_{X_0}\tau=0$, then g is a critical point of $\mathcal{F}$. Conversely assume that g is critical for $\mathcal{F}$. We put $k=\nabla_{X_0}\tau$, then by Proposition 3.1 k is symmetric, $k(X_0, \cdot)=0$, and $k(\phi X, Y)=-k(X, \phi Y)$. Consequently, by [6] p. 304 (see also [15]), $g(t)=ge^{tk^*}$, $-\varepsilon<t<\varepsilon$, is a smooth curve in $\mathcal{M}(\omega)$ such that $g(0)=g$ where $k^*=(k^i_j)$ and $g(t)(X, Y)=g(X, e^{tk^*}Y)$. Applying (3.6) to this deformation, we get $\nabla_{X_0}\tau=0$.

Remark 3.1. (i) Blair and Ledger [2] proved, in general dimension, that a metric g in $\mathcal{M}(\omega)$ is a critical point of $\mathcal{F}$ if and only if the Ricci operator and ϕ when restricted to the contact distribution, commute.

(ii) $\nabla_{X_0}\tau=0$ is the condition incorrectly obtained in [7] (cf. Theorem 5.4) for a metric $\mathcal{E}_C$-critical. Therefore, by our Theorem 3.2, the main result of [9] holds replacing the assumption g $\mathcal{E}_C$-critical by g $\mathcal{F}$-critical.

(iii) The condition $\nabla_{X_0}\tau=0$, in general dimension, was studied by Tanno (see [15] § 7) because it is related to some interesting properties. For example he proved that the conditions: $\nabla_{X_0}\tau=0$, $\nabla_{X_0}\nabla X_0=0$ and $\nabla_{X_0}T^*=0$, are equivalent, where T^* is the torsion tensor of the generalized Tanaka connection. From Proposition 3.1 we obtain

$$\nabla_{X_0}\tau=0 \text{ iff } K(X_0, E')-K(X_0, E)=|hE|^2-|hE'|^2 \text{ for } E, E' \text{ in } B,\ |E|=|E'|=1.$$

Hence, when $\nabla_{X_0}\tau=0$, we have at a given point

$$K(X_0, E')>K(X_0, E) \text{ (resp. } =) \text{ iff } |hE|>|hE'| \text{ (resp. } =).$$

In particular if E' is an unit vector of the plane generated by $\{E, \phi E\}$, we have $|hE'|=|hE|$. So, for three-dimensional manifolds, the condition $\nabla_{X_0}\tau=0$ is equivalent to the condition that the sectional curvatures of all planes at a given point perpendicular to B be equal.

Remark 3.2. Let (M, ω) be a compact contact three-manifold. Chern and Hamilton [7] studied also the following energy

$$\mathcal{E}_w(g)=\int_M W dv \qquad g\in\mathcal{M}(\omega)$$

where $W=\frac{1}{8}\left(r+\frac{c^2}{2}+2\right)$ is the Webster scalar curvature (see [7] p. 284). If $g(t)$ is a smooth curve in $\mathcal{M}(\omega)$ with $g(0)=g$, then $8\mathcal{E}_w(g(t))=\mathcal{F}(g(t))+\frac{1}{2}\mathcal{E}_C(g(t))+2\mathrm{vol}(M, g)$ and hence

$$8(d\mathcal{E}_w/dt)(0)=(d\mathcal{F}/dt)(0)+\frac{1}{2}(d\mathcal{E}_C/dt)(0). \tag{3.7}$$

Tanno proved (see [15] § 5) that

$$(d\mathcal{E}_C/dt)(0)=-\int_M \langle k, \nabla_{X_0}\tau-2\tau\cdot\phi\rangle dv \tag{3.8}$$

From (3.6), (3.7) and (3.8), we get

$$(d\mathcal{E}_w/dt)(0)=\frac{1}{8}\int_M \langle k, \tau\cdot\phi\rangle dv. \tag{3.9}$$

If $\tau=0$, then g is a critical point of $\mathcal{E}_w$. Conversely assume that g is a critical point of $\mathcal{E}_w$, defining $k=\tau\cdot\phi$, $g(t)=ge^{tk*}$ is a smooth curve in $\mathcal{M}(\omega)$ (see [6] p. 304 or [15]) with $g(0)=g$. Applying (3.9) to this deformation we have $\tau=0$. Therefore we obtain the following.

THEOREM 3.3 (Chern-Hamilton). *Let (M, ω) be a compact contact three-manifold. Then a metric g in $\mathcal{M}(\omega)$ is a critical point of $\mathcal{E}_w$ if and only if the characteristic vector field X_0 is of Killing with respect to g.*

This Theorem was obtained in [7] (see Theorem 5.2) where $\mathcal{E}_w$ was studied as a functional on $\mathcal{M}(\omega)$ regarded as the set of "CR-structures" on M.

Examples of critical metrics. Let (ω, g) be a contact Riemannian structure on a compact three-manifold M.

(i) Note that: M is K-contact if and only if g is a critical metric for $\mathcal{F}$ and $\mathcal{E}_C$.

(ii) If g is ω-Einstein (in particular if g is of constant sectional curvature), then g is critical for $\mathcal{F}$ (see Theorem 4.3). If g is of constant sectional curvature $K=0$, then it is not K-contact and so this metric is critical for $\mathcal{F}$ but not for $\mathcal{E}_C$.

(iii) If g is locally symmetric, then g is a critical metric for $\mathcal{F}$. In fact, since g is locally symmetric, by Lemma 1 of [5] we have $\nabla_{X_0}h=0$, and so by formula (3.2) we get $\nabla_{X_0}\tau=0$.

(iv) The natural contact Riemannian structure of the tangent sphere bundle of a compact Riemannian 2-manifold with constant curvature -1 is critical for $\mathcal{E}_C$ but not for $\mathcal{F}$ (combine (i), Theorem of [4] and a result of Tashiro [3] p. 136).

(v) Let N be a compact orientable surface of constant negative curvature -1. Let (θ^1, θ^2) be an orthonormal coframe and Ω_2^1 the connection 1-form. Chern and Hamilton [7] defined on the unit tangent bundle T_1N a contact Riemannian structure (ω, g') by

$$\omega=\frac{1}{2}\Omega_2^1 \quad \text{and} \quad g'=\frac{1}{4}\{\theta^1\otimes\theta^1+\theta^2\otimes\theta^2+4\omega\otimes\omega\}.$$

It is not difficult to see that the Ricci curvature in the direction of X_0 and the scalar curvature of (T_1N, ω, g') are given by

$$S(X_0, X_0)=2 \quad \text{and} \quad r=\text{const.}=-10.$$

Recall that a contact Riemannian three-manifold is K-contact iff $S(X_0, X_0)=2$ (see [3] p. 65). Also recall the result of Tanno [14] that a locally symmetric K-contact manifold is of constant curvature. So the metric g' is critical for $\mathcal{F}$ and $\mathcal{E}_C$ but is not locally symmetric.

4. Contact ω-Einstein spaces of dimension three. Let M be a (2n+1)-dimensional manifold with contact metric structure (ω, g, X_0, ϕ). M is said to be *ω-Einstein* if the Ricci tensor S is of the form

$$S=ag+b\omega\otimes\omega \tag{4.1}$$

where a and b are functions on M. It is known that if M is a K-contact ω-Einstein $(2n+1)$-manifold, with $n>1$, then the functions a and b are constant. Moreover every K-contact three-manifold is ω-Einstein and the Ricci tensor is given by

$$S=\left(\frac{r}{2}-1\right)g+\left(-\frac{r}{2}+3\right)\omega\otimes\omega.$$

However, we know nothing about the geometry of contact ω-Einstein three-manifolds. Note that the connected sum of two non-simply connected closed three-manifolds has never K-contact structure (see [13]), while every compact orientable three-manifold has a contact structure (see [12]). In this section we give a characterization of contact ω-Einstein three-manifolds in terms of critical metrics of $\mathcal{F}$. Moreover we give some properties of a tensor S_1 which measures the deviation from the ω-Einstein structure.

PROPOSITION 4.1. *Let M be a $(2n+1)$-dimensional manifold with contact metric structure (ω, g, X_0, ϕ). If M is ω-Einstein, then the Ricci tensor is given by*

$$S=\left\{\frac{r}{2n}-\left(1-\frac{c^2}{4n}\right)\right\}g+\left\{-\frac{r}{2n}+(2n+1)\left(1-\frac{c^2}{4n}\right)\right\}\omega\otimes\omega \tag{4.2}$$

If, in addition, $n=1$, then the curvature tensor is given by

$$\begin{aligned}R(X, Y)Z=&\left\{\frac{r}{2}-2\left(1-\frac{c^2}{4}\right)\right\}\cdot\{g(Y, Z)X-g(X, Z)Y\}+\left\{3\left(1-\frac{c^2}{4}\right)-\frac{r}{2}\right\}\cdot\\ &\cdot\{g(Y, Z)\omega(X)X_0+\omega(Y)\omega(Z)X-g(X, Z)\omega(Y)X_0-\omega(X)\omega(Z)Y\}.\end{aligned} \tag{4.3}$$

Proof. Let $(X_0, E_\iota, \phi E_\iota)$ be an orthonormal ϕ-basis. From (4.1), $S(X_0, X_0)=a+b$. Besides (see [3])

$$\begin{aligned}S(X_0, X_0)&=2n-\mathrm{trace}\left(\frac{1}{2}L_{X_0}\phi\right)^2=2n-\frac{1}{4}|L_{X_0}\phi|^2\\ &=2n-\frac{1}{4}|L_{X_0}g|^2=2n\left(1-\frac{c^2}{4n}\right).\end{aligned}$$

Consequently

$$a+b=2n-c^2/2. \tag{4.4}$$

Moreover (4.1) implies

$$r=S(X_0, X_0)+2\sum_1^n S(E_\iota, E_\iota)=(2n+1)a+b. \tag{4.5}$$

From (4.4) and (4.5) we get

$$a=\frac{r}{2n}-1+\frac{c^2}{4n} \quad\text{and}\quad b=-\frac{r}{2n}+(2n+1)\left(1-\frac{c^2}{4n}\right).$$

Finally, when $n=1$, the curvature tensor is given by

$$\begin{aligned}R(X, Y)Z=&S(Y, Z)X+g(Y, Z)Q(X)-S(X, Z)Y-g(X, Z)Q(Y)\\ &-\frac{r}{2}\{g(Y, Z)X-g(X, Z)Y\}\end{aligned} \tag{4.6}$$

where $Q=S^*=(S^i_j)$ is the Ricci curvature operator. So (4.3) follows from (4.2) and (4.6).

Remark 4.1. If M is an Einstein contact manifold, by (4.2) the scalar curvature $r\leqq 2n(2n+1)$ where the equality holds if and only if M is a K-contact Einstein manifold.

PROPOSITION 4.2. *Let M be a three-manifold with contact metric structure (ω, g, X_0, ϕ). Let S_1 be the tensor defined by*

$$S_1 = S - ag - b\omega \otimes \omega, \tag{4.7}$$

where

$$a = \left(\frac{r}{2} - 1 + \frac{1}{4}c^2\right) \quad and \quad b = \left(-\frac{r}{2} + 3 - \frac{3}{4}c^2\right).$$

Then

(j) $|S_1|^2 = 2|\sigma|^2 + \frac{1}{4}|\nabla_{X_0}\tau|^2$ *where* $\sigma = S(X_0, \cdot)_{|B}$;

(jj) $\langle S_1, \tau\rangle = \langle S, \tau\rangle = -\frac{1}{2}\langle \nabla_{X_0}\tau, \tau\rangle$;

(jjj) *if* $\nabla_{X_0}\tau = 0$ *or* $\nabla_{X_0}\tau = 2\tau\cdot\phi$ *holds, then* S *and* S_1 *are perpendicular to* τ;

(jv) $\langle S_1, \nabla_{X_0}\tau\rangle = \langle S, \nabla_{X_0}\tau\rangle = -\frac{1}{2}|\nabla_{X_0}\tau|^2$.

Proof. (j) By a direct computation we get

$$|S_1|^2 = |S|^2 + 3a^2 + b^2 - 2ar - 4b\left(1 - \frac{1}{4}c^2\right) + 2ab$$

and hence

$$|S_1|^2 = |S|^2 - \frac{1}{2}\left(r - 2 + \frac{1}{2}c^2\right)^2 - 4\left(1 - \frac{1}{4}c^2\right)^2. \tag{4.8}$$

If $(E, \phi E, X_0)$ is an arbitrary ϕ-basis, from (3.4) we get

$$S(\phi E, \phi E) - S(E, E) = (\nabla_{X_0}\tau)(E, E). \tag{4.9}$$

From (3.5), by putting $E' = \phi E$, we obtain

$$S(E, \phi E) = -\frac{1}{2}(\nabla_{X_0}\tau)(E, \phi E) \tag{4.10}$$

Moreover the scalar curvature is given by

$$r = S(E, E) + S(\phi E, \phi E) + S(X_0, X_0) = 2S(E, E) + (\nabla_{X_0}\tau)(E, E) + S(X_0, X_0),$$

from which

$$S(E, E) = \frac{r}{2} - \left(1 - \frac{1}{4}c^2\right) - \frac{1}{2}(\nabla_{X_0}\tau)(E, E)$$

$$S(\phi E, \phi E) = \frac{r}{2} - \left(1 - \frac{1}{4}c^2\right) + \frac{1}{2}(\nabla_{X_0}\tau)(E, E).$$

It follows that

$$|S|^2=S(X_0, X_0)^2+S(E, E)^2+S(\phi E, \phi E)^2+2S(E, \phi E)^2+2S(X_0, E)^2+2S(X_0, \phi E)^2$$
$$=4\left(1-\frac{1}{4}c^2\right)^2+\frac{1}{2}\left(r-2+\frac{1}{2}c^2\right)^2+\frac{1}{4}|\nabla_{X_0}\tau|^2+2|\sigma|^2.$$

So, by (4.8) we obtain (j).

(jj) Let $(X_0, E, \phi E)$ be an arbitrary ϕ-basis. Using (2.1), (2.2), (4.7), (4.9) and (4.10) we get

$$\langle S_1, \tau\rangle=\langle S, \tau\rangle-a\langle g, \tau\rangle-b\langle\omega\otimes\omega, \tau\rangle=\langle S, \tau\rangle$$
$$=S(E, E)\tau(E, E)+2S(E, \phi E)\tau(E, \phi E)+S(\phi E, \phi E)\tau(\phi E, \phi E)$$
$$=-\tau(E, E)\{S(\phi E, \phi E)-S(E, E)\}+2S(E, \phi E)\tau(E, \phi E)$$
$$=-\tau(E, E)(\nabla_{X_0}\tau)(E, E)-\tau(E, \phi E)(\nabla_{X_0}\tau)(E, \phi E)=-\frac{1}{2}\langle\nabla_{X_0}\tau, \tau\rangle.$$

(jjj) is a consequence of (jj) and (2.2).

(jv) is obtained like (jj) by using (i)-(iii) of Proposition 3.1.

Combining Theorem 3.2, Proposition 4.1 and (j) of Proposition 4.2 we obtain the following result.

THEOREM 4.3. *Let M be a compact three-manifold with contact metric structure (ω, g). Then g is ω-Einstein if, and only if, g is a critical point of $\mathcal{F}$ and* $\sigma=0$.

Remark 4.2. The condition $\sigma=0$ means (see [9] p. 372) that in the space $\mathcal{R}$ of all Riemannian metrics on M, the tangent vector $\psi\in T_g(\mathcal{R})$, $\psi(X, Y)=-\tau(X, \phi Y)$, is perpendicular to the orbit of g under the group of diffeomorphisms of M.

THEOREM 4.4. *Let (M, ω, g) be a contact Riemannian three-manifold. Then $\sigma=0$ if and only if $S_1=-\frac{1}{2}\nabla_{X_0}\tau$, that is, the Ricci tensor is given by*

$$S=-\frac{1}{2}\nabla_{X_0}\tau+\left(\frac{r}{2}-1+\frac{1}{4}c^2\right)g+\left(-\frac{r}{2}+3-\frac{3}{4}c^2\right)\omega\otimes\omega. \tag{4.11}$$

Proof. Let T be the tensor defined by

$$T=S_1+\frac{1}{2}\nabla_{X_0}\tau.$$

Then

$$|T|^2=|S_1|^2+\frac{1}{4}|\nabla_{X_0}\tau|^2+\langle\nabla_{X_0}\tau, S_1\rangle,$$

and hence (j) and (jv) of Proposition 4.2 imply

$$|T|^2=2|\sigma|^2.$$

So Theorem 4.4 follows from (4.7).

THEOREM 4.5. *Let* (M, ω, g) *be a compact contact Riemannian three-manifold. Then* g *is a critical metric for* $\mathcal{E}_C$ *and* $\sigma=0$ *if and only if* $S_1=\psi$, *that is, the Ricci tensor is given by*

$$S=\psi+\left(\frac{r}{2}-1+\frac{1}{4}c^2\right)g+\left(-\frac{r}{2}+3-\frac{3}{4}c^2\right)\omega\otimes\omega. \tag{4.12}$$

Proof. If g is a critical metric for $\mathcal{E}_C$ and $\sigma=0$, from Theorem 4.4 above and Theorem 5.1 of [15] we get (4.12). Conversely assume (4.12), since $\psi(X_0, \cdot)=0$, we have $\sigma=0$. So (4.11), (4.12) and Theorem 5.1 of [15] imply that g is critical for $\mathcal{E}_C$.

5. Curvature of K-contact three-manifolds. In [10] the following was proved.

THEOREM 5.1. *Let* M *be a compact three-manifold with K-contact metric structure* (ω, g). *If the scalar curvature* $r>-2$ *or the Webster curvature* $W>0$, *then* M *admits a K-contact metric structure* $(\tilde{\omega}=a\omega, \tilde{g}=ag+(a^2-a)\omega\otimes\omega)$ *of positive sectional curvature for some* a, $0<a\leqq 1$.

This result is relative to the question posed by S. S. Chern (cf. appendix of [7]) of determining those compact three-manifolds admitting a contact metric structure (ω, g) for which the torsion invariant $|\tau|$ is identically zero (i.e. the contact metric structure is K-contact). In this section we extend Theorem 5.1; precisely we give the following.

THEOREM 5.2. *Let* M *be a compact three-manifold with K-contact metric structure* (ω, g). *If one of the following four conditions holds:*
(a) $W>0$, (b) $r>-2$, (c) $S+2g>0$, (d) *the* ϕ*-sectional curvature* $H>-3$,
then M *admits a K-contact metric structure* $(\tilde{\omega}, \tilde{g})$ *of positive sectional curvature.*

This Theorem is a consequence of Theorem 5.1 and of the following Proposition.

PROPOSITON 5.3. *Let* M *be a contact* ω*-Einstein three-manifold with contact metric structure* (ω, g, X_0, ϕ). *Then, if* $c<2$, *the following five conditions are equivalent:*
(a) $W>c^2/8$, (b) $r+2>c^2/2$, (c) $S+2g>(c^2/2)g$,
(d) *the sectional curvature* $K>-3(1-c^2/4)$,
(e) *the* ϕ*-sectional curvature* $H>-3(1-c^2/4)$.

Proof. Since $8W=r+2+c^2/2$, (a) and (b) are equivalent. If X is vertical, that is, if $X=tX_0$, then

$$\{S+2(1-c^2/4)g\}(X, X)=4t^2(1-c^2/4)>0.$$

If X is horizontal, that is, if $\omega(X)=0$, then by (4.2)

$$\{S+2(1-c^2/4)g\}(X, X)=(r/2+1-c^2/4)g(X, X)>0.$$

On the other hand $S(X_0, \cdot)_{|B}=\sigma=0$, so (b) and (c) are equivalent. For each point $x\in M$, we consider an arbitrary plane P of $T_x(M)$ and an orthonormal basis (X, Y) of P with $Y=P\cap B$. Then, by (4.3), the sectional curvature $K(P)$ at x is given by

$$K(P)=g(R(X, Y)Y, X)=(r/2-2+c^2/2)+(-r/2+3-3c^2/4)g(X_0, X)^2$$
$$=(r/2-2+c^2/2)\sin^2(X, X_0)+(1-c^2/4)\cos^2(X, X_0)$$

and hence

(5.1) $$K(P)=(r/2-2+c^2/2)\cos^2\alpha+(1-c^2/4)\sin^2\alpha$$

where α is the angle between P and B. By (5.1) we get that (b) implies (d). The converse is trivial. Moreover the scalar curvature at x is given by

$$r=\operatorname{trace} S=2S(X_0, X_0)+2g(R(E, \phi E)\phi E, E),$$

where E is an unit vector of B, that is

(5.2) $$r=4(1-c^2/4)+2H.$$

Therefore (b) and (e) are equivalent.

Remark 5.1. (i) The main result of [7] says that every contact structure on a compact orientable three-manifold has a contact metric whose Webster curvature W is either >0 or $W=\text{const.}\leqq 0$.

(ii) Hamilton [11] showed that a metric g of positive Ricci curvature on a compact three-manifold can be deformed to a metric of (positive) constant curvature. Hence in Theorem 5.2 if, in addition, M is simply-connected, then it is diffeomorphic with the three-sphere. This extends Corollary of [8] (cf. p. 654).

(iii) The formula (5.2) holds for every metric $g\in\mathcal{M}(\omega)$. So the conditions on the scalar curvature given in [10] can be replaced by conditions on the ϕ-sectional curvature H.

(iv) Let (M, g) be a compact Riemannian manifold and $\mathcal{S}^2$ the space of all symmetric tensor fields of type (0, 2). Berger and Ebin (cf. [1] § 6) introduced a zero-order differential operator $\mathcal{K}:\mathcal{S}^2\to\mathcal{S}^2$ which is related to the rough Laplacian and to the Lichnerowicz operator. They proved that the operator $\mathcal{K}$ is positive definite on $TZ=\{D\in\mathcal{S}^2 : \operatorname{trace} D=0\}$ if (M, g) is of strictly positive sectional curvature. Now observe that if (M, ω, g) is a compact three-manifold as in Theorem 5.2, then M admits a contact metric structure $(\tilde{\omega}, \tilde{g})$ for which the corresponding operator $\tilde{\mathcal{K}}$ is positive definite on $T_{\tilde{g}}(\mathcal{N}(\tilde{\omega}))=TZ$, where $\mathcal{N}(\tilde{\omega})$ is the set of all Riemannian metrics on M which have the same volume element of the metrics of $\mathcal{M}(\tilde{\omega})$.

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