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A CERTAIN PROPERTY OF GEODESICS OF THE FAMILY OF RIEMANNIAN MANIFOLDS $O_n^2(X)$

Dedicated to Professor Shiing-Shen Chern on his 77th birthday

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§0. **Introduction.** This is a continuation of Part (IX) ([22]) with the same title by the present author which proved the following conjecture is true for $2.4 \leqq n \leqq 4.5$ and exactly the final one of the series (I)-(X). He will show that this conjecture is also true for $2 \leqq n \leqq 2.4$ in the present paper by developing a new method which is applicable for all values of $n \geqq 2.4$. As stated at the end of the previous one, the principle used until now could not hold for $2 \leqq n \leqq 2.38$. We shall also use the same notation in the previous papers (I)-(IX). Any geodesic of the 2-dimensional Riemannian manifolds O_n^2 defined on the unit disk $u^2+v^2<1$ of the uv-plane by the metric:

$$ds^2=(1-u^2-v^2)^{n-2}\{(1-v^2)du^2+2uvdudv+(1-u^2)dv^2\}$$

has the support function $x(t)$ which is a solution of the non linear differential equation of order 2([6]):

$$\text{(E)} \qquad nx(1-x^2)\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+(1-x^2)(nx^2-1)=0$$

When the parameter $n>1$, any non-constant solution $x(t)$ of (E) such that $x^2+x'^2<1$ is periodic and its period T is given by the improper integral ([10]):

$$(0.1) \qquad T=\sqrt{nc}\int_{x_0}^{x_1}\frac{dx}{x\sqrt{(n-x)\{x(n-x)^{n-1}-c\}}},$$

where $x_0=n\{\min x(t)\}^2$, $x_1=n\{\max x(t)\}^2$, $0<x_0<1<x_1<n$ and $c=x_0(n-x_0)^{n-1}=x_1(n-x_1)^{n-1}$.

CONJECTURE C. *The period T as a function of $\tau=(x_1-1)/(n-1)$ and n is monotone decreasing with respect to $n(>2)$ for any fixed $\tau(0<\tau<1)$.*

§1. Preliminaries.

Setting $T=\Omega(\tau, n)$, we have the formulas

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(1.1) $$\frac{\partial\Omega(\tau, n)}{\partial n}=-\frac{\sqrt{c}}{2(B-c)n\sqrt{n}}\int_{x_0}^{1}\frac{(1-x)\sqrt{x(n-x)^{n-1}-c}}{x^2(n-x)^n}V(x, x_1)dx$$

((7.4) and Proposition 3 in (III) and (1.12), (1.16) in (IX)), where $B=(n-1)^{n-1}$ and

(1.2) $$V(x, x_1)=\frac{x^2\sqrt{n-x}F_2(x)}{(1-x)^5}\{\tilde{\lambda}(x)-\tilde{\lambda}(x_1)\}$$

$$+\frac{X^2\sqrt{n-X}F_2(X)}{(X-1)^5}\{\tilde{\lambda}(X)-\tilde{\lambda}(x_1)\}-\eta(x)+\eta(X),$$

(1.3) $$F_2(z)=-BP_2(z)+(n-z)^{n-1}P_3(z),$$

(1.4) $$P_2(z)=(2n+1)z^2-2(2n^2+5n-4)z+(4n-1)(4n-3),$$

(1.5) $$P_3(z)=-(n-1)z^3+(2n^2-7n+8)z^2+(n-3)(4n-1)z+3n(2n-1),$$

(1.6) $$\eta(z)=\frac{z\sqrt{n-z}}{(n-1)(z-1)^3}\{-B\tilde{Q}_1(z)+(n-z)^{n-1}\tilde{Q}_2(z)\},$$

(1.7) $$\tilde{Q}_1(z)=-(4n^2+2n-3)z+(4n-1)(4n-3),$$

(1.8) $$\tilde{Q}_2(z)=(n-1)(2n-3)z^2+(4n^2-10n+3)z+3n(2n-1),$$

(1.9) $$\tilde{\lambda}(z)=\log(n-z)+\frac{nz-1}{(n-1)z}, \quad \phi(z)=z(n-z)^{n-1},$$

and $X=X_n(x)$, $0<x<1<X<n$, defined by $\phi(x)=\phi(X)$.

By Lemma 8.1 in (III), we know the following:

FACT 1. *For $0<x<1$, $V(x, x_1)$ is increasing with respect to x_1 in $X_n(x)\leqq x_1<n$;*

FACT 2. *We have*

(1.10) $$\frac{1}{B}V(x, X_n(x))=\frac{n(4n^2-10n+3)}{(n-1)^3}\sqrt{n-X_n(x)}\{1+O(n-X_n(x))\}$$

near $x=0$, which implies $V(x, X_n(x))$ is negative near $x=0$, when $2<n<(5+\sqrt{13})/4=2.151387819\cdots$ ((8.10) in III));

FACT 3. *We have*

(1.11) $$\frac{1}{B}V(x, X_n(x))=\frac{n(2n-1)(n^2-n-3)}{6(n-1)^2\sqrt{n-1}}(1-x)\{1+O(1-x)\}$$

near $x=1$, which implies $V(x, X_n(x))$ is negative near $x=1$, when $2<n<(1+\sqrt{13})/2=2.302775638\cdots$ ((8.22) in (III)).

We have shown that Conjecture C is true for $2.4\leqq n<\infty$ by proving that

$V(x, X_n(x))>0$ for $0<x<1$ by Fact 1 in (I)-(IX). Now, we show that it is still alive for $2\leqq n<2.4$ by the following

PROPOSITION 1.

$$\left.\frac{\partial\Omega(\tau, n)}{\partial n}\right|_{n=2}<0 \qquad \textit{for } 0<x<1.$$

Proof. By (1.3)-(1.9) we have $B=1$ and

$$P_2(z)=5z^2-28z+35, \qquad P_3(z)=-z^3+2z^2-7z+18,$$

$$F_2(z)=-P_2(z)+(2-z)P_3(z)=(z-1)^4,$$

$$\tilde{Q}_1(z)=-17z+35, \qquad \tilde{Q}_2(z)=z^2-z+18,$$

$$-\tilde{Q}_1(z)+(2-z)\tilde{Q}_2(z)=-(z-1)^3, \qquad \eta(z)=-z\sqrt{2-z},$$

$$\tilde{\lambda}(z)-\tilde{\lambda}(x_1)=\log\frac{2-z}{2-x_1}-\frac{1}{z}+\frac{1}{x_1}$$

and

$$V(x, x_1)=\frac{x^2\sqrt{2-x}}{1-x}\left\{\log\frac{2-x}{2-x_1}-\frac{1}{x}+\frac{1}{x_1}\right\}$$
$$+\frac{X^2\sqrt{2-X}}{X-1}\left\{\log\frac{2-X}{2-x_1}-\frac{1}{X}+\frac{1}{x_1}\right\}+x\sqrt{2-x}-X\sqrt{2-X}.$$

Since we have $X=2-x$, $x_1=2-x_0$, we have

$$V(x, x_1)=\frac{x^2\sqrt{2-x}}{1-x}\left\{\log\frac{2-x}{x_0}-\frac{1}{x}+\frac{1}{2-x_0}\right\}$$
$$+\frac{(2-x)^2\sqrt{x}}{1-x}\left\{\log\frac{x}{x_0}-\frac{1}{2-x}+\frac{1}{2-x_0}\right\}+x\sqrt{2-x}-(2-x)\sqrt{x}$$
$$=\frac{\sqrt{x(2-x)}}{1-x}\left[x^{3/2}\left(\log\frac{2-x}{x_0}-\frac{1-x_0}{2-x_0}\right)+(2-x)^{3/2}\left(\log\frac{x}{x_0}-\frac{1-x_0}{2-x_0}\right)\right]$$

and hence

$$\left.\frac{\partial\Omega(\tau, n)}{\partial n}\right|_{n=2}=-\frac{\sqrt{c}}{4\sqrt{2}(1-c)}\int_{x_0}^{1}\frac{\sqrt{x(2-x)-c}}{(x(2-x))^{3/2}} \tag{1.12}$$
$$\times\left[x^{3/2}\left(\log\frac{2-x}{x_0}-\frac{1-x_0}{2-x_0}\right)+(2-x)^{3/2}\left(\log\frac{x}{x_0}-\frac{1-x_0}{2-x_0}\right)\right]dx$$
$$=-\frac{\sqrt{c}}{4\sqrt{2}(1-c)}\int_{x_0}^{2-x_0}\frac{\sqrt{x(2-x)-c}}{(x(2-x))^{3/2}}x^{3/2}\left(\log\frac{2-x}{x_0}-\frac{1-x_0}{2-x_0}\right)dx.$$

Putting $x_0=1-a$, and $x=1+t$, we obtain

(1.13) $$I(a)=\int_{x_0}^{2-x_0}\frac{\sqrt{x(2-x)-c}}{(x(2-x))^{3/2}}x^{3/2}\left(\log\frac{2-x}{x_0}-\frac{1-x_0}{2-x_0}\right)dx$$
$$=\int_{-a}^{a}\frac{\sqrt{a^2-t^2}}{(1-t)^{3/2}}\left(\log\frac{1-t}{1-a}-\frac{a}{1+a}\right)dt$$

where $0<a<1$. It is sufficient to prove $I(a)>0$.

Since we have

$$\left\{\frac{2}{\sqrt{1-t}}\left(\log\frac{1-t}{1-a}+\frac{2+a}{1+a}\right)\right\}'=\frac{1}{(1-t)^{3/2}}\left(\log\frac{1-t}{1-a}-\frac{a}{1+a}\right),$$

we obtain from (1.13)

(1.14) $$I(a)=2\int_{-a}^{a}\frac{t}{\sqrt{(1-t)(a^2-t^2)}}\left(\log\frac{1-t}{1-a}+\frac{2+a}{1+a}\right)dt$$
$$=2\int_{0}^{a}\frac{t}{\sqrt{a^2-t^2}}\left\{\frac{1}{\sqrt{1-t}}\left(\log\frac{1-t}{1-a}+\frac{2+a}{1+a}\right)\right.$$
$$\left.-\frac{1}{\sqrt{1+t}}\left(\log\frac{1+t}{1-a}+\frac{2+a}{1+a}\right)\right\}dt$$

by the integration by parts.

Setting for a fixed a $(0<a<1)$

(1.15) $$f(t):=\frac{1}{1-t}\left(\log\frac{1-t}{1-a}+\frac{2+a}{1+a}\right)^2 \qquad \text{for } -a\leqq t\leqq a\,,$$

we prove that

(1.16) $$f(t)>f(-t) \qquad \text{for small } t>0\,.$$

Since we have

$$f'(t)=\frac{1}{(1-t)^2}\left(\log\frac{1-t}{1-a}+\frac{2+a}{1+a}\right)\left(\log\frac{1-t}{1-a}-\frac{a}{1+a}\right)$$

and

$$f'(0)=\left(\log\frac{1}{1-a}+\frac{2+a}{1+a}\right)\left(\log\frac{1}{1-a}-\frac{a}{1+a}\right)>0$$

which implies (1.16).

Next we prove that

(1.17) $$f(a)>f(-a)\,.$$

Since we have

$$f(a)=\frac{1}{1-a}\left(\frac{2+a}{1+a}\right)^2 \quad\text{and}\quad f(-a)=\frac{1}{1+a}\left(\log\frac{1+a}{1-a}+\frac{2+a}{1+a}\right)^2$$

and so $f(a)>f(-a)$ is equivalent to

$$\frac{2+a}{1+a}\left(\sqrt{\frac{1+a}{1-a}}-1\right)>\log\frac{1+a}{1-a} \qquad \text{for } 0<a<1 .$$

Setting $b=(1+a)/(1-a)$ and

$$g(b):=\frac{2+a}{1+a}\left(\sqrt{\frac{1+a}{1-a}}-1\right)-\log\frac{1+a}{1-a}=\frac{3b+1}{2b}(\sqrt{b}-1)-\log b ,$$

we have $b>1$ for $0<a<1$ and

$$g(1)=0 ,$$

$$g'(b)=\frac{1}{4b^2}(\sqrt{b}-1)^2(3\sqrt{b}+2)>0 \qquad \text{for } b>1 ,$$

which implies (1.17).

Last, we prove

$$f(t)>f(-t) \qquad \text{for } 0<t<a . \tag{1.18}$$

If this inequality does not hold, there exist $0<t_1\leqq t_2<a$ such that

$$f(t_1)=f(-t_1) \quad \text{and} \quad f'(t_1)+f'(-t_1)\leqq 0$$

and

$$f(t_2)=f(-t_2) \quad \text{and} \quad f'(t_2)+f'(-t_2)\geqq 0 .$$

Since we obtain easily from (1.15)

$$f'(t)+f'(-t)=\frac{1}{(1-t)^2}\left(\log\frac{1-t}{1-a}\right)^2-\frac{2}{(1-t)^2}\left(\log\frac{1-t}{1-a}+\frac{2+a}{1+a}\right)$$
$$+\frac{1}{(1+t)^2}\left(\log\frac{1+t}{1-a}+\frac{2+a}{1+a}\right)^2-\frac{2}{(1+t)^2}\left(\log\frac{1+t}{1-a}+\frac{2+a}{1+a}\right)$$

which can be written at $t=t_1$ and t_2 as

$$=\frac{2h(t)}{(1-t)^2(1+t)}\left(\log\frac{1-t}{1-a}+\frac{2+a}{1+a}\right).$$

where

$$h(t):=\log\frac{1-t}{1-a}+\frac{2+a}{1+a}-\frac{(1+t)^{3/2}+(1-t)^{3/2}}{\sqrt{1+t}} .$$

Regarding $h(t)$, we have

$$h(0)=\log\frac{1}{1-a}-\frac{a}{1+a}>0$$

and

$$h'(t)=-\frac{2-t}{1-t}+\frac{3}{2}\sqrt{\frac{1-t}{1+t}}+\frac{1}{2}\left(\frac{1-t}{1+t}\right)^{3/2} ,$$

which can be written as

$$-\frac{1+3u}{2u}+\frac{3}{2}u^{1/2}+\frac{1}{2}u^{3/2}$$

$$=\frac{1}{2u}(u^{1/2}-1)(u^2+u^{3/2}+4u+u^{1/2}+1),$$

where $u=(1-t)/(1+t)$. Since $0<u\leqq 1$ for $0\leqq t<1$, we have $h'(t)<0$ for $0<t<1$. Hence $h(t)$ is decreasing in $0<t<a$ and

$$h(a)=\frac{2+a}{1+a}-\frac{(1+a)^{3/2}+(1-a)^{3/2}}{\sqrt{1+a}}:=G(a).$$

Regarding $G(a)$, we have

$$G(0)=0$$

and

$$G'(a)=-\frac{1}{(1+a)^2}-1+\frac{3}{2}\sqrt{\frac{1-a}{1+a}}+\frac{1}{2}\left(\frac{1-a}{1+a}\right)^{3/2}$$

$$=-\frac{1}{4}(1-b^{1/2})(5-b^{1/2}+b-b^{3/2}),$$

where $b=(1-a)/(1+a)$ and $0<b<1$ for $0<a<1$. Hence we obtain

$$G'(a)<0 \quad \text{and} \quad G(a)<0 \qquad \text{for } 0<a<1\,.$$

Thus we see that there exists a unique $t=\gamma(a)$ such that

$$h(t)>0 \qquad \text{for } 0\leqq t<\gamma(a)$$

$$h(t)<0 \qquad \text{for } \gamma(a)<t<a\,.$$

Since we have $h(t_1)\leqq 0$ and $h(t_2)\geqq 0$, it must be

$$t_1=t_2=\gamma(a)\,,$$

and hence there

$$f(t)=f(-t) \quad \text{and} \quad f'(t)+f'(-t)=0\,,$$

from which we obtain

$$\log\frac{1+t}{1-t}=\{(1+t)^{3/2}+(1-t)^{3/2}\}\left\{\frac{1}{\sqrt{1-t}}-\frac{1}{\sqrt{1+t}}\right\}$$

at $t=t_1=t_2$. Using $v=(1+t)/(1-t)=x^2$, we have

$$\{(1+t)^{3/2}+(1-t)^{3/2}\}\left\{\frac{1}{\sqrt{1-t}}-\frac{1}{\sqrt{1+t}}\right\}-\log\frac{1+t}{1-t}$$

$$=\frac{2}{v+1}(v^{3/2}+1)\left(1-\frac{1}{v^{1/2}}\right)-\log v$$

$$=2\left[\frac{(x-1)(x^3+1)}{x(x^2+1)}-\log x\right].$$

Since we have easily

$$\left(\frac{(x-1)(x^3+1)}{x(x^2+1)}-\log x\right)'=\frac{(x-1)^2}{x^2(x^2+1)^2}(x^4+x^3+4x^2+x+1)$$

and $1<x<\infty$ for $0<t<1$, it must be

$$\{(1+t)^{3/2}+(1-t)^{3/2}\}\left\{\frac{1}{\sqrt{1-t}}-\frac{1}{\sqrt{1+t}}\right\}>\log\frac{1+t}{1-t} \qquad \text{for } 0<t<1,$$

which implies a contradiction for t_1 and t_2. Thus we obtain (1.18), which implies $I(a)>0$. Q.E.D.

In the following, we shall use the methods appeared in the proof of Proposition 1.

§2. The fundamental principle.

By (7.10) in (III), we have

(2.1) $$V(x, x_1)=\frac{x^2N(x, x_1)}{(1-x)^5\sqrt{n-x}}+\frac{X^2N(X, x_1)}{(X-1)^5\sqrt{n-X}},$$

where

(2.2) $$N(z, x_1):=(n-z)F_2(z)\{\lambda(z)-\tilde{\lambda}(x_1)\}+3(z-1)^2f_0(z)$$
$$-2n(z-1)^3\{B-\phi(z)\},$$

(2.3) $$f_0(z):=(2n-1-z)B-(n-z)^{n-1}\{n-z+(n-1)z^2\},$$

(2.4) $$\lambda(z):=\log(n-z)+\frac{n-1}{n-z}$$

and $X=X_n(x)$ for $0\leqq x\leqq 1$. Since we have

$$\phi'(x)=n(1-x)(n-x)^{n-2} \quad \text{and} \quad \frac{1-x}{x(n-x)}dx=-\frac{X-1}{X(n-X)}dX,$$

we obtain

(2.5) $$I=\int_{x_0}^{1}\frac{(1-x)\sqrt{\phi(x)-c}}{x^2(n-x)^n}V(x, x_1)dx$$
$$=\int_{x_0}^{x_1}\frac{\sqrt{\phi(x)-c}N(x, x_1)}{(x-1)^4(n-x)^{n+1/2}}dx.$$

Regarding $N(x, x_1)$, we know that $F_2(z)$, $f_0(z)$ and $B-\phi(z)$ are of order 4, 3 and 2 in $z-1$ at $z=1$ by (8.14) and (8.17) in (III) and Lemma 4.1 in (I), respectively, and hence the integrand of the right hand side of (2.5) is real analytic at $x=1$. Therefore we set

$$M(x, x_1):=\int_1^x \frac{N(t, x_1)dt}{(t-1)^4(n-t)^{n+1/2}} \tag{2.6}$$

and hence by the integration by parts we have

$$\begin{aligned} I&=[\sqrt{\phi(x)-c}M(x, x_1)]_{x_0}^{x_1}-\frac{n}{2}\int_{x_0}^{x_1}\frac{(1-x)(n-x)^{n-2}}{\sqrt{\phi(x)-c}}M(x, x_1)dx \\ &=-\frac{n}{2}\int_{x_0}^{x_1}\frac{(1-x)(n-x)^{n-2}}{\sqrt{\phi(x)-c}}M(x, x_1)dx . \end{aligned}$$

Since we have

$$\begin{aligned} &\int_1^{x_1}\frac{(1-x)(n-x)^{n-2}}{\sqrt{\phi(x)-c}}M(x, x_1)dx \\ &=\int_1^{x_1}\frac{(1-X)(n-X)^{n-2}}{\sqrt{\phi(X)-c}}M(X, x_1)dX \\ &=-\int_1^{x_0}\frac{(1-X)(n-X)^{n-2}}{\sqrt{\phi(x)-c}}M(X, x_1)\cdot\frac{X(n-X)}{X-1}\cdot\frac{1-x}{x(n-x)}dx \\ &=-\int_{x_0}^1\frac{(1-x)(n-x)^{n-2}}{\sqrt{\phi(x)-c}}M(X, x_1)dx , \end{aligned}$$

we obtain

$$I=\frac{n}{2}\int_{x_0}^1\frac{(1-x)(n-x)^{n-2}}{\sqrt{\phi(x)-c}}\{M(X, x_1)-M(x, x_1)\}dx . \tag{2.7}$$

From this formula, we see that if we can prove

$$M(X_n(x), x_1)-M(x, x_1)>0 \qquad \text{for } x_0\leqq x<1 , \tag{\#}$$

then we obtain $I>0$. When $n=2$, we can easily check that

$$\begin{aligned} M(X, x_1)-M(x, x_1)=&\frac{2}{\sqrt{1-t}}\left(\log\frac{1-t}{1-a}+\frac{2+a}{1+a}\right) \\ &-\frac{2}{\sqrt{1+t}}\left(\log\frac{1+t}{1-a}+\frac{2+a}{1+a}\right), \end{aligned}$$

where $a=1-x_0$, $t=1-x$, and so (#) corresponds to (1.18).

Now, we set

$$\rho(x, x_1):=M(X_n(x), x_1)-M(x, x_1)=\int_x^X\frac{N(t, x_1)dt}{(t-1)^4(n-t)^{n+1/2}} \tag{2.9}$$

where $X=X_n(x)$.

LEMMA 2.1. i) $\rho(1, x_1)=0$,

ii) $\rho(x, x_1)>0$ *and* $\searrow$ *for* $x<1$ *near* $x=1$.

Proof. i) is evident. By (8.14), (8.17) in (III), near $t=1$ we have

$$N(t, x_1)=(n-t)\{\lambda(t)-\tilde{\lambda}(x_1)\}$$
$$\times\left\{\frac{n(n^2-n+1)B}{6(n-1)}+O(1-t)\right\}(1-t)^4$$
$$-\left\{\frac{n(2n-1)B}{2}+O(1-t)\right\}(1-t)^5$$
$$+2n\left\{\frac{n(n-1)^{n-2}}{2}+O(1-t)\right\}(1-t)^5$$

and hence

$$\frac{N(t, x_1)}{(t-1)^4(n-t)^{n+1/2}}=\frac{n(n^2-n+1)B}{6(n-1)}\{\lambda(t)-\tilde{\lambda}(x_1)\}\frac{1}{(n-t)^{n-1/2}}+O(1-t)$$
$$=\frac{n(n^2-n+1)}{6(n-1)^{3/2}}\{\lambda(1)-\tilde{\lambda}(x_1)\}+O(1-t).$$

For sufficiently small $\varepsilon>0$, we obtain

$$\rho(1-\varepsilon, x_1)=\frac{n(n^2-n+1)}{6(n-1)^{3/2}}\{\lambda(1)-\tilde{\lambda}(x_1)\}\{X(1-\varepsilon)-(1-\varepsilon)\}$$
$$+\left[\frac{(1-t)^2}{2}\right]_{1-\varepsilon}^{X(1-\varepsilon)}\times O(1).$$

Since we have

$$X(1-\varepsilon)=1+\varepsilon+\frac{2(n-2)}{3(n-1)}\varepsilon^2+\cdots$$

by (8.12) in (III),

$$\rho(1-\varepsilon, x_1)=\frac{n(n^2-n+1)}{6(n-1)^{3/2}}\{\lambda(1)-\tilde{\lambda}(x_1)\}$$
$$\times 2\varepsilon\left(1+\frac{n-2}{3(n-1)}\varepsilon+\cdots\right)+\frac{1}{2}\varepsilon^3\left[\frac{4(n-2)}{3(n-1)}+\cdots\right]\times O(1),$$

which implies $\rho(1-\varepsilon, x_1)>0$ for sufficiently small $\varepsilon>0$ and $\rho(x, x_1)\searrow$ near $x=1$, because

$$\lambda(1)-\tilde{\lambda}(x_1)>0$$

by Lemma 7.1 in (III). Q. E. D.

This lemma corresponds to (1.16) for $n=2$.

Next, we get

$$(2.10) \qquad g(x_0) := \rho(x_0, x_1) = \int_{x_0}^{x_1} \frac{N(x, x_1)dx}{(x-1)^4(n-x)^{n+1/2}}$$

and want to show that $g(x_0)>0$ which corresponds to (1.17) for $n=2$. From the evaluation of $N(t, x_1)$ in the proof of Lemma 2.1, we can easily see that

$$(2.11) \qquad \lim_{x_0 \to 1} g(x_0) = 0 .$$

From (2.10), we have

$$\begin{aligned} g'(x_0) &= \frac{\partial x_1}{\partial x_0} \frac{N(x_1, x_1)}{(x_1-1)^4(n-x_1)^{n+1/2}} - \frac{N(x_0, x_1)}{(1-x_0)^4(n-x_0)^{n+1/2}} \\ &\quad + \frac{\partial x_1}{\partial x_0} \int_{x_0}^{x_1} \frac{1}{(x-1)^4(n-x)^{n+1/2}} \frac{\partial N(x, x_1)}{\partial x_1} dx \\ &= -\frac{1-x_0}{x_0(n-x_0)} \cdot \frac{x_1(n-x_1)}{x_1-1} \cdot \frac{N(x_1, x_1)}{(x_1-1)^4(n-x_1)^{n+1/2}} \\ &\quad - \frac{N(x_0, x_1)}{(1-x_0)^4(n-x_0)^{n+1/2}} - \frac{1-x_0}{x_0(n-x_0)} \cdot \frac{x_1(n-x_1)}{x_1-1} \\ &\quad \times \int_{x_0}^{x_1} \frac{(n-x)F_2(x)}{(x-1)^4(n-x)^{n+1/2}} \frac{(x_1-1)\{n+(n-1)x_1\}}{(n-1)x_1^2(n-x_1)} dx \end{aligned}$$

by means of

$$\frac{d\tilde{\lambda}(x)}{dx} = -\frac{(x-1)\{n+(n-1)x\}}{(n-1)x^2(n-x)} .$$

Since we have

$$\begin{aligned} &\frac{1-x_0}{x_0(n-x_0)} \cdot \frac{x_1(n-x_1)}{x_1-1} \cdot \frac{1}{(x_1-1)^4(n-x_1)^{n+1/2}} \\ &\quad = \frac{1-x_0}{x_0^2(n-x_0)^n} \cdot \frac{x_1^2}{(x_1-1)^5(n-x_1)^{1/2}} \end{aligned}$$

by $x_0(n-x_0)^{n-1} = x_1(n-x_1)^{n-1}$, we obtain

$$\begin{aligned} g'(x_0) &= -\frac{1-x_0}{x_0^2(n-x_0)^n} \left[\frac{x_1^2 N(x_1, x_1)}{(x_1-1)^5\sqrt{n-x_1}} + \frac{x_0^2 N(x_0, x_1)}{(1-x_0)^5\sqrt{n-x_0}} \right] \\ &\quad - \frac{1-x_0}{(n-1)x_0(n-x_0)} \frac{n+(n-1)x_1}{x_1} \int_{x_0}^{x_1} \frac{F_2(x)dx}{(x-1)^4(n-x)^{n-1/2}} \end{aligned}$$

i.e.

$$\begin{aligned} (2.12) \qquad g'(x_0) &= -\frac{1-x_0}{x_0^2(n-x_0)^n} V(x_0, x_1) \\ &\quad - \frac{1-x_0}{(n-1)x_0(n-x_0)} \frac{n+(n-1)x_1}{x_1} \int_{x_0}^{x_1} \frac{F_2(x)dx}{(x-1)^4(n-x)^{n-1/2}} . \end{aligned}$$

LEMMA 2.2. *When* $n \geqq 2.4$, *we have*

$$g'(x_0)<0 \qquad for\ 0<x_0<1.$$

Proof. Since we proved the inequality $V(x_0, x_1)>0$ for $n \geqq 84$ in (IV), for $16 \leqq n \leqq 84$ in (V), for $9.7 \leqq n \leqq 16$ in (VI), for $5 \leqq n \leqq 9.7$ in (VII), for $4.5 \leqq n \leqq 5$ in (VIII), for $2.4 \leqq n \leqq 4.5$ in (IX) and we have $F_2(x)>0$ for $0<x<n$, $x \neq 1$, with $n \geqq 2$ by Proposition 1 in (II), we obtain

$$g'(x_0)<0 \qquad \text{for } 0<x_0<1 \text{ with } n \geqq 2.4. \tag{2.13}$$

From (2.12), we see that $g'(x_0)<0$ is equivalent to

$$-V(x_0, x_1)<\frac{\{n+(n-1)x_1\}(n-x_1)^{n-1}}{n-1}\int_{x_0}^{x_1}\frac{F_2(x)dx}{(x-1)^4(n-x)^{n-1/2}}$$

and so

$$\eta(x_0, n)-\eta(x_1, n)<\frac{x_0^2\sqrt{n-x_0}F_2(x_0)}{(1-x_0)^5}\{\tilde{\lambda}(x_0)-\tilde{\lambda}(x_1)\}$$

$$+(n-x_1)^{n-1}\left(x_1+\frac{n}{n-1}\right)\int_{x_0}^{x_1}\frac{F_2(x)dx}{(x-1)^4(n-x)^{n-1/2}}.$$

By Lemma 7.1 in (III) and Lemma 2.1 in (IV), we have

$$\tilde{\lambda}(x_0)-\tilde{\lambda}(x_1)=\frac{1}{n-1}\int_{x_0}^{x_1}\frac{(x-1)\{n+(n-1)x\}dx}{x^2(n-x)}$$

and

$$\eta(x_1, n)-\eta(x_0, n)=\frac{1}{2(n-1)}\int_{x_0}^{x_1}\frac{\{B\tilde{Q}_3(x)-(n-x)^{n-1}\tilde{Q}_4(x)\}dx}{(x-1)^4\sqrt{n-x}},$$

where

$$\tilde{Q}_3(z)=(4n^2+2n-3)z^3-8(n-1)(n^2+5n-3)z^2 +3(16n^3-40n^2+24n-3)z+2n(4n-1)(4n-3), \tag{2.14}$$

$$\tilde{Q}_4(z)=(n-1)(2n-1)(2n-3)z^4+(2n-3)(2n^2-13n+8)z^3 +3(8n^3-26n^2+21n-3)z^2+n(28n^2-52n+15)z+6n^2(2n-1). \tag{2.15}$$

Using these in the above inequality and introducing

$$G(x, x_0, n):=\frac{B\tilde{Q}_3(x)-(n-x)^{n-1}\tilde{Q}_4(x)}{2(n-1)(x-1)^4\sqrt{n-x}} +\frac{x_0^2\sqrt{n-x_0}F_2(x_0)}{(1-x_0)^5}\frac{x-1}{x^2(n-x)}\left(x+\frac{n}{n-1}\right) +(n-x_1)^{n-1}\left(x_1+\frac{n}{n-1}\right)\frac{F_2(x)}{(x-1)^4(n-x)^{n-1/2}}, \tag{2.16}$$

$g'(x_0)<0$ is equivalent to

(2.17) $$J:=\int_{x_0}^{x_1} G(x,\ x_0,\ n)dx>0\ .$$

We can prove that

$$\int_{x_0}^{x_1} G(x,\ x_0,\ 2)dx>0 \qquad \text{for } 0<x_0<1\ .$$

Now looking over the expression of $G(x,\ x_0,\ n)$, we divide the above integral J into the three parts as follows:

$$J=J_1+J_2+J_3\ ,$$

(2.18) $$J_1=\frac{x_0^2\sqrt{n-x_0}F_2(x_0)}{(1-x_0)^5}\int_{x_0}^{x_1}\frac{x-1}{x^2(n-x)}\left(x+\frac{n}{n-1}\right)dx$$
$$=\frac{x_0^2\sqrt{n-x_0}F_2(x_0)}{(1-x_0)^5}\left\{\log\frac{n-x_0}{n-x_1}-\frac{1}{n-1}\left(\frac{1}{x_0}-\frac{1}{x_1}\right)\right\}\ .$$

(2.19) $$J_2=\frac{1}{2(n-1)}\int_{x_0}^{x_1}\frac{1}{(x-1)^4\sqrt{n-x}}\{B\tilde{Q}_3(x)-(n-x)^{n-1}\tilde{Q}_4(x)\}dx$$
$$=\eta(x_1,\ n)-\eta(x_0,\ n)\ ,$$

(2.20) $$J_3=(n-x_1)^{n-1}\left(x_1+\frac{n}{n-1}\right)\int_{x_0}^{x_1}\frac{F_2(x)dx}{(x-1)^4(n-x)^{n-1/2}}\ .$$

In order to make easy the handling of J_2 we use the parameter $y=x-1$, then we obtain by Lemma 2.2 in (IX)

(2.21) $$\tilde{Q}_3(1+y)=\sum_{i=1}^{3} a_i(n)y^i\ ,$$

where

$$a_0=72n^3-180n^2+144n-36=36(n-1)^2(2n-1)\ ,$$
$$a_1=32n^3-172n^2+206n-66=2(n-1)(16n^2-70n+33)\ ,$$
$$a_2=-8n^3-20n^2+70n-33\ ,\qquad a_3=4n^2+2n-3$$

and

(2.22) $$\tilde{Q}_4(1+y)=\sum_{i=0}^{4} b_i(n)y^i\ ,$$

where

$$b_0=72n^3-180n^2+144n-36=36(n-1)^2(2n-1)\ ,$$
$$b_1=104n^3-352n^2+350n-102=2(n-1)(52n^2-124n+51)\ ,$$
$$b_2=60n^3-246n^2+294n-99=3(20n^3-82n^2+98n-33)\ ,$$

$$b_3=20n^3-80n^2+99n-36\,,$$

$$b_4=4n^3-12n^2+11n-3=(n-1)(4n^2-8n+3)\,.$$

Analogously, we have for $P_2(x)$ and $P_3(x)$

$$\text{(2.23)}\quad \begin{cases} P_2(1+y)=12(n-1)^2-2(n-1)(2n+5)y+(2n+1)y^2\,, \\ P_3(1+y)=12(n-1)^2+2(n-1)(4n-11)y+(2n^2-10n+11)y^2-(n-1)y^3\,. \end{cases}$$

Regarding J_2 and J_3, we have the following Lemmas 2.3—Lemma 2.6, which will be proved in another paper ([24]).

LEMMA 2.3. *Regarding* $\tilde{Q}_3(1+y)$ *we have the following*:

i) $\tilde{Q}_3(1+y)$ *is* $\nearrow\searrow$ *in* $-1<y<0$ *and* $\searrow$ *in* $0<y<n-1$ *with* $2\leqq n\leqq 3.5$,
and >0 *for* $-1\leqq y\leqq n-1$ *with* $2\leqq n\leqq(5+\sqrt{13})/4$,
and >0 *for* $-1\leqq y\leqq 1$ *and* <0 *at* $y=n-1$ *with* $(5+\sqrt{13})/4<n\leqq 3.5$;

ii) $\tilde{Q}_3(1+y)/y^4\sqrt{1-\dfrac{y}{n-1}}$ *is* $\nearrow$ *in* $-1<y<0$
and positive for $-1\leqq y<0$, *tends to* $+\infty$ *as* $y\to-0$ *with* $2\leqq n\leqq 2.65$,
and $\searrow\nearrow$ *in* $0<y<n-1$, *positive for* $0<y<n-1$,
tends to $+\infty$ *as* $y\to+0$ *or* $y\to n-1$ *with* $2\leqq n<(5+\sqrt{13})/4$,
and $\searrow$ *in* $0<y<n-1$, *positive for* $0<y\leqq 1$,
tends to $+\infty$ *as* $y\to+0$ *and* $-\infty$ *as* $y\to n-1$.
with $(5+\sqrt{13})/4<n\leqq 2.65$ (*see Fig. 1*).

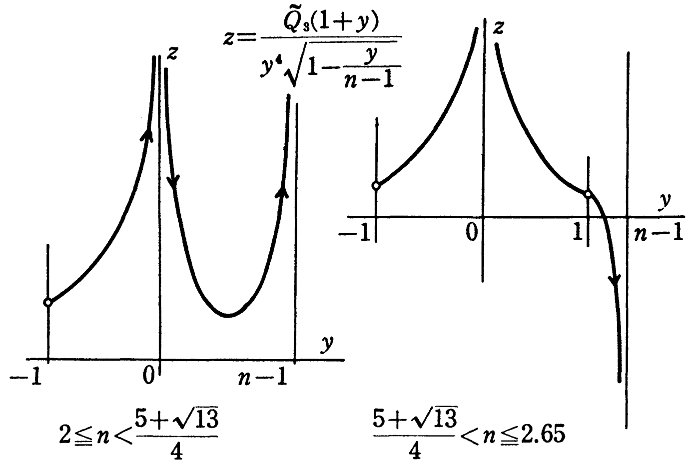


Fig. 1.

LEMMA 2.4. *Regarding* $\tilde{Q}_4(1+y)$ *we have the following*:

i) $\tilde{Q}_4(1+y)>0$ *for* $-1\leqq y\leqq n-1$ *with* $n\geqq 2$;

ii) $\frac{1}{y^4}\left(1-\frac{y}{n-1}\right)^{n-3/2}\tilde{Q}_4(1+y)$ *is* $\nearrow$ *in* $-1<y<0$, $\searrow$ *in* $0<y<n-1$, *positive for* $-1\leqq y\leqq n-1$, $\neq 0$, *tends to* $+\infty$ *as* $y\to 0$, *with* $2\leqq n\leqq 2.5$.

LEMMA 2.5. *Regarding* $P_2(1+y)$ *we have the following*:

i) *as a quadratic polynomial of* y, *its* y-*coordinate of symmetric axis* $(n-1)(2n+5)/(2n+1)>n-1$ *with* $n\geqq 2$, *it takes its values* $12(n-1)^2$ *at* $y=0$ *and* $(n-1)^2(3-2n)$ *at* $y=n-1$;

ii) $\dfrac{P_2(1+y)}{y^4\left(1-\frac{y}{n-1}\right)^{n-1/2}}$ *is* $\nearrow$ *in* $-1<y<0$, *positive for* $-1\leqq y<0$, *tends to* $+\infty$ *as* $y\to -0$, *and* $\searrow$ *in* $0<y<n-1$ *with* $n\geqq 2$.

LEMMA 2.6. *Regarding* $P_3(1+y)$ *we have the following*:

i) $P_3(1+y)$ *is* $\searrow$ *in* $-1<y<n-1$, *positive for* $-1\leqq y\leqq n-1$ *with* $2\leqq n\leqq 2.722$;

ii) $\dfrac{P_3(1+y)}{y^4\sqrt{1-\frac{y}{n-1}}}$ *is* $\nearrow$ *in* $-1<y<0$, *tends to* $+\infty$ *as* $y\to -0$, *and* $\searrow\nearrow$ *in* $0<y<n-1$, *tends to* $+\infty$ *as* $y\to +0$ *or* $y\to n-1$ *with* $2\leqq n\leqq 2.64$.

Now, regarding the integral J_2 and J_3 we have

$$J_2=\frac{(n-1)^{n-5/2}}{2}\left[\int_{-1}^{n-1}\frac{\tilde{Q}_3(1+y)dy}{y^4\sqrt{1-\frac{y}{n-1}}}-\int_{-1}^{n-1}\frac{1}{y^4}\left(1-\frac{y}{n-1}\right)^{n-3/2}\tilde{Q}_4(1+y)dy\right]$$

and

$$J_3=\frac{(n-x_1)^{n-1}}{\sqrt{n-1}}\left(x_1+\frac{n}{n-1}\right)\left[-\int_{-1}^{n-1}\frac{P_2(1+y)dy}{y^4\left(1-\frac{y}{n-1}\right)^{n-1/2}}+\int_{-1}^{n-1}\frac{P_3(1+y)dy}{y^4\sqrt{1-\frac{y}{n-1}}}\right]$$

in which the integrands in the brackets all tend to $+\infty$ as $y\to 0$. Therefore in order to evaluate J_2 and J_3 we need the following lemmas, which will be proved also in [24].

LEMMA 2.7. *Regarding* J_2 *we have*

$$\int_{0.99}^{1.01}\frac{1}{(x-1)^4\sqrt{n-x}}\{B\tilde{Q}_3(x)-(n-x)^{n-1}\tilde{Q}_4(x)\}dx$$

$$>\frac{4}{3}(n-1)^{n+1/2}(b'-b)+\frac{1}{3}(n-1)^{n-2}n(2n-1)(n^2-n-3)$$

$$\times(\sqrt{n-0.99}-\sqrt{n-1.01})+\frac{4}{3}(n-1)^{n-1}$$

$$\times\left\{\sqrt{n-0.99}\left(n-\frac{201}{200}\right)b-\sqrt{n-1.01}\left(n-\frac{199}{200}\right)b'\right\},$$

where

$$b=b_5 \text{ for } 2\leqq n\leqq 2.24,\ b=b_6 \text{ for } 2.24<n\leqq 2.40 \text{ and } b'=b_7,$$

$$b_5=b_4-\frac{n-2}{2(n-1)}b_3-\frac{(n-2)(3-n)}{6(n-1)^2}\left(\frac{n-1}{n-0.99}\right)^{4-n}b_2$$

$$-\frac{(n-2)(3-n)(4-n)}{24(n-1)^3}\left(b_1-\frac{1}{50}b_2+\frac{3}{10000}b_3-\frac{1}{250000}b_4\right)$$

$$-\frac{(n-2)(3-n)(4-n)(5-n)}{120(n-1)^4}b_0,$$

$$b_6=b_4-\frac{n-2}{2(n-1)}b_3-\frac{(n-2)(3-n)}{6(n-1)^2}b_2-\frac{(n-2)(3-n)(4-n)}{24(n-1)^3}$$

$$\times 1.00016\times b_1-\frac{(n-2)(3-n)(4-n)(5-n)}{120(n-1)^4}b_0,$$

$$b_7=\left(\frac{n-1.01}{n-1}\right)^{n-2}b_4-\frac{n-2}{2(n-1)}\left(\frac{n-1}{n-1.01}\right)^{3-n}\left(b_3+\frac{1}{25}b_4\right)$$

$$-\frac{(n-2)(3-n)}{6(n-1)^2}\left(\frac{n-1}{n-1.01}\right)^{4-n}\left(b_2+\frac{3}{100}b_3+\frac{3}{5000}b_4\right)\times h$$

$$-\frac{(n-2)(3-n)(4-n)}{24(n-1)^3}\left(\frac{n-1}{n-1.01}\right)^{5-n}k\times b_1$$

$$-\frac{(n-2)(3-n)(4-n)(5-n)}{120(n-1)^4}\left(\frac{n-1}{n-1.01}\right)^{6-n}$$

$$\times\left(b_0+\frac{1}{10^2}b_1+\frac{1}{10^4}b_2+\frac{1}{10^6}b_3+\frac{1}{10^8}b_4\right),$$

$$\begin{cases} k=1 & \text{for } 2\leqq n\leqq 2.2413\cdots, \\ \quad 1.002 & \text{for } 2.2413\cdots<n<2.32, \\ \quad 1.0034 & \text{for } 2.32\leqq n\leqq 2.4, \end{cases}$$

$$\begin{cases} h=\left(\frac{n-1.01}{n-1}\right)^{4-n} & \text{for } 2\leqq n<2.2413\cdots, \\ \quad 1 & \text{for } 2.2413\cdots<n\leqq 2.4. \end{cases}$$

LEMMA 2.8. *Reading J_3 we have*

$$\int_{0.99}^{1.01}\frac{F_2(x)dx}{(1-x)^4(n-x)^{n-1/2}}>\frac{n(n^2-n+1)}{21\sqrt{n-1}}\left\{1-\left(\frac{n-1}{n-0.99}\right)^{7/2}\right\}$$

$$+\frac{n}{24(n-1)^{5/2}(n-3/2)}\left\{\left(\frac{n-1}{n-1.01}\right)^{n-3/2}-1\right\}\times b_{10},$$

where

$$b_{10}=3.9202n^4-11.780001n^3+15.680498n^2$$
$$-11.740299n+3.919002 \quad \text{for } n\geqq 2\,.$$

PROPOSITION 2. *Regarding* $\rho(x, x_1)$ *defined by* (2.9) *we have*

$$\rho(x_0, x_1)>0 \qquad \text{for } 0<x_0<1<x_1<n,\ x_1=X_n(x_0)$$

with $n\geqq 2$.

Proof. By (2.11), it is sufficient to show

$$g'(x_0)<0 \qquad \text{for } 0<x_0<1\,,$$

which is held by Lemma 2.1 for $n\geqq 2.4$. For $2\leqq n\leqq 2.4$, we proved this inequality by showing that (2.17) holds by means of numerical evaluation of J by computors in which we computed J_1, J_2 and J_2 for $0.01\leqq x_0\leqq 0.99$ and $2\leqq n\leqq 2.41$ with step 1/100 and obtained $J>0$ for these values of x_0 and n, from which we see that this inequality holds for $0<x_0<1$ and $2\leqq n\leqq 2.4$ in general, by taking the properties of J_1, J_2, J_3 by Lemma 2.3—Lemma 2.8 into consideration. Q. E. D.

As the next step, we wish to prove $\rho(x, x_1)>0$ for $x_0<x<1$, which corresponds to (1.18) in the proof of Proposition 1. If this inequality does not hold, then there exist ξ_1, ξ_2 such that $x_0<\xi_1\leqq\xi_2<1$,

$$\rho(\xi_1, x_1)=\rho(\xi_2, x_1)=0\,,$$

and

$$\frac{\partial\rho(x, x_1)}{\partial x}\leqq 0 \text{ at } x=\xi_1 \quad\text{and}\quad \frac{\partial\rho(x, x_1)}{\partial x}\geqq 0 \text{ at } x=\xi_2$$

by Lemma 2.1 and Proposition 2. From (2.1) and (2.9) we obtain

$$\begin{aligned}\frac{\partial\rho(x, x_1)}{\partial x}&=\frac{N(X, x_1)}{(X-1)^4(n-X)^{n+1/2}}\frac{\partial X}{\partial x}-\frac{N(x, x_1)}{(1-x)^4(n-x)^{n+1/2}}\\&=-\frac{1-x}{x(n-x)}\left[\frac{XN(X, x_1)}{(X-1)^5(n-X)^{n-1/2}}+\frac{xN(x, x_1)}{(1-x)^5(n-x)^{n-1/2}}\right]\\&=-\frac{1-x}{x^2(n-x)^2}\left[\frac{X^2N(X, x_1)}{(X-1)^5\sqrt{n-X}}+\frac{x^2N(x, x_1)}{(1-x)^5\sqrt{n-x}}\right],\end{aligned}$$

i. e.

(2.24) $$\frac{\partial p(x, x_1)}{\partial x}=-\frac{1-x}{x^2(n-x)^n}V(x, x_1) \qquad \text{for } 0<x<1\,,$$

by (7.10) in (III). Hence, the above inequalities for ξ_1 and ξ_2 become

(2.25) $$V(\xi_1, x_1)\geqq 0 \quad\text{and}\quad V(\xi_2, x_1)\leqq 0\,.$$

Therefore, if we can prove that

$$\frac{\partial V(x, x_1)}{\partial x}>0 \qquad \text{for } x_0<x<1, \tag{2.26}$$

then it implies that

$$\xi_1=\xi_2 \quad \text{and} \quad V(\xi_1, x_1)=0$$

and

$$\rho(x, x_1)>0 \qquad \text{for } x_0<x<\xi_1 \text{ and } \xi_1<x<1,$$

which also implies $I>0$ by (2.7) and (2.9). In the following we shall try to prove (2.26).

§ 3. $\partial V(x, x_1)/\partial x$.

LEMMA 3.1. *Setting $x=1-t$ and $X_n(x)=1+s$ near $x=1$, we have*

$$s=t+\frac{2(n-2)}{3(n-1)}t^2+\frac{4(n-2)^2}{9(n-1)^2}t^3+\cdots. \tag{3.1}$$

Proof. From $x(n-x)^{n-1}=X(n-X)^{n-1}$, we have

$$(1-t)\left(1+\frac{t}{n-1}\right)^{n-1}=(1+s)\left(1-\frac{s}{n-1}\right)^{n-1},$$

i. e.

$$1-\frac{n}{2(n-1)}t^2-\frac{n(n-2)}{3(n-1)^2}t^3-\frac{n(n-2)(n-3)}{8(n-1)^3}t^4-\cdots$$

$$=1-\frac{n}{2(n-1)}s^2+\frac{n(n-2)}{3(n-1)^2}s^3-\frac{n(n-2)(n-3)}{8(n-1)^3}s^4+\cdots.$$

By (8.12) in (III), we can put

$$s=t+\frac{2(n-2)}{3(n-1)}t^2+bt^3+\cdots$$

and substituting this expression into the above equality we obtain easily

$$b=\frac{4(n-2)^2}{9(n-1)^2}.$$

Q. E. D.

LEMMA 3.2. *Near $x=1$, we have*

$$\frac{1}{B}V(x, x_1)=\frac{n(n^2-n+1)C}{3\sqrt{n-1}(1-x)}-\frac{n(n-2)(n^2-n+1)C}{9(n-1)^{3/2}} \tag{3.2}$$

$$-\frac{n\{144(2n-1)+(n^2-n+1)(32n^2-128n+137)C\}}{216(n-1)^{5/2}}(1-x)+\cdots,$$

where $C=\log(n-1)+1-\tilde{\lambda}(x_1)=\lambda(1)-\tilde{\lambda}(x_1)>0.$

Proof. Setting $x=1-t$ and $X=X_n(x)=1+s$, we obtain

$$\frac{1}{B}F_2(1-t)=\frac{n(n^2-n+1)}{6(n-1)}t^4+\frac{n(n-2)(6n^2-7n+7)}{60(n-1)^2}t^5$$
$$+\frac{n(n-2)(n-3)(4n^2-6n+5)}{120(n-1)^3}t^6+\cdots$$

by an analogous computation as the one in §8 of (III), and

$$\frac{1}{B}f_0(1-t)=-\frac{n(2n-1)}{6(n-1)}t^3-\frac{n(n-2)(3n-1)}{12(n-1)^2}t^4+\cdots,$$

$$\frac{1}{B}\{B-\phi(1-t)\}=\frac{n}{2(n-1)}t^2+\frac{n(n-2)}{3(n-1)^2}t^3+\cdots,$$

$$\frac{x^2}{\sqrt{n-x}}=\frac{1}{\sqrt{n-1}}\left\{1-\frac{4n-3}{2(n-1)}t+\frac{8n^2-8n+3}{8(n-1)^2}t^2+\cdots\right\},$$

$$\lambda(1-t)=\log(n-1)+1+\frac{1}{2(n-1)^2}t^2-\frac{2}{3(n-1)^3}t^3+\cdots.$$

Hence, we obtain

$$\frac{x^2N(x,\,x_1)}{B(1-x)^5\sqrt{n-x}}=\frac{1}{\sqrt{n-1}}\left\{1-\frac{4n-3}{2(n-1)}t+\frac{8n^2-8n+3}{8(n-1)^2}t^2+\cdots\right\}$$
$$\times\Big[\frac{n-1+t}{t}\Big\{\frac{n(n^2-n+1)}{6(n-1)}+\frac{n(n-2)(6n^2-7n+7)}{60(n-1)^2}t$$
$$+\frac{n(n-2)(n-3)(4n^2-6n+5)}{120(n-1)^3}t^2+\cdots\Big\}\Big\{C+\frac{1}{2(n-1)^2}t^2$$
$$-\frac{2}{3(n-1)^2}t^3+\cdots\Big\}-\frac{n(2n-1)}{2(n-1)}-\frac{n(n-2)(3n-1)}{4(n-1)^2}t+\cdots$$
$$+2n\Big\{\frac{n}{2(n-1)}+\frac{n(n-2)}{3(n-1)^2}t+\cdots\Big\}\Big]$$
$$=\frac{1}{\sqrt{n-1}}\left\{1-\frac{4n-3}{2(n-1)}t+\frac{8n^2-8n+3}{8(n-1)^2}t^2+\cdots\right\}$$
$$\times\Big[\frac{n(n^2-n+1)C}{6t}+\frac{n(6n^3-9n^2+11n-4)C}{60(n-1)}+\frac{n}{2(n-1)}$$
$$+\Big(\frac{n(4n-5)}{12(n-1)^2}+\frac{n(n-2)(4n^3-6n^2+9n-1)C}{120(n-1)^2}\Big)t+\cdots\Big],$$

i. e.

$$\frac{\sqrt{n-1}\,x^2N(x,\,x_1)}{B(1-x)^5\sqrt{n-x}}=\frac{n(n^2-n+1)C}{6t}+\frac{n}{2(n-1)} \tag{3.3}$$
$$-\frac{n(14n^3-26n^2+24n-11)C}{60(n-1)}-\left\{\frac{n(2n-1)}{3(n-1)^2}+\frac{n(n^2-n+1)C}{48(n-1)^2}\right\}t+\cdots$$

We obtain analogously

$$\frac{\sqrt{n-1}\,X^2N(X,\,x_1)}{B(X-1)^5\sqrt{n-X}}=\frac{n(n^2-n+1)C}{6s}-\frac{n}{2(n-1)} \tag{3.3$'$}$$
$$+\frac{n(14n^3-26n^2+24n-11)C}{60(n-1)}-\left\{\frac{n(2n-1)}{3(n-1)^2}+\frac{n(n^2-n+1)C}{48(n-1)^2}\right\}s+\cdots.$$

Therefore, using (3.1) and (2.1) we obtain

$$\frac{\sqrt{n-1}}{B}V(x,\,x_1)=\frac{n(n^2-n+1)C}{6}\left(\frac{1}{t}+\frac{1}{s}\right)$$
$$-\left\{\frac{n(2n-1)}{3(n-1)^2}+\frac{n(n^2-n+1)C}{48(n-1)^2}\right\}(t+s)+\cdots$$
$$=\frac{n(n^2-n+1)C}{6t}\left\{2-\frac{2(n-2)}{3(n-1)}t-\frac{8(n-2)^2}{9(n-1)^2}t^2+\cdots\right\}$$
$$-\left\{\frac{n(2n-1)}{3(n-1)^2}+\frac{n(n^2-n+1)C}{48(n-1)^2}\right\}(2t+\cdots)$$
$$=\frac{n(n^2-n+1)C}{3t}-\frac{n(n-2)(n^2-n+1)C}{9(n-1)}$$
$$-\frac{n\{144(2n-1)+(n^2-n+1)(32n^2-128n+137)C\}}{216(n-1)^2}t+\cdots,$$

which implies this lemma. Q. E. D.

LEMMA 3.3. *We have for* $1<x_1<n$

i) $\lim_{x\to 1-0} V(x,\,x_1)=+\infty$ *with* $n>1$;

ii) $\lim_{x\to 1-0}(1-x)V(x,\,x_1)=\dfrac{Bn(n^2-n+1)C}{3\sqrt{n-1}}$ *with* $n>1$;
and $(1-x)V(x,\,x_1)$ *is* $\nearrow$ *at* $x=1$ *with* >2;

iii) $\lim_{x\to 1-0}\dfrac{\partial\rho(x,\,x_1)}{\partial x}=-\dfrac{n(n^2-n+1)C}{3(n-1)^{3/2}}<0$,

where $C=\lambda(1)-\tilde{\lambda}(x_1)=\log\dfrac{n-1}{n-x_1}-\dfrac{x_1-1}{(n-1)x_1}>0$.

Proof. i) and ii) are evident from Lemma 3.2. Next, from (2.24) we obtain

$$\lim_{x\to1-0}\frac{\partial\rho(x,\,x_1)}{\partial x}=-\frac{1}{(n-1)^n}\lim_{x\to1-0}\{(1-x)V(x,\,x_1)\}$$
$$=-\frac{n(n^2-n+1)C}{3(n-1)^{3/2}}<0.$$

Q.E.D.

Now, we compute $\partial V(x,\,x_1)/\partial x$. Regarding the functions in (1.2), we have

$$\left(\frac{x^2\sqrt{n-x}}{(1-x)^5}\right)'=\frac{x\{4n+(6n-5)x-5x^2\}}{2(1-x)^6\sqrt{n-x}},$$

$$F_2'(x)=-2B\{(2n+1)x-(2n^2+5n-4)\}-(n-x)^{n-2}Q_3(x),$$

where

$$Q_3(x)=2n^2(n+2)+n(n-13)x+2(n^3-n^2-n+4)x^2-(n-1)(n+2)x^3$$

((3.1) and (3.2) in (VII)),

$$\tilde{\lambda}'(x)=-\frac{(x-1)\{n+(n-1)x\}}{(n-1)x^2(n-x)}$$

(Lemma 7.1 in (III)), from which we obtain

$$\left(\frac{x^2\sqrt{n-x}F_2(x)}{(1-x)^5}\{\tilde{\lambda}(x)-\tilde{\lambda}(x_1)\}\right)' \tag{3.4}$$
$$=\frac{\{n+(n-1)x\}F_2(x)}{(n-1)(1-x)^4\sqrt{n-x}}+\frac{xW(x,\,n)\{\tilde{\lambda}(x)-\tilde{\lambda}(x_1)\}}{2(1-x)^6\sqrt{n-x}},$$

where

$$W(x,\,n):=-BW_4(x,\,n)+(n-x)^{n-1}W_5(x,\,n)(=W(x)), \tag{3.5}$$

$$\begin{aligned}W_4(x,\,n):=&4n(4n-1)(4n-3)+(72n^3-236n^2+146n-15)x\\&-(16n^3+76n^2-190n+71)x^2+(16n^2+14n-33)x^3\\&-(2n+1)x^4(=W_4(x)),\end{aligned} \tag{3.6}$$

$$\begin{aligned}W_5(x,\,n):=&12n^2(2n-1)+3n(16n^2-36n+9)x\\&+3(12n^3-50n^2+52n-5)x^2+(8n^3-70n^2+130n-71)x^3\\&+(4n^3-18n^2+44n-33)x^4-(n-1)(2n-1)x^5(=W_5(x)).\end{aligned} \tag{3.7}$$

By Lemma 2.1 in (IX) we have

$$\eta'(x,\,n)=\frac{1}{2(n-1)(1-x)^4\sqrt{n-x}}\{B\tilde{Q}_3(x,\,n)-(n-x)^{n-1}\tilde{Q}_4(x,\,n)\},$$

$$\tilde{Q}_3(x, n)=2n(4n-1)(4n-3)+3(16n^3-40n^2+24n-3)x$$
$$-8(n-1)(n^2+5n-3)x^2+(4n^2+2n-3)x^3(=\tilde{Q}_3(x)),$$
$$\tilde{Q}_4(x, n)=6n^2(2n-1)+n(28n^2-52n+15)x$$
$$+3(8n^3-26n^2+21n-3)x^2+(2n-3)(2n^2-13n+8)x^3$$
$$+(n-1)(2n-1)(2n-3)x^4(=\tilde{Q}_4(x))$$

and hence

$$V'(x, x_1)=\frac{\partial}{\partial x}V(x, x_1)=\frac{\{n+(n-1)x\}F_2(x)}{(n-1)(1-x)^4\sqrt{n-x}}$$
$$+\frac{xW(x)\{\tilde{\lambda}(x)-\tilde{\lambda}(x_1)\}}{2(1-x)^6\sqrt{n-x}}-\frac{B\tilde{Q}_3(x)-(n-x)^{n-1}\tilde{Q}_4(x)}{2(n-1)(1-x)^4\sqrt{n-x}}$$
$$+\Big[\frac{\{n+(n-1)X\}F_2(X)}{(n-1)(X-1)^4\sqrt{n-X}}+\frac{XW(X)\{\tilde{\lambda}(X)-\tilde{\lambda}(x_1)\}}{2(X-1)^6\sqrt{n-X}}$$
$$-\frac{B\tilde{Q}_3(X)-(n-X)^{n-1}\tilde{Q}_4(X)}{2(n-1)(X-1)^4\sqrt{n-X}}\Big]\frac{1-x}{x(n-x)}\cdot\frac{X(n-X)}{X-1}.$$

Carefully arranging the right hand side, we obtain the following important formula:

$$\text{(3.8)}\qquad \frac{x(n-x)}{1-x}\frac{\partial V(x, x_1)}{\partial x}=\frac{x^2\sqrt{n-x}W(x, n)}{2(1-x)^7}\{\tilde{\lambda}(x)-\tilde{\lambda}(x_1)\}$$
$$+\frac{x\sqrt{n-x}S(x, n)}{2(n-1)(1-x)^5}+\frac{X^2\sqrt{n-X}W(X, n)}{2(X-1)^7}\{\tilde{\lambda}(X)-\tilde{\lambda}(x_1)\}$$
$$+\frac{X\sqrt{n-X}S(X, n)}{2(n-1)(X-1)^5},$$

where $X=X_n(x)$ and

$$\text{(3.9)}\qquad S(x, n):=-BS_3(x, n)+(n-x)^{n-1}S_4(x, n)(=S(x)),$$

$$\text{(3.10)}\qquad S_3(x, n):=4n(4n-1)(4n-3)+3(24n^3-68n^2+42n-5)x$$
$$-2(8n^3+20n^2-15n+20)x^2+(8n^2-5)x^3(=S_3(x)),$$

$$\text{(3.11)}\qquad S_4(x, n):=12n^2(2n-1)+3n(16n^2-32n+9)x$$
$$+3(12n^3-42n^2+37n-5)x^2+(8n^3-52n^2+87n-40)x^3$$
$$+(n-1)(4n^2-10n+5)x^4.$$

Looking over the right hand side of (3.8), we see that if we have

$$\text{(3.12)}\qquad W(x, n)>0 \qquad \text{for } 0<x<n,\ x\neq 1,$$

then it is increasing with respect to x_1, since

$$\tilde{\lambda}'(x_1) = -\frac{(x_1-1)\{n+(n-1)x_1\}}{(n-1)x_1^2(n-x_1)} < 0 \qquad \text{for } 1<x_1<n\,.$$

Furthermore, if (3.12) holds, we shall obtain

$$\frac{\partial V(x,\, x_1)}{\partial x} > 0$$

by proving that

$$\text{(3.13)} \quad \Psi(x,\, n) := -\frac{x^2\sqrt{n-x}\,W(x,\, n)}{(1-x)^7}\{\tilde{\lambda}(X_n(x))-\tilde{\lambda}(x)\} + \frac{x\sqrt{n-x}\,S(x,\, n)}{(n-1)(1-x)^5} + \frac{X\sqrt{n-X}\,S(X,\, n)}{(n-1)(X-1)^5} > 0 \quad \text{for } 0<x<1\,.$$

Remark. By Lemma 4.1 in (IX), $\tilde{\lambda}(X_n(x))-\tilde{\lambda}(x)$ is decreasing in $0<x<1$ and positive there and

$$\lim_{x\to+0} \tilde{\lambda}(X_n(x))-\tilde{\lambda}(x) = +\infty\,, \qquad \tilde{\lambda}(X_n(1))-\tilde{\lambda}(1)=0\,.$$

§ 4. Some properties of $W(x, n)$.

LEMMA 4.1. $W(0, n)>0$ *for* $n\geqq 2$.

Proof. From (3.5), (3.6) and (3.7) we have

$$W(0,\, n) = -4Bn(4n-1)(4n-3)+12n^{n+1}(2n-1)\,,$$

and we shall prove the following inequality:

$$\text{(4.1)} \qquad e_{n-1} := \left(\frac{n}{n-1}\right)^{n-1} > h(n) := \frac{(4n-1)(4n-3)}{3n(2n-1)} \qquad \text{for } n\geqq 2\,.$$

We see easily that (4.1) holds for $n=2$ and as $n\to\infty$ and e_{n-1} and $h(n)$ are increasing in $2<n<\infty$. If (4.1) does not hold, there exist n_1 and n_2 such that $2<n_1\leqq n_2<\infty$,

$$e_{n-1}=h(n) \qquad \text{at } n=n_1,\ n_2 \text{ and}$$

$$\log\frac{n}{n-1}-\frac{1}{n} \leqq \frac{h'}{h} = \frac{32n^2-32n+10}{(2n-1)(4n-1)(4n-3)}-\frac{1}{n} \qquad \text{at } n=n_1$$

$$\geqq \qquad \text{at } n=n_2\,.$$

On the other hand, since we have

$$\log\frac{n}{n-1} = \frac{1}{n}+\frac{1}{2n^2}+\frac{1}{3n^3}+\cdots < \frac{6n^2-3n-1}{6n^2(n-1)} \qquad \text{for } n>1\,,$$

we consider the inequality:

$$\frac{6n^2-3n-1}{6n^2(n-1)} \leqq \frac{32n^2-32n+10}{(2n-1)(4n-1)(4n-3)},$$

which is equivalent to

$$8n^3-24n^2+13n-3 \geqq 0.$$

We see easily that this cubic inequality in n holds for $n \geqq 2.4$, which implies $n_2 < 2.4$. By numerical computation we have the following data:

n	e_{n-1}	$h(n)$
2	2	$35/18=1.94$
2.1	$2.036616\cdots$	$39.96/3\times2.1\times3.2=1.982142\cdots$
2.2	$2.069615\cdots$	$45.24/3\times2.2\times3.4=2.016042\cdots$
2.3	$2.099519\cdots$	$50.84/3\times2.3\times3.6=2.046698\cdots$
2.4	$2.126750\cdots$	$56.76/3\times2.4\times3.8=2.074561\cdots$

which show that $e_{n-1} > h(n)$ for $2 \leqq n \leqq 2.4$, and so $2.4 < n_1$. Thus, we reach a contradiction and (4.1) has to hold. Q. E. D.

LEMMA 4.2. *$W(n, n) > 0$ for $n > 3/2$.*

Proof. From (3.5) and (3.6) we have

$$W(n, n) = -BW_4(n, n) = Bn(n-1)^3(2n-3) > 0 \quad \text{for } n > 3/2.$$

Q. E. D.

LEMMA 4.3. *Near $x=1$, we have*

$$W(x, n) = \frac{B}{3} n(n^2-n+1)(x-1)^4 + \cdots.$$

Proof. Setting $x=1+y$, we obtain from (3.6) and (3.7)

$$\begin{aligned} W_4(1+y, n) &= 120n^3+20(n-1)^2(2n-13)y \\ &\quad -4(n-1)(4n^2+11n-44)y^2+(16n^2+6n-37)y^3-(2n+1)y^4, \end{aligned} \tag{4.2}$$

$$\begin{aligned} W_5(1+y, n) &= 120(n-1)^3+20(n-1)^2(8n-19)y \\ &\quad +4(n-1)(21n^2-101n+109)y^2+3(8n^3-54n^2+112n-71)y^3 \\ &\quad +(n-2)(4n^2-20n+19)y^4-(n-1)(2n-1)y^5. \end{aligned} \tag{4.3}$$

Hence, we obtain

$$\frac{1}{B}W(1+y,\, n)=-W_4(1+y,\, n)+\left(1-\frac{y}{n-1}\right)^{n-1}W_5(1+y,\, n)$$

$$=-W_4(1+y)+\left\{1-y+\frac{n-2}{2(n-1)}y^2-\frac{(n-2)(n-3)}{6(n-1)^2}y^3\right.$$

$$\left.+\frac{(n-2)(n-3)(n-4)}{24(n-1)^3}y^4+\cdots\right\}W_5(1+y)$$

$$=\frac{n(n^2-n+1)}{3}y^4+\cdots,$$

which implies this lemma. Q. E. D.

LEMMA 4.4. *$W_5(x,\, n)>0$ for $0\leqq x\leqq n$ with $n\geqq 2$.*

Proof. Since we have (3.7) and (4.3)

$$\text{(4.4)}\quad \begin{cases} W_5(0)=12n^2(2n-1), & W_5(1)=120(n-1)^3, \\ W_5(n)=2n^7-7n^6+9n^5-5n^4+n^3=n^3(n-1)^3(2n-1), \end{cases}$$

which are all positive for $n>1$. Therefore, we shall show that $W_5(x)>0$ for $0\leqq x\leqq n$ with $n\geqq 2$. For simplicity, setting

$$\text{(4.5)}\quad \begin{cases} a_0=n^2(2n-1), \qquad a_1=n(16n^2-36n+9) \\ a_2=12n^3-50n^2+52n-5=(2n-5)(6n^2-10n+1), \\ a_3=8n^3-70n^2+130n-71, \qquad a_4=4n^3-18n^2+44n-33, \\ a_5=(n-1)(2n-1)=2n^2-3n+1, \end{cases}$$

we have

$$\text{(4.6)}\qquad W_5(x,\, n)=12a_0+3a_1x+3a_2x^2+a_3x^3+a_4x^4-a_5x^5$$

First, we consider the quadratic polynomial of x:

$$\frac{1}{6}W_5^{(3)}(x)=a_3+4a_4x-10a_5x^2,$$

We have $a_5>0$ for $n>1$ and $a_4>0$ for $n\geqq 2$, and easily can show

$$0<\frac{a_4}{5a_5}<n \qquad \text{for } n\geqq 2\,.$$

We have

$$\text{(4.7)}\qquad D(n):=(\text{its discriminant})/8=2a_4^2+5a_3a_5$$

$$=32n^6-208n^5+532n^4-1306n^3+3238n^2-4093n+1823$$

and $D'(n)$ is $\searrow\nearrow$ in $2<n<\infty$, and $D'(2)=-285$, $D'(3)=-55$ and $D'(3.1)=647.65392$. Hence $D(n)$ is $\searrow\nearrow$ in $2<n<\infty$. Since we have

$$D(2)=45,\quad D(2.11)=3.8156\cdots,\quad D(2.12)=-0.9263\cdots,$$

$$D(3.39)=-16.1886\cdots,\quad D(3.40)=28.1905\cdots,$$

hence

$$D(n)>0 \qquad \text{for } 2\leqq n<2.11\cdots \text{ and } 3.39\cdots<n<\infty,$$

$$D(n)<0 \qquad \text{for } 2.11\cdots<n<3.39\cdots.$$

We see that $a_3(n)$ is $\searrow\nearrow$ in $2<n<\infty$ and $a_3(2)=-27$, $a_3(6.4)=-9.048$, $a_3(6.5)=13.5$ and hence

$$a_3(n)<0 \qquad \text{for } 2\leqq n<6.4\cdots \text{ and}$$

$$a_3(n)>0 \qquad \text{for } n>6.4\cdots \text{ and}$$

$$W_5^{(3)}(n)=-6(4n^4+34n^3-96n^2+2n+71)<0 \qquad \text{for } n\geqq 2.$$

i) Case $2\leqq n\leqq 2.11\cdots$. From the above argument, the cubic polynomial of x:

$$\frac{1}{2}W_5''(x)=3a_2+3a_3x+6a_4x^2-10a_5x^3$$

is $\searrow\nearrow\searrow$ in $0<x<n$. Since $a_2(n)<0$ for $2\leqq n<2.5$, we consider its value at the root γ of $W_5^{(3)}(x)=0$:

$$\gamma=\frac{2a_4+\sqrt{2(2a_4a_4+5a_3a_5)}}{10a_5},$$

which becomes

$$\frac{1}{5a_5}[4a_4^3+15a_3a_4a_5+75a_2a_5^2+\sqrt{2}(2a_4^2+5a_3a_4)^{3/2}],$$

and, using that $a_3(n)$ is $\searrow$ in $a<n<4$, we can prove the quantity in the brackets is negative for $2\leqq n\leqq 2.11\cdots$. Thus, we see that $W_5''(x)<0$ for $0\leqq x\leqq n$ and so the graph of $W_5(x)$ is convex upward and so (4.4) implies $W_5(x)>0$ for $0\leqq x\leqq n$.

ii) Case $2.11\cdots<n<3.39\cdots$ $(D(n)<0)$. $W_5^{(3)}(x)<0$ for $-\infty<x<+\infty$ and so $W_5''(x)$ is $\searrow$ in $0<x<n$. When $2.11\cdots<n\leqq 2.5$, $W_5''(x)<0$ for $0<x\leqq n$ and so we have $W_5(x)>0$. When $2.5<n<3.39\cdots$, $W_5'(x)$ is $\nearrow\searrow$ or $\nearrow$ in $0<x<n$. Since we have

$$W_5'(0)=3a_1>0 \qquad \text{for } n\geqq 2,$$

$$W_5'(n)=6n^6-33n^5+33n^4+6n^3-9n^2-3n$$

$$=3n(n-1)^2(2n^3-7n^2-5n-1)$$

and $2n^3-7n^2-5n-1$ is ↘↗ in $2<n<\infty$, -23 at $n=2$, $-1.1583\cdots$ at $n=4.13$, $0.2386\cdots$ at $n=4.14$. Therefore $W_5'(n)<0$ for $2\leqq n<4.13\cdots$. Hence we see that $W_5(x)$ is ↗↘ in $0<x<n$ with $2.5<n<3.39\cdots$ and so (4.4) implies also $W_5(x)>0$ for $0\leqq x\leqq n$.

iii) Case $3.39\cdots\leqq n<\infty$. $W_5''(x)$ is ↘↗↘ or ↗↘ in $0<x<n$ with $3.39\cdots\leqq n<6.4\cdots$ or $6.4\cdots\leqq n<\infty$ respectively. We have

$$W_5''(0)=6a_2>0 \qquad \text{for } n>2.5\,,$$

and

$$W_5''(n)=2(4n^5-54n^4+80n^3+42n^2-57n-15) \text{ is ↘↗ in } 2<n<\infty,\ <0$$

$$\text{for } 2\leqq n<11.7\cdots \text{ and } >0 \quad \text{for } n>11.7\cdots.$$

Now, for the case $3.39\cdots\leqq n\leqq 6.4\cdots$, we consider the value of $W_5''(x)$ at the other root of $W_5^{(3)}(x)=0$:

$$\delta=\frac{2a_4-\sqrt{2(2a_4^2+5a_3a_5)}}{10a_5}\,,$$

which becomes

$$\frac{1}{5a_5}[4a_4^3+15a_3a_4a_5+75a_2a_5^2-\sqrt{2}(2a_4^2+5a_3a_4)^{3/2}]\,.$$

Since we have that

$$\begin{aligned}&4a_4^3+15a_3a_4a_5+75a_2a_5^2\\&=256n^9-2496n^8+13440n^7-47976n^6+111984n^5-208542n^4\\&\quad+368072n^3-490365n^2+364497n-108978\,,\end{aligned}$$

which is ↗ in $3.3<n<\infty$, and $D(n)$ is ↗ in $3.1<n<\infty$, and computed the values of $4a_4^3+15a_3a_4a_5+75a_2a_5^2$ and $\sqrt{2}D^{3/2}$ for $3.39\leqq n\leqq 6.5$ with step 1/100, we obtain

$$4a_4^3+15a_3a_4a_5+75a_2a_5^2-\sqrt{2}D^{3/2}>0 \qquad \text{for } 3.39\cdots\leqq n\leqq 6.5\,.$$

Therefore, $W_5'(x)=3a_1+6a_2x+3a_3x^2+4a_4x^3-5a_5x^4$ is ↗↘ in $0<x<n$. We have $a_1>0$ for $n\geqq 2$. Therefore $W_5(x)$ is ↗↘ or ↗ in $0<x<n$, which implies $W_5(x)>0$ for $0\leqq x\leqq n$ by (4.4).

Last, we consider the case $6.4\cdots<n<\infty$. In this case $W_5''(x)$ is ↗↘ in $0<x<n$ and $W_5''(0)>0$. Hence $W_5'(x)$ is ↗↘ or ↗ in $0<x<n$. We have $W_5'(0)>0$ and $W_5'(n)>0$. Therefore $W_5'(x)>0$ for $0\leqq x\leqq n$, which implies $W_5(x)>0$ for $0\leqq x\leqq n$ by (4.4). Thus, we have proved

$$W_5(x,n)>0 \qquad \text{for } 0\leqq x\leqq n \text{ with } n\geqq 2\,.$$

Q. E. D.

LEMMA 4.5. *We have*

$$W(x,\ n)=-BW_4(x,\ n)+(n-x)^{n-1}W_5(x,\ n)>0$$
$$\text{for } 0\leqq x\leqq n,\ x\neq 1,\ \text{with } n\geqq 2.$$

Proof. If the above inequality does not hold, then there exist ξ_1, ξ_2 and η_1, η_2 such that

$$(4.8)\quad \begin{cases} 0<\xi_1\leqq\xi_2<1 \text{ and } 1<\eta_1\leqq\eta_2<n \\ W(\xi_i)=W(\eta_i)=0, \quad i=1, 2, \\ W'(\xi_1)\leqq 0,\ W'(\xi_2)\geqq 0 \text{ and } W'(\eta_1)\leqq 0,\ W'(\eta_2)\geqq 0, \end{cases}$$

by Lemma 4.1—Lemma 4.3. By Lemma 4.4, the last conditions are equivalent to

$$(n-1)W_4(x)W_5(x)-(n-x)(W_4(x)W_5'(x)-W_4'(x)W_5(x))$$
$$\geqq 0 \text{ at } x=\xi_1,\ \eta_1 \text{ and } \leqq 0 \text{ at } x=\xi_2,\ \eta_2.$$

Now, for simplicity, setting

$$(4.9)\quad \begin{cases} b_0=n(4n-1)(4n-3),\quad b_1=72n^3-236n^2+146n-15, \\ b_2=16n^3+76n^2-190n+71,\quad b_3=16n^2+14n-33, \\ b_4=2n+1, \end{cases}$$

we have

$$(4.10)\quad W_4(x,\ n)=4b_0+b_1x-b_2x^2+b_3x^3-b_4x^4.$$

From (4.5), (4.6) and (4.9), (4.10), we obtain

$$\begin{aligned}
&(n-1)W_4W_5-(n-x)(W_4W_5'-W_4'W_5) \\
&=12\{4(n-1)b_0a_0-n(b_0a_1-b_1a_0)\}+12\{nb_0a_1+(n-2)b_1a_0 \\
&\quad -2n(b_0a_2+b_2a_0)\}x+3\{4(n+1)b_0a_2+(n-1)b_1a_1-4(n-3)b_2a_0 \\
&\quad -4nb_0a_3-n(b_1a_2+b_2a_1)+12nb_3a_0\}x^2+\{4(n+2)b_0a_3 \\
&\quad +3nb_1a_2-3(n-2)b_2a_1+12(n-4)b_3a_0-2n(8b_0a_4+b_1a_3-3b_3a_1 \\
&\quad +24b_4a_0\}x^3+\{4(n+3)b_0a_4+(n+1)b_1a_3-3(n-1)b_2a_2 \\
&\quad +3(n-3)b_3a_1-12(n-5)b_4a_0+n(20b_0a_5-3b_1a_4+b_2a_3+3b_3a_2 \\
&\quad -9b_4a_1)\}x^4+\{-4(n+4)b_0a_5+(n+2)b_1a_4-nb_2a_3+3(n-2)b_3a_2 \\
&\quad -3(n-4)b_4a_1+2n(2b_1a_5+b_2a_4-3b_4a_2)\}x^5+\{-(n+3)b_1a_5 \\
&\quad -(n+1)b_2a_4+(n-1)b_3a_3-3(n-3)b_4a_2-n(3b_2a_5+b_3a_4 \\
&\quad +b_4a_3)\}x^6+\{(n+2)b_2a_5+nb_3a_4-(n-2)b_4a_3+2nb_3a_5\}x^7 \\
&\quad +\{-(n+1)b_3a_5-(n-1)b_4a_4-nb_4a_5\}x^8+nb_4a_5x^9
\end{aligned}$$

$$=n(n-1)[96n^3(2n^2-2n+3)-96n^2(6n^3+5n+8)x$$
$$+2n(240n^4+672n^3+60n^2+1140n+351)x^2$$
$$+(160n^5-2080n^4-400n^3-2444n^2-2388n-225)x^3$$
$$-2(240n^5-480n^4-500n^3-608n^2-1585n-403)x^4$$
$$+(288n^5+192n^4-608n^3-260n^2-2200n-1135)x^5$$
$$-(64n^5+320n^4-40n^3+152n^2-930n-820)x^6$$
$$+(96n^4+80n^3+140n^2-212n-335)x^7$$
$$-(40n^3+16n^2+2n-70)x^8+(4n^2-1)x^9]$$
$$=n(n-1)(x-1)^3W_6(x, n),$$

where

(4.10) $$W_6(x, n):=c_6x^6-c_5x^5+c_4x^4-c_3x^3+6c_2x^2+192c_1x-96c_0,$$

(4.11) $$\begin{cases} c_6=4n^2-1, \quad c_5=40n^3+4n^2+2n-67, \\ c_4=96n^4-40n^3+116n^2-218n-131, \\ c_3=64n^5+32n^4-40n^3-212n^2-282n-225, \\ c_2=n(16n^4-32n^3-68n^2+4n-117), \\ c_1=n^2(3n^2-2n+4), \quad c_0=n^3(2n^2-2n+3). \end{cases}$$

From the above argument, the last relations of (4.8) can be replaced by

(4.12) $$W_6(\xi_1)\leqq 0, \quad W_6(\xi_2)\geqq 0 \quad \text{and} \quad W_6(\eta_1)\geqq 0, \quad W_6(\eta_2)\leqq 0.$$

In the following, we shall prove the inequality:

$$W_6(x, n)<0 \qquad \text{for } 0\leqq x\leqq n \text{ with } n\geqq 2.$$

We see easily the following:

(4.13) $$\begin{cases} W_6(0)=-96c_0<0 \qquad \text{for } n>0, \\ W_6(n)=-n^3(4n^5-20n^4+39n^3-37n^2+17n-3)<0 \qquad \text{for } n\geqq 2. \end{cases}$$

First, we consider the quadratic polynomial of x:

$$\frac{1}{24}W_6^{(4)}(x)=15c_6x^2-5c_5x+c_4.$$

We see easily that $c_6>0$ for $n>1/2$, c_4 is $\nearrow$ in $2<n<\infty$, $\geqq 1113$, and $5c_5/(2\times 15c_6)=c_5/6c_6>n$ for $n\geqq 2$, and

$$W_6^{(4)}(n)=24(15c_6n^2-5c_5n+c_4)$$
$$=-24(44n^4+60n^3-91n^2-117n+131)<0 \quad \text{for } n\geqq 2.$$

Therefore, the cubic polynomial of x:

$$\frac{1}{6}W_6^{(3)}(x)=20c_6x^3-10c_5x^2+4c_4x-c_3$$

is ↗↘ in $0<x<n$. We see easily that c_3 is ↗ in $2<n<\infty$, $\geqq 603$, hence $W_6^{3}(0)<0$, and

$$W_6^{(3)}(n)=6(20c_6n^3-10c_5n^2+4c_4n-c_3)$$
$$=-6(232n^4-464n^3-10n^2+242n-225)<0 \quad \text{for } n\geqq 2,$$

since $232n^4-464n^3-10n^2+242n-225$ is ↗ in $2<n<\infty$, $\geqq 219$ for $n\geqq 2$. Then, we have

$$W_6^{(3)}\left(\frac{3n}{4}\right)=\frac{3}{8}(135c_6n^3-90c_5n^2+48c_4n-16c_3)$$
$$=\frac{3}{8}(524n^5-2792n^4+5893n^3-1042n^2-1776n+3600),$$

which is ↗ in $2<n<\infty$, $\geqq(3/8)\times 15120=5670$ for $n\geqq 2$. Therefore, the polynomial of x of 4th order:

$$\frac{1}{2}W_6''(x)=15c_6x^4-10c_5x^3+6c_4x^2-3c_3x+6c_2$$

is ↘↗↘ in $0<x<n$. We can easily prove that

$$W_6''(0)=12c_2<0 \quad \text{for } 2\leqq n<3.40\cdots \text{ and}$$
$$>0 \quad \text{for } 3.40\cdots<n<\infty,$$

since $16n^5-32n^4-68n^3+4n^2-117n$ is ↘↗ in $2<n<\infty$, -762 at $n=2$, $-30.8393\cdots$ at 3.4, $1.6173\cdots$ at 3.41. Then, we have

$$W_6''(n)=2(15c_6n^4-10c_5n^3+6c_4n^2-3c_3n+6c_2)$$
$$=2n(44n^5-280n^4+589n^3-410n^2+84n-27)>0 \quad \text{for } n\geqq 2,$$

and

$$W_6''(1)=2(15c_6-10c_5+6c_4-3c_3+6c_2)$$
$$=-16(12n^5-36n^4+116n^3-172n^2+148n-68)<0 \quad \text{for } n\geqq 2.$$

Therefore, the polynomial of x of 5th order:

$$W_6'(x)=6c_6x^5-5c_5x^4+4c_4x^3-3c_3x^2+12c_2x+192c_1$$

is ↘↗ in $1<x<n$ with $n\geqq 2$, and ↘ in $0<x<1$ when $2\leqq n\leqq 3.40\cdots$, and ↗↘ in $0<x<1$ when $3.40\cdots<n<\infty$. We have

$$W_6'(0)=192c_1>0 \qquad \text{for } n>0\,,$$

$$\begin{aligned}W_6'(1)&=480(n^4-3n^3+4n^2-3n+1)\\ &=480(n-1)^2(n^2-n+1)>0 \qquad \text{for } n>1\,,\end{aligned}$$

$$\begin{aligned}W_6'(n)&=16n^7-84n^6+184n^5-141n^4-14n^3+39n^2\\ &=n^2(n-1)^2(16n^3-52n^2+64n+39)>0 \qquad \text{for } n>1\,.\end{aligned}$$

Thus, we see that $W_6'(x)>0$ for $0\leqq x\leqq 1$ with $n\geqq 2$, and hence $W_6(x)<0$ for $0\leqq x\leqq 1$ with $n\geqq 2$ by (4.13).

Next, we consider the interval $1\leqq x\leqq n$. $W_6'(x)$ takes its minimum in $1\leqq x\leqq n$ at the root α of $W_6''(x)=0$. Since we have

$$\begin{aligned}\frac{1}{6}W_6^{(3)}\left(\frac{5n}{6}\right)=\frac{1}{(1.2)^3}(&42.368n^5-333.696n^4+693.28n^3\\ &-85.344n^2-267.264n+388.8)>0 \qquad \text{for } n\geqq 2\,,\end{aligned}$$

we obtain

$$W_6^{(3)}(x)>0 \qquad \text{for } \frac{3}{4}n\leqq x\leqq\frac{5}{6}n$$

from the facts stated above. Then, we have

$$\begin{aligned}\frac{1}{2}W_6''\left(1+\frac{3}{10}n\right)=&-16.074n^6+96.6n^5-96.4215n^4-390.25n^3+809.06n^2\\ &-868.1n+544 \searrow \text{ in } 2<n<\infty,\ <0 \qquad \text{for } n\geqq 2\,,\end{aligned}$$

which implies

$$1+\frac{3}{10}n<\alpha \qquad \text{for } n\geqq 2\,.$$

We have also

$$\begin{aligned}\frac{1}{2}W_6''\left(1+\frac{n}{3}\right)=-\frac{1}{27}(&360n^6-2792n^5+3481n^4+9218n^3-24084n^2\\ &+22491n-14688>0 \qquad \text{for } 2\leqq n<4.93\cdots\\ &<0 \qquad \text{for } 4.93\cdots<n<\infty\,,\end{aligned}$$

which implies

$$\alpha<1+\frac{n}{3} \qquad \text{for } 2\leqq n<4.93\cdots\,,$$

$$1+\frac{n}{3}<\alpha \qquad \text{for } n>4.93\cdots\,.$$

We have $1+n/3\leqq 5n/6$ for $n\geqq 2$ and

$$\begin{aligned}\frac{1}{6}W_6^{(3)}\left(1+\frac{3}{10}n\right)=&17.36n^5+82n^4-335.14n^3+497.3n^2-383.2n+351\\ &\text{is} \nearrow \text{ in } 2<n<\infty,\ \geqq 760.2 \qquad \text{for } n\geqq 2\,.\end{aligned}$$

Hence we have

$$W_6^{(3)}(x)>0 \qquad \text{for } 1+\frac{3}{10}n\leqq x\leqq\frac{5}{6}n \text{ with } n\geqq 2.$$

When $2\leqq n<4.93\cdots$, taking account of these facts we have

$$W_6'(\alpha)\geqq W_6'\Big(1+\frac{3}{10}n\Big)+W_6''\Big(1+\frac{3}{10}n\Big)\times\Big(\frac{1}{3}n-\frac{3}{10}n\Big)$$

$$=W_6'\Big(1+\frac{3}{10}n\Big)+\frac{n}{30}W_6''\Big(1+\frac{3}{10}n\Big)$$

$$=-\frac{1}{15}(143.1792n^7-281.55n^6-640.8648n^5-1025.2475n^4$$

$$+10571.26n^3-18697.45n^2+16160n-7200) \text{ is } \nearrow\searrow$$

in $2<n<\infty$, >0 for $2\leqq n<2.62\cdots$

and <0 for $2.62\cdots<n<4.93\cdots$,

which implies

$$W_6'(x)>0 \qquad \text{for } 1\leqq x\leqq n \text{ with } 2\leqq n<2.62\cdots,$$

and so

$$W_6(x)<0 \qquad \text{for } 1\leqq x\leqq n \text{ with } 2\leqq n<2.62\cdots.$$

Next, we have

$$W_6'\Big(1+\frac{3}{10}n\Big)=-(8.47368n^7-12.33n^6-49.15242n^5-94.3665n^4$$

$$+758.688n^3-1304.37n^2+1113.6n-480),$$

which is $\searrow$ in $2.7<n<\infty$, <0 for $2.711\cdots<n<\infty$. Therefore, for $n>2.711\cdots$ we consider the root β of $W_6'(x)=0$ in $1<x<\alpha$ and the graph of $W_6(x)$ as shown in Fig. 2. The x-sections of the tangent lines at $(1, W_6(1))$ and $(1+(3/10)n, W_6(1+(3/10)n))$ are

$$1-\frac{W_6(1)}{W_6'(1)} \quad \text{and} \quad 1+\frac{3}{10}n-\frac{W_6\Big(1+\frac{3}{10}n\Big)}{W_6'\Big(1+\frac{3}{10}n\Big)}$$

respectively. The condition:

$$-\frac{W_6(1)}{W_6'(1)}\geqq\frac{3}{10}n-\frac{W_6\Big(1+\frac{3}{10}n\Big)}{W_6'\Big(1+\frac{3}{10}n\Big)}$$

is equivalent to

$$W_6'(1)W_6\Big(1+\frac{3}{10}n\Big)\leqq W_6'\Big(1+\frac{3}{10}n\Big)\times\Big\{W_6(1)+\frac{3}{10}nW_6'(1)\Big\}.$$

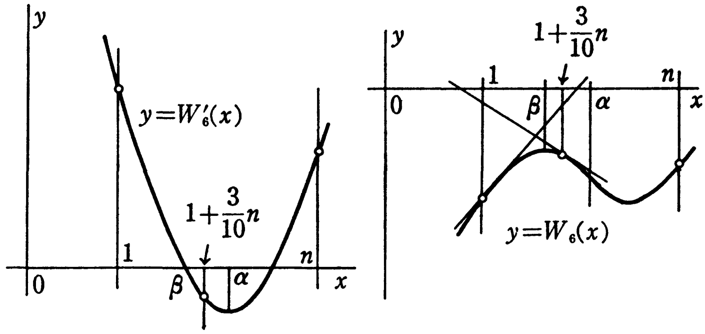


Fig. 2.

Since we have

$$W_6(1)=-160(n-1)^3(n^2-n+1),\quad W_6'(1)=480(n-1)^2(n^2-n+1),$$

the above condition becomes

$$30W_6\left(1+\frac{3}{10}n\right)\leqq(10-n)W_6'\left(1+\frac{3}{10}n\right).$$

Computing carefully we obtain

$$(10-n)W_6'\left(1+\frac{3}{10}n\right)-30W_6\left(1+\frac{3}{10}n\right)=n\times\Xi(n),$$

where

$$\Xi(n)=39.8142n^7-66.1248n^6-469.99275n^5+2853.7548n^4$$
$$-7804.674n^3+10341.24n^2-7957.5n+3264,$$

which is $\nearrow$ in $2.7<n<\infty$, $\geqq 3797.356219$ for $n\geqq 2.7$. Therefore, the above condition is satisfied for $n\geqq 2.7$. Hence, we obtain

$$W_6(x)<0 \qquad \text{for } 1<x<n \text{ with } n>2.711\cdots.$$

Finally, on the rest interval $2.62\cdots<n<2.711\cdots$, we can show $W_6(x)<0$ for $1\leqq x\leqq n$ by means of numerical computations by computors as before. Thus, we have proved

$$W_6(x, n)<0 \qquad \text{for } 0\leqq x\leqq n \text{ with } n\geqq 2,$$

which contradicts to (4.12). Q. E. D.

LEMMA 4.6. *$W(x, n)$ is $\searrow$ in $0<x<1$ with $n\geqq 2$.*

Proof. For simplicity, setting $n=m+1$, $x=1+y$, we obtain from (4.2) and (4.3)

$$W_4(1+y)=120m^3+20m^2(2m-11)y-4m(4m^2+19m-29)y^2$$
$$+(16m^2+38m-15)y^3-(2m+3)y^4,$$
$$W_5(1+y)=120m^3+20m^2(8m-11)y+4m(21m^2-59m+29)y^2$$
$$+3(8m^3-30m^2+28m-5)y^3+(4m^3-16m^2+15m-3)y^4$$
$$-m(2m+1)y^5.$$

By Lemma 4.3, we have

$$W(1)=W'(1)=W''(1)=W^{(3)}(1)=0,$$

and hence

$$W'(1+y)=\frac{1}{6}y^3W^{(4)}(1+\theta y), \qquad 0<\theta<1.$$

From $W(1+y)=-BW_4(1+y)+(m-y)^mW_5(1+y)$, we obtain

$$\frac{1}{B}W^{(4)}(1+y)=24(2m+3)+\left(1-\frac{y}{m}\right)^{m-4}$$
$$\times\left\{\left(1-\frac{y}{m}\right)^4W_5^{(4)}-4\left(1-\frac{y}{m}\right)^3W_5^{(3)}+\frac{6(m-1)}{m}\left(1-\frac{y}{m}\right)^2W_5''\right.$$
$$\left.-\frac{4(m-1)(m-2)}{m^2}\left(1-\frac{y}{m}\right)W_5'+\frac{(m-1)(m-2)(m-3)}{m^3}W_5\right\}$$
$$=24(2m+3)+\frac{1}{m^4}\left(1-\frac{y}{m}\right)^{m-4}\{8m^4(m-2)(m+2)^2$$
$$-4m^3(8m^4+29m^3+54m^2+4m-59)y$$
$$+4m^2(m+2)(21m^4+43m^3+68m^2+68m-38)y^2$$
$$-m(m+2)(m+3)(40m^4+114m^3+110m^3+123m-33)y^3$$
$$+(m+2)(m+3)(m+4)(4m^4+28m^3+19m^2+12m-3)y^4$$
$$-m(m+2)(m+3)(m+4)(m+5)(2m+1)y^5\}.$$

Since the polynomials of m:

$$8m^4+29m^3+54m^2+4m-59,\ 21m^4+43m^3+68m^2+68m-38,$$
$$40m^4+114m^3+110m^2+123m-33,\ 4m^4+28m^3+19m^2+12m-3$$

are all positive for $m\geqq 1$, we see easily that

$$W^{(4)}(1+y)>0 \qquad \text{for } -1\leqq y<0 \text{ with } m\geqq 2.$$

When $1\leqq m<2$, we have

$$0<\left(1-\frac{y}{m}\right)^{m-4}<1 \qquad \text{for } y<0,$$

and hence for $-1<y<0$

$$\frac{1}{B}W^{(4)}(1+y)>24(2m+3)+\frac{1}{m^4}\left(1-\frac{y}{m}\right)^{m-4}$$
$$\times 8m^4(m-2)(m+2)^2$$
$$>24(2m+3)+8(m-2)(m+2)^2$$
$$=8(m^3+2m^2+2m+1)>0.$$

Thus, we have proved

$$W'(x, n)<0 \qquad \text{for } 0\leqq x<1 \text{ with } n\geqq 2.$$

Q. E. D.

PROPOSITION 3. *$W(x, n)$ is positive for $0\leqq x\leqq n$, $x\neq 1$ and decreasing in $0<x<1$ with $n\geqq 2$.*

§ 5. Some properties of $S(x, n)$.

In this section, setting

$$(5.1) \qquad \begin{cases} a_0=n(4n-1)(4n-3), & a_1=24n^3-68n^2+42n-5, \\ a_2=8n^3+20n^2-51n+20, & a_3=8n^2-5 \end{cases}$$

and

$$(5.2) \qquad \begin{cases} b_0=n^2(2n-1), \quad b_1=n(16n^2-32n+9), \\ b_2=12n^3-42n^2+37n-5, \quad b_3=8n^3-52n^2+87n-40, \\ b_4=(n-1)(4n^2-10n+15)=4n^3-14n^2+15n-5, \end{cases}$$

we have

$$(5.3) \qquad \begin{cases} S_3(x, n)=4a_0+3a_1x-2a_2x^2+a_3x^3, \\ S_4(x, n)=12b_0+3b_1x+3b_2x^2+b_3x^3+b_4x^4, \\ S(x, n)=-BS_3(x, n)+(n-x)^{n-1}S_4(x, n). \end{cases}$$

LEMMA 5.1. *$S(0, n)>0$ and $S(n, n)>0$ for $n\geqq 2$.*

Proof. We have first

$$S(0, n)=-4Ba_0+12n^{n-1}b_0$$
$$=-4Bn(4n-1)(4n-3)+12n^{n+1}(2n-1)$$
$$=12n^2(2n-1)B\left\{e_{n-1}-\frac{(4n-1)(4n-3)}{3n(2n-1)}\right\}>0 \qquad \text{for } n\geqq 2$$

by (4.1). Next, we have

$$S(n, n)=-BS_3(n, n)=B(8n^5-32n^4+43n^3-22n^2+3n)$$
$$=Bn(n-1)^2(8n^2-16n+3)>0 \qquad \text{for } n\geqq 2.$$

Q. E. D.

LEMMA 5.2. *Near $x=1$, we have*

$$S(x, n)=\frac{Bn(2n-1)(n^2-n+5)}{6(n-1)}(x-1)^4+\cdots.$$

Proof. Setting $x=1+y$, we obtain from (5.1)-(5.3)

(5.4) $$S_3(1+y, n)=60(n-1)^2(2n-1)+10(n-1)(4n^2-22n+11)y$$
$$-(16n^3+16n^2-102n+55)y^2+(8n^2-5)y^3,$$

(5.5) $$S_4(1+y, n)=60(n-1)^2(2n-1)+10(n-1)(16n^2-40n+17)y$$
$$+3(28n^3-122n^2+154n-55)y^2$$
$$+3(8n^3-36n^2+49n-20)y^3$$
$$+(n-1)(4n^2-10n+5)y^4.$$

Hence, we obtain

$$\frac{1}{B}S(1+y, n)=-S_3(1+y, n)+\left\{1-y+\frac{n-2}{2(n-1)}y^2-\frac{(n-2)(n-3)}{6(n-1)^2}y^3\right.$$
$$\left.+\frac{(n-2)(n-3)(n-4)}{24(n-1)^3}y^4+\cdots\right\}\times S_4(1+y, n)$$
$$=\frac{n(2n-1)(n^2-n+5)}{6(n-1)}y^4+\cdots,$$

which implies this lemma. Q. E. D.

LEMMA 5.3. *$S_4(x, n)>0$ for $0\leqq x\leqq n$ with $n\geqq 2$.*

Proof. From (5.2) and (5.5) we obtain

(5.6) $$\begin{cases} S_4(0)=12n^2(2n-1)>0 & \text{for } n>1/2, \\ S_4(1)=60(n-1)^2(2n-1)>0 & \text{for } n>1, \\ S_4(n)=4n^7-6n^6-n^5+4n^4-n^3 & \\ \qquad =n^3(n-1)^2(4n^2+2n-1)>0 & \text{for } n>1. \end{cases}$$

We consider the quadratic polynomial of x:

$$\frac{1}{6}S_4''(x)=2b_4x^2+b_3x+b_2.$$

We have its discriminant

(5.7) $D(n)=b_3^2-8b_2b_4$

$$=-(320n^6-1856n^5+3232n^4-136n^3-5049n^2+4880n-1400)$$

which is $\nearrow\searrow$ in $2<n<\infty$, 124 at $n=2$ and

$$D(2.424)=0.5304\cdots, \qquad D(2.425)=-1.9072\cdots,$$

and hence $D(n)<0$ for $n>2.42\cdots$. Therefore, we have

$$S_4''(x)>0 \quad \text{for } -\infty<x<\infty \text{ with } n>2.42\cdots.$$

Next, we see that

$$-\frac{b_3}{4b_4}=-\frac{8n^3-52n^2+87n-40}{4(n-1)(4n^2-10n+5)} \text{ is } \searrow \text{ in } 2<n<\infty,$$

because

$$\left\{\frac{8n^3-52n^2+87n-40}{(n-1)(4n^2-10n+5)}\right\}'=\frac{3(32n^4-152n^3+266n^2-200n+55)}{(n-1)^2(4n^2-10n+5)^2}>0 \quad \text{for } n\geqq 2.$$

We consider the condition:

$$-\frac{b_3}{4b_4}>n, \qquad \text{i.e. } 4nb_4+b_3<0,$$

which is equivalent to

$$16n^4-48n^3+8n^2+67n-40<0.$$

Since the left hand side is $\nearrow$ in $2<n<\infty$, we see that this condition is equivalent to $2\leqq n<2.04\cdots$. Then, we consider the condition:

$$-\frac{b_3}{4b_4}<1, \qquad \text{i.e. } 4b_4+b_3>0,$$

which is equivalent to

$$24n^3-108n^2+147n-60>0.$$

Since the left hand side is $\nearrow$ in $2<n<\infty$, we see that this condition is equivalent to $n>2.33\cdots$.

On the other hand, we have

$$S_4''(0)=6b_2=6(12n^3-42n^2+37n-5)\nearrow \text{ in } 2<n<\infty,$$

$$<0 \quad \text{for } 2\leqq n<2.16\cdots \text{ and}$$

$$>0 \quad \text{for } n>2.16\cdots$$

and

$$S_4''(n)=6(8n^5-20n^4-10n^3+35n^2-3n-5)\nearrow \text{ in } 2<n<\infty,$$

$$<0 \quad \text{for } 2\leqq n<2.24\cdots \text{ and}$$

$$>0 \quad \text{for } n>2.24\cdots.$$

Taking account of these facts, we consider our subject in the following cases.

i) Case $2 \leqq n \leqq 2.16 \cdots$.

We see easily $S_4''(x) < 0$ for $0 \leqq x \leqq n$. Therefore the graph of $S_4(x)$ is convex upward and so (5.6) implies

$$S_4(x) > 0 \qquad \text{for } 0 \leqq x \leqq n .$$

ii) Case $2.16 \cdots < n \leqq 2.24 \cdots$.

$$S_4'(x) = 3b_1 + 6b_2 x + 3b_3 x^2 + 4b_4 x^3 \text{ is } \nearrow\searrow \text{ in } 0 < x < n .$$

We have

$$S_4'(0) = 3b_1 = 3n(16n^2 - 32n + 9) > 0 \qquad \text{for } n \geqq 2 ,$$

$$S_4'(n) = 16n^6 - 32n^5 - 24n^4 + 37n^3 + 6n^2 - 3n$$

$$= n(n-1)(16n^4 - 16n^3 - 40n^2 - 3n + 3) ,$$

and $16n^4 - 16n^3 - 40n^2 - 3n + 3$ is $\nearrow$ in $2 < n < \infty$,

$$< 0 \qquad \text{for } 2 < n < 2.17 \cdots \text{ and}$$

$$> 0 \qquad \text{for } n > 2.17 \cdots .$$

Therefore $S_4(x)$ is $\nearrow\searrow$ or $\nearrow$ in $0 < x < n$. Then, (5.6) also implies $S_4(x) > 0$ for $0 \leqq x \leqq n$.

iii) Case $2.42 \cdots < n < \infty$ (i. e. $D(n) < 0$).

$S_4'(x)$ is $\nearrow$ in $-\infty < n < \infty$ and so $S_4'(x) > 0$ for $x \geqq 0$, hence $S_4(x)$ is $\nearrow$ in $0 < x < n$, and $S_4(x) > 0$ for $0 \leqq x \leqq n$ by (5.6).

iv) Case $2.24 \cdots < n < 2.42 \cdots$.

$$S_4'(x) \text{ is } \nearrow\searrow\nearrow \text{ in } 0 < x < n .$$

Let α be the root of $S_4''(x) = 0$:

$$\alpha(n) = \frac{-b_3 + \sqrt{b_3 b_3 - 8 b_2 b_4}}{4 b_4} .$$

In this range of n, evaluating $\alpha(n)$ by computor, we presumed it as

$$17.9 - 7n < \alpha(n) < 18.1 - 7n \qquad \text{for } 2.2 \leqq n < 2.42 \cdots ,$$

and we have

$$\frac{1}{6} S_4''(17.9 - 7n) = 392n^5 - 3432.8n^4 + 11569.28n^3$$

$$-18561.28n^2 + 13992.6n - 3925.1 \text{ is } \nearrow\searrow\nearrow$$

$$\text{in } 2.2 < n < 2.4 \text{ and } \nearrow \text{ in } 2.4 < n < \infty ,$$

$$< 0 \qquad \text{for } 2.2 \leqq n < 2.422 \cdots$$

and

$$\frac{1}{6}S_4''(18.1-7n)=392n^5-3455.2n^4+11706.88n^3$$
$$-18857.28n^2+14254n-4005.1 \text{ is } \nearrow\searrow\nearrow$$
$$\text{in } 2.2<n<2.4 \text{ and } \nearrow \text{ in } 2.4<n<\infty,$$
$$>0 \quad \text{for } n\geqq 2.2.$$

Hence we obtain

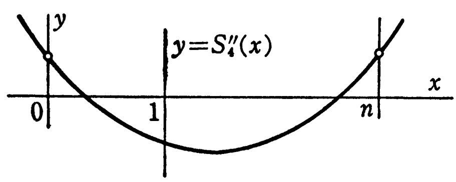
$$(5.8)\qquad \begin{cases} 17.9-7n<\alpha(n) & \text{for } 2.2\leqq n<2.422\cdots, \\ \alpha(n)<18.1-7n & \text{for } 2.2\leqq n<2.42\cdots \end{cases}$$

(here, $2.42\cdots$ means the root of $D(n)=0$). Regarding the symmetric axis of the quadratic polynomial $2b_4x^2+b_3x+b_2$, the condition:

$$-\frac{b_3}{4b_4}=-\frac{8n^3-52n^2+87n-40}{4(n-1)(4n^2-10n+5)}<17.9-7n$$

is equivalent to

$$112n^4-686.4n^3+1474.7n^2-1301n+398<0.$$



The left hand side is $\searrow\nearrow$ in $2.2<n<2.4$ and $\nearrow$ in $2.4<n<\infty$, and negative for $2.2\leqq n\leqq 2.43$. Hence we see that

$$(5.9)\qquad -\frac{b_3}{4b_4}<17.9-7n \qquad \text{for } 2.2\leqq n\leqq 2.43.$$

Since we have

$$S_4'(17.9-7n)=-5488n^6+62484.8n^5-289752.96n^4$$
$$+695978.824n^3-904952.944n^2+596602.35n-153692.98,$$

taking account of (5.8) and (5.9), we obtain

$$S_4'(\alpha)>S_4'(17.9-7n)+0.2\times S_4''(17.9-7n)$$
$$=-5488n^6+62955.2n^5-293872.32n^4+709861.96n^3$$
$$-927226.48n^2+613393.47n-158403.1:=\mu(n).$$

Since $\mu^{(4)}(n)<0$ for $2.24<n<\infty$, and $\mu''(n)<0$ for $2.24<n<\infty$, hence $\mu(n)$ is $\nearrow\searrow$ in $2.24<n<\infty$. Since $\mu(2.24)=50.32\cdots$ and $\mu(2.42)=197.31\cdots$, and so $\mu(n)>0$ for $2.24\leqq n\leqq 2.42$, which implies $S_4'(\alpha)>0$. Since $S_4'(0)=3b_1>0$, we have $S_4'(x)>0$ for $0\leqq x\leqq n$. Hence, we obtain also

$$S_4(x)>0 \qquad \text{for } 0\leqq x\leqq n\,.$$

Regarding the rest very narrow range $2.42<n<2.42\cdots$ (the root of $D(n)=0$), we can drow the same result by numerical treatment by computor. Thus, we have proved

$$S_4(x,\ n)>0 \qquad \text{for } 0\leqq x\leqq n \text{ with } n\geqq 2\,.$$

Q. E. D.

PROPOSITION 4. *We have*

$$S(x,\ n)=-BS_3(x,\ n)+(n-x)^{n-1}S_4(x,\ n)>0$$
$$\textit{for } 0\leqq x\leqq n,\ x\neq 1 \textit{ with } n\geqq 2\,.$$

Proof. If the above inequality does not hold, then there exist ξ_1, ξ_2 and η_1, η_2 such that

$$(5.10)\qquad \begin{cases} 0<\xi_1\leqq\xi_2<1 \quad\text{and}\quad 1<\eta_1\leqq\eta_2<n \\ S(\xi_i)=S(\eta_i)=0, \qquad i=1,\,2\,, \\ S'(\xi_1)\leqq 0,\quad S'(\xi_2)\geqq 0 \quad\text{and}\quad S'(\eta_1)\leqq 0,\quad S'(\eta_2)\geqq 0 \end{cases}$$

by Lemma 5.1 and Lemma 5.2. By Lemma 5.3, the last conditions are equivalent to

$$(n-1)S_3(x)S_4(x)-(n-x)(S_3(x)S_4'(x)-S_3'(x)S_4(x))$$
$$\geqq 0 \text{ at } x=\xi_1,\ \eta_1 \quad\text{and}\quad \leqq 0 \text{ at } x=\xi_2,\ \eta_2\,.$$

By (5.1)-(5.3), working out the computation very carefully, we obtain

$$\begin{aligned}
&(n-1)S_3S_4-(n-x)(S_3S_4'-S_3'S_4)\\
&=n(n-1)[n^3(192n^2-192n+192)\\
&\quad -n^2(576n^3-192n^2-288n+624)x\\
&\quad +n(480n^4+768n^3-1680n^2+720n+660)x^2\\
&\quad +(160n^5-1600n^4+1240n^3+1540n^2-1550n-225)x^3\\
&\quad -(480n^5-1120n^4-680n^3+2720n^2-640n-700)x^4\\
&\quad +(288n^5-288n^4-1032n^3+1080n^2+780n-750)x^5\\
&\quad -(64n^5+32n^4-392n^3+16n^2+580n-300)x^6\\
&\quad +(32n^4-80n^3+20n^2+50n-25)x^7]\\
&=n(n-1)(x-1)^3T_4(x,\ n)\,,
\end{aligned}$$

where

$$(5.11)\qquad T_4(x,\ n)=c_4x^4-c_3x^3+12c_2x^2+48c_1x-192c_0\,,$$

$$(5.12)\qquad \begin{cases} c_4=32n^4-80n^3+20n^2+50n-25\,, \\ c_3=64n^5-64n^4-152n^3-44n^2+430n-225\,, \\ c_2=n(8n^4-16n^3-28n^2+96n-55)\,, \\ c_1=n^2(8n^2-18n+13), \qquad c_0=n^3(n^2-n+1)\,. \end{cases}$$

From the above relation, the last relations of (5.10) are equivalent to

$$(5.13)\qquad T_4(\xi_1)\leqq 0,\quad T_4(\xi_2)\geqq 0 \quad\text{and}\quad T_4(\eta_1)\geqq 0,\quad T_4(\eta_2)\leqq 0\,.$$

In the following, we shall prove the inequality:

$$T_4(x,\,n)<0 \qquad \text{for } 0\leqq x\leqq n \text{ with } n\geqq 2\,.$$

We see easily the following:

$$(5.14)\qquad \begin{cases} T_4(0)=-192c_0<0 \qquad \text{for } n>0\,, \\ T_4(1)=-40(4n^5-12n^4+33n^3-46n^2+26n-5) \\ \qquad =-40(n-1)(2n-1)(2n^3-3n^2+11n-5)<0 \quad \text{for } n>1\,, \\ T_4(n)=-n^3(n-1)(32n^4-48n^3-28n^2+22n-3)<0 \quad \text{for } n\geqq 2\,, \end{cases}$$

and $c_4=32n^4-80n^3+20n^2+50n-25$ is ↗ in $2<n<\infty$, $\geqq 27$ for $n\geqq 2$. Now, we consider the quadratic polynomial of x:

$$\frac{1}{6}T_4''(x)=2c_4x^2-c_3x+4c_2\,.$$

Since we have

$$c_2=n(8n^4-16n^3-28n^2+96n-55)\nearrow \text{ in } 2<n<\infty,$$
$$\geqq 50 \qquad \text{for } n\geqq 2\,,$$
$$\frac{1}{6}T_4''(1)=-32n^5+64n^4-120n^3+468n^2-550n+175$$
$$\searrow \text{ in } 2<n<\infty,\ \leqq -13 \qquad \text{for } n\geqq 2\,,$$
$$\frac{1}{6}T_4''(n)=-n(64n^4-128n^3-32n^2+96n-5)$$
$$\searrow \text{ in } 2<n<\infty,\ \leqq -118 \qquad \text{for } n\geqq 2\,,$$

we see that $T_4''(x)<0$ for $1\leqq x\leqq n$ and

$$T_4'(x)=4c_4x^3-3c_3x^2+24c_2x+48c_1$$
$$\nearrow\searrow \text{ in } 0<x<1$$
$$\text{and } \searrow \text{ in } 1<x<n \text{ with } n\geqq 2\,.$$

Then, we have

$$(5.15)\quad \begin{cases} T_4'(0)=48c_1=48n^2(8n^2-18n+13)>0 \qquad \text{for } n\geqq 2\,, \\ T_4'(1)=5(64n^4-280n^3+628n^2-482n+115) \\ \qquad \nearrow \text{ in } 2<n<\infty,\ \geqq 2235 \qquad \text{for } n\geqq 2\,, \\ T_4'(n)=-n^2(64n^5-64n^4-152n^3-44n^2-50n+21) \\ \qquad \searrow \text{ in } 2<n<\infty,\ >0 \qquad \text{for } 2\leqq n<2.24\cdots \\ \qquad \text{and } <0 \qquad \text{for } n>2.24\cdots\,. \end{cases}$$

Therefore, we have

$$T_4'(x)>0 \qquad \text{for } 0\leqq x\leqq 1 \text{ with } n\geqq 2\,,$$

and hence

$$T_4(x)<0 \qquad \text{for } 0\leqq x\leqq 1 \text{ with } n\geqq 2$$

by (5.14). When $2\leqq n\leqq 2.24\cdots$, we see that

$$T_4'(x)>0 \qquad \text{for } 1<x<n$$

and $T_4(x)$ is $\nearrow$ in $1<x<n$. Thus we obtain

$$T_4(x)<0 \qquad \text{for } 1\leqq x\leqq n\,.$$

When $2.24\cdots<n<\infty$, we see that

$$T_4(x) \nearrow\searrow \text{ in } 1<x<n \text{ and convex upward}.$$

Let α_1 and α_n be the x-coordinate sections of the tangent lines of the graph of $T_4(x)$ at $(1, T_4(1))$ and $(n, T_4(n))$, respectively. Since

$$\alpha_1=1-\frac{T_4(1)}{T_4'(1)}, \qquad \alpha_n=n-\frac{T_4(n)}{T_4'(n)},$$

we have

$$\begin{aligned} \alpha_1-\alpha_n &= \frac{T_4(n)}{T_4'(n)}-\frac{T_4(1)}{T_4'(1)}-(n-1) \\ &= \frac{n(n-1)(32n^4-48n^3-28n^2+22n-3)}{64n^5-64n^4-152n^3-44n^2-50n+21} \\ &\quad +\frac{8(n-1)(4n^4-8n^3+25n^2-21n+5)}{64n^4-280n^3+628n^2-482n+115}-(n-1) \\ &= \frac{n-1}{2}\left[-\frac{32n^4-96n^3-88n^2-44n+21}{64n^5-64n^4-152n^3-44n^2-50n+21}\right. \\ &\quad \left.+\frac{152n^3-228n^2+146n-35}{64n^4-280n^3+628n^2-482n+115}\right]. \end{aligned}$$

Since the denominators of the both fractional expressions in the brackets are positive for $n>2.24\cdots$, the condition: $\alpha_1 \geqq \alpha_n$ is equivalent to

$$(64n^5-64n^4-152n^3-44n^2-50n+21)(152n^3-228n^2+146n-35)$$
$$-(64n^4-280n^3+628n^2-482n+115)(32n^4-96n^3-88n^2-44n+21)\geqq 0\,,$$

i. e.

$$1280n^8-1536n^7-6752n^6+11712n^5-4312n^4$$
$$+2604n^3-5804n^2+3333n-525\geqq 0\,,$$

whose left hand side is $\nearrow$ in $2<n<\infty$, 8493 at $n=2$. Hence, we see that $\alpha_1>\alpha_n$ for $n>2.24\cdots$, which implies

$$T_4(x)<0 \qquad \text{for } 1\leqq x\leqq n\,.$$

thus, we have proved $T_4(x)<0$ for $0\leqq x\leqq n$ with $n\geqq 2$. Therefore (5.13) contradicts this fact, which implies this proposition. Q. E. D.

The following proposition will be proved also in [24].

PROPOSITION 5. *$S(x, n)$ is decreasing in $0<x<1$ with $n\geqq 2$ and increasing in $1<x<n$ with $2\leqq n\leqq (11+\sqrt{77})/4=4.9437410\cdots$.*

§ 6. Proof of Main Theorem.

All the facts expected to obtain $\partial V(x, x_1)/\partial x>0$ described at the end of § 3 are satisfied by Propositions 3, 4 and 5 for $2\leqq n\leqq (11+\sqrt{77})/4$.

Regarding $\Psi(x, n)$ defined by (3.13), we have

$$\Psi(x, n)=\frac{x\sqrt{n-x}}{(n-1)(1-x)^5}\Big[-\frac{(n-1)x}{(1-x)^2}\{\tilde{\lambda}(X)-\tilde{\lambda}(x)\}W(x, n)$$
$$+S(x, n)+\Big(\frac{1-x}{X-1}\Big)^5\frac{X\sqrt{n-X}}{x\sqrt{n-x}}S(X, n)$$
$$=\frac{x\sqrt{n-x}}{(n-1)(1-x)^5}\Phi(x, n),$$

where $X=X_n(x)$ and

$$\Phi(x, n)=-\frac{(n-1)x}{(1-x)^2}\{\tilde{\lambda}(X)-\tilde{\lambda}(x)\}W(x, n)$$
$$+S(x, n)+\Big(\frac{1-x}{X-1}\Big)^5\Big(\frac{n-x}{n-X}\Big)^{n-3/2}S(X, n). \tag{6.1}$$

In the following, we shall prove $\Phi(x, n)>0$ for $0<x<1$ with $2\leqq n<(11+\sqrt{77})/4$ by means of numerical evaluation by computors partially, which implies our Main Theorem.

LEMMA 6.1. *We have*

$$\text{(i)}\quad \lim_{x\to 0} \Phi(x, n)=\infty ,$$

$$\text{(ii)}\quad \Phi(x, n) \text{ is } \searrow 0 \text{ near } x=1 .$$

Proof. (i) Since we have

$$\lim_{x\to 0} x\{\tilde{\lambda}(X)-\tilde{\lambda}(x)\}=\lim_{x\to 0}\frac{\tilde{\lambda}(X)-\tilde{\lambda}(x)}{1/x}$$

$$=\lim_{x\to 0}\frac{1}{-1/x^2}\frac{n(1-x)}{(n-1)x(n-x)}\left(\frac{1}{X}-\frac{1}{x}\right)$$

$$=\lim_{x\to 0}\frac{n(1-x)}{(n-1)(n-x)}\left(1-\frac{x}{X}\right)=\frac{1}{n-1}$$

by Lemma 4.1 in (IX), we obtain

$$\lim_{x\to 0}\Phi(x, n)=-W(0, n)+S(0, n)+S(n, n)\lim_{x\to 0}\left(\frac{n-x}{n-X}\right)^{n-3/2}=\infty .$$

(ii) By Lemma 7.1 in (III) we have

$$\tilde{\lambda}(x)=\tilde{\lambda}(1)-\frac{2n-1}{2(n-1)^2}(x-1)^2+\frac{3n^2-6n+2}{3(n-1)^3}(x-1)^3+\cdots .$$

Setting $x=1-t$ and $X=1+s$, we obtain

$$\tilde{\lambda}(X)-\tilde{\lambda}(x)=-\frac{2n-1}{2(n-1)^2}(s^2-t^2)+\frac{3n^2-6n+2}{3(n-1)^3}(s^3+t^3)+\cdots$$

$$=-\frac{2n-1}{2(n-1)^2}\left(\frac{4(n-2)}{3(n-1)}t^3+\cdots\right)+\frac{3n^2-6n+2}{3(n-1)^3}(2t^3+\cdots)+\cdots$$

$$=\frac{2n}{3(n-1)^2}t^3+\cdots$$

by Lemma 3.1. Therefore, by Lemma 4.3 and Lemma 5.2 we obtain

$$\Phi(x, n)=-(n-1)\cdot\frac{2n}{3(n-1)^2}\cdot\frac{B}{3}n(n^2-n+1)t^5+\cdots$$

$$+\left(\frac{Bn(2n-1)(n^2-n+5)}{6(n-1)}t^4+\cdots\right)$$

$$+\left(\frac{Bn(2n-1)(n^2-n+5)}{6(n-1)}s^4+\cdots\right),$$

which implies (ii). Q.E.D.

Since we have $\lim_{x\to 0}\tilde{\lambda}(X)-\tilde{\lambda}(x)=\infty$ by Lemma 4.1 in (IX), we arrange the following lemma to make easy evaluations at the both ends of $0\leqq x\leqq 1$.

LEMMA 6.2. *We have*

$$x^2\{\tilde{\lambda}(X)-\tilde{\lambda}(x)\}<\frac{k_n}{n-1}x(1-x)^3 \qquad for\ 0<x<1,$$

where $X=X_n(x)$, $k_n=n/2$ *for* $n\geqq 3$ *and* $=n^2/3(n-1)$ *for* $2\leqq n<3$.

Proof. Considering the following expression:

$$\frac{x^2\{\tilde{\lambda}(X)-\tilde{\lambda}(x)\}}{x(1-x)^3}=\frac{\tilde{\lambda}(X)-\tilde{\lambda}(x)}{\frac{1}{x}(1-x)^3},$$

and noticing Cauchy's mean value theorem and a formula in the proof of Lemma 4.1 in (IX) we consider

$$\frac{\frac{n(1-x)}{(n-1)x(n-x)}\left(\frac{1}{X}-\frac{1}{x}\right)}{-\frac{1}{x^2}+3-2x}=\frac{n}{n-1}\cdot\frac{1}{(n-x)(2x+1)X}\cdot\frac{X-x}{1-x}.$$

By Proposition 4 in (IV), $(X-1)/(1-x)$ is ↘ 1 as $x\to 1$, therefore the above expression

$$<\frac{n^2}{n-1}\cdot\frac{1}{(n-x)(2x+1)X}<\frac{n^2}{n-1}\cdot\frac{1}{(n-x)(2x+1)(2-x)}.$$

On the other hand, the cubic polynomial of x:

$$(n-x)(2x+1)(2-x)=2n+(3n-2)x-(2n+3)x^2+2x^3$$

is ↗↘ in $0<x<1$ and equal to $2n$ at $x=0$ and $3(n-1)$ at $x=1$. Therefore, we obtain

$$\frac{x^2\{\tilde{\lambda}(X)-\tilde{\lambda}(x)\}}{x(1-x)^3}<\frac{k_n}{n-1} \qquad \text{for } 0<x<1.$$

Q. E. D.

Remark. Regarding the expressions in $\Phi(x, n)$, we have the following facts.

(i) $\frac{x}{(1-x)^2}$ is ↗ in $0<x<1$;

(ii) $\tilde{\lambda}(X)-\tilde{\lambda}(x)$ is ↘ in $0<x<1$, from ∞ to 0 (Lemma 4.1 in (IX));

(iii) $\frac{1-x}{X-1}$ is ↗ in $0<x<1$, from $\frac{1}{n-1}$ to 1;

(iv) $\left(\frac{n-x}{n-X}\right)^{n-3/2}$ is ↘ in $0<x<1$, from ∞ to 1.

We computed the values of $\Phi(x, n)$ for $0<x<1$ and $2\leqq n<(11+\sqrt{77})/4$ with step 1/100 by a personal computor, fully taking account of Propositions 3, 4, 5, Lemmas 6.1, 6.2 and the above Remark, and obtained the inequality

$$\Phi(x, n)>0$$

for these values of x and n, from which we can have the same inequality for all pairs of x and n of the above intervals of x and n, thus we have proved the following

PROPOSITION 6. *We have*

$$\Psi(x, n)>0 \qquad \text{for } 0<x<1 \text{ and } 2\leqq n\leqq\frac{11+\sqrt{77}}{4}.$$

By means of the arguments developed in §2 and §3, we have proved the following

PROPOSITION 7. *Regarding* $\rho(x, x_1)$ *defined by* (2.9) *we have*

$$\rho(x, x_1)>0 \qquad \text{for } x_0=X_n^{-1}(x_1)\leqq x<1 \text{ with } 2\leqq n\leqq\frac{11+\sqrt{77}}{4}.$$

Thus, we have proved Main Theorem, combining the results in (IV)-(IX).

MAIN THEOREM. *The period T as a function of $\tau=(x_1-1)/(n-1)$ and n is monotone decreasing with respect to $n(\geqq 2)$ for any fixed $\tau(0<\tau<1)$.*

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