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ON MEROMORPHIC FUNCTIONS WITH REGIONS FREE OF POLES AND ZEROS

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Introduction. In this note we improve one of the results of Edrei and Fuchs [2]. We shall adopt the terminology, notations and conventions of [2]. We shall write, for instance, [2, Lemma 5] to denote Lemma 5 of [2].

The aim of this investigation is to prove the following

THEOREM. *Let the s B-regular curves*

$$L_j:\ z=te^{i\alpha_j(t)}\quad (t\geqq t_0>0,\quad j=1, 2, \cdots, s;$$
$$\alpha_1(t)<\alpha_2(t)<\cdots<\alpha_s(t)<\alpha_1(t)+2\pi=\alpha_{s+1}(t))$$

divide $|z|\geqq t_0$ *into s sectors, each of which has opening* $\geqq c>0$.

Suppose that all but a finite number of zeros and poles of the meromorphic function $f(z)$ *lie on the curves* L_j.

If some τ $(\tau\neq 0,\ \tau\neq\infty)$ *is a deficient value (in the sense of R. Nevanlinna) of the function* $f(z)$, *then the order* λ *of* $f(z)$ *does not exceed* λ_1, *where*

$$\lambda_1=\begin{cases}\dfrac{eB(B+1)}{2\sin\dfrac{c}{4}}-1 & \left(\sin\dfrac{c}{4}>\dfrac{2B+1}{2(B+1)}\right),\\[2ex] \dfrac{eB\left(\sqrt{B^2+\sin^2\dfrac{c}{4}}+\sin\dfrac{c}{4}\right)}{2\sin\dfrac{c}{4}}-1 & \left(\sin\dfrac{c}{4}\leqq\dfrac{2B+1}{2(B+1)}\right).\end{cases}$$

Proof of Theorem.

1. [2, Theorem 2] implies that λ is finite. We prove Theorem by deducing from the assumption

$$(+\infty>)\lambda>\lambda_1 \tag{1.1}$$

the contradiction that $f(z)$ is a constant.

Choose a positive number ε such that $\lambda-\varepsilon>\lambda_1$. Using [1, Lemma 1] with $\phi(t)=T(t, f)/t^{\lambda-\varepsilon}$ and $\psi(t)=t^{2\varepsilon}$, we find a sequence $\{r_n\}_1^\infty$ of values tending to infinity such that

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$$\frac{T(t, f)}{T(r_n, f)} \leqq \left(\frac{t}{r_n}\right)^{\lambda-\varepsilon} \quad (t_0 \leqq t \leqq r_n) \tag{1.2}$$

and

$$\frac{T(t, f)}{T(r_n, f)} \leqq \left(\frac{t}{r_n}\right)^{\lambda+\varepsilon} \quad (t \geqq r_n). \tag{1.3}$$

In view of (1.2), $\limsup_{n\to\infty} T(r_n, f)/r_n^{\lambda-\varepsilon} = +\infty$, so by relabelling suitably if necessary, we may assume that

$$\frac{T(r_n, f)}{r_n^{\lambda-\varepsilon}} \geqq 1 \quad (n=1, 2, 3, \cdots). \tag{1.4}$$

Since τ is a deficient value of $f(z)$, there is at least one index $k=k(r)$ such that

$$m\left(r, \frac{1}{f-\tau}; J_k(r)\right) > \kappa T(r, f) \quad \left(r > t_1(\geqq t_0); \ \kappa = \frac{\delta(\tau, f)}{s+1}\right).$$

When $r \to \infty$ through the values of the sequence $\{r_n\}_1^\infty$ satisfying (1.2)-(1.4), at least one value of $k(r)$ must be taken infinitely often. Without loss of generality we may assume it to be $k=1$. Thus by relabelling appropriately again if necessary, we may assume that

$$m\left(r_n, \frac{1}{f-\tau}; J_1(r_n)\right) > \kappa T(r_n, f) \quad (n=1, 2, 3, \cdots). \tag{1.5}$$

Now, we apply [2, Lemma C] to the function $(f-\tau)^{-1}$ with $R'=2r$ and $I(r)=I_1(r, 2\delta)$ $(0<\delta<c/4)$. This yields

$$m\left(r, \frac{1}{f-\tau}; I_1(r, 2\delta)\right) \leqq 22T\left(2r, \frac{1}{f-\tau}\right) 4\delta\left(1+\log^+\frac{1}{4\delta}\right).$$

Using the first fundamental theorem and (1.3), we have

$$m\left(r_n, \frac{1}{f-\tau}; I_1(r_n, 2\delta)\right) \leqq 90\cdot 2^{\lambda+\varepsilon} T(r_n, f)\delta\left(1+\log^+\frac{1}{4\delta}\right) < \frac{\kappa}{2} T(r_n, f)$$

provided $0<\delta<\delta_1 \equiv \delta_1(\kappa, \lambda, \varepsilon) < c/4$, $n \geqq n_0$. Hence, by (1.5)

$$m\left(r_n, \frac{1}{f-\tau}; J_1(r_n, 2\delta)\right) > \frac{\kappa}{2} T(r_n, f) \quad (n \geqq n_0, \ 0<\delta<\delta_1).$$

Combining this and [2, Lemma B], we obtain

$$\begin{aligned} m(r_n, f/f', J_1(r_n, 2\delta)) &> m(r_n, 1/(f-\tau); J_1(r_n, 2\delta)) - m(r_n, f'/f) \\ &\quad - m(r_n, f'/(f-\tau)) - O(1) > \frac{\kappa}{2} T(r_n, f) - O(\log r_n) > \frac{\kappa}{3} T(r_n, f) \\ &\qquad (n \geqq n_1(\geqq n_0), \ 0<\delta<\delta_1). \end{aligned} \tag{1.6}$$

2. By the definition of λ_1 and the inequality $\lambda-\varepsilon>\lambda_1$, we are able to choose $\delta\in(0, \delta_1)$ so that if $\sin(c/4)>(2B+1)/2(B+1)$, then

$$\lambda-\varepsilon>\frac{eB(B+1)}{2\sin\left(\frac{c}{4}-\delta\right)}-1,\quad \sin\left(\frac{c}{4}-\delta\right)\geqq\frac{2B+1}{2(B+1)}, \tag{2.1}$$

and if $\sin(c/4)\leqq(2B+1)/2(B+1)$, then

$$\lambda-\varepsilon>\frac{eB\left\{\sqrt{B^2+\sin^2\left(\frac{c}{4}-\delta\right)}+\sin\left(\frac{c}{4}-\delta\right)\right\}}{2\sin\left(\frac{c}{4}-\delta\right)}-1. \tag{2.1$'$}$$

From now on we assume that δ has been chosen in this way and we shall make no further changes in the choice of δ. Using [2, Lemma 4] with $H=1$, $q=0$, and $R'=2r$, we have

$$|f'(z)/f(z)|<A\{T(2r, f)\}^A \quad (|z|=r)$$

on $D_1=\{z=re^{i\theta};\ (t_1\leqq)t_2\leqq r<+\infty,\ \alpha_1(r)+\delta\leqq\theta\leqq\alpha_2(r)-\delta\}$. Therefore, since $f(z)$ is of finite order, we can find a positive integer $h=h(\lambda)$ such that

$$|z^{-h}f'(z)/f(z)|<1 \quad (z\in D_1). \tag{2.2}$$

The function $g(z)\equiv z^{-h}f'(z)/f(z)$ is regular in D_1. By (1.6)

$$m(r_n, 1/g;\ J_1(r_n, 2\delta))>\frac{\kappa}{3}T(r_n, f) \quad (n\geqq n_2(\geqq n_1)).$$

It follows from this and [2, Lemma 5] with $\beta=1/40B$ that

$$\log|g(z)|<-KT(r_n, f) \quad (z\in\Gamma_n,\ n\geqq n_3(\geqq n_2)), \tag{2.3}$$

where $\Gamma_n=\{z=r_ne^{i\theta};\ \theta\in J_1(r_n, 2\delta)\}$ and $K=K(\kappa, B, \delta)$ is a positive constant. Since $\log|g(z)|$ is subharmonic on $D_1(r_n)=\{z=re^{i\theta};\ t_2\leqq r\leqq r_n,\ \theta\in J_1(r, 2\delta)\}$, from (2.2) and (2.3) we have

$$\log|g(z)|<-K\omega(z)T(r_n, f) \quad (z\in D_1(r_n),\ n\geqq n_3), \tag{2.4}$$

where $\omega(z)$ is the harmonic measure of Γ_n with respect to $D_1(r_n)$ at the point z.

3. We denote by $\mathcal{L}_n$ the B-regular path $z(t)=te^{i\left(\alpha_1(t)+\frac{c}{2}\right)}$ $(2t_2\leqq t\leqq r_n)$, and by $s=s(z(t))$ the arc length of $\mathcal{L}_n$ from $z(2t_2)$ to $z(t)$ $(2t_2\leqq t\leqq r_n)$. Let $\rho(s)$ be the shortest distance from $\partial D_1(r_n)\setminus\Gamma_n$ to the point $z(t)\in\mathcal{L}_n$, and let $m(s)$ stand for the minimum of $\omega(z)$ in $\{z\in D_1(r_n);\ |z-z(t)|\leqq\rho(s)/e\}$.

If we take a point $z(t)$ $(\neq z(2t_2))$ on $\mathcal{L}_n$, then for a sufficiently small $\Delta t>0$ we have

$$m(s-\Delta s)\geqq m(s)\log\frac{\rho(s)}{\Delta r+\rho(s-\Delta s)/e},$$

where $\Delta s=s(z(t))-s(z(t-\Delta t))$ (>0) and $\Delta r=|z(t)-z(t-\Delta t)|$. (For the proof, confer [3, p. 82].) Therefore

$$m(s-\Delta s)-m(s)\geqq m(s)\log\frac{\rho(s)/e}{\Delta r+\rho(s-\Delta s)/e}$$
$$=-m(s)\log\left\{1+\frac{\rho(s-\Delta s)-\rho(s)}{\rho(s)}+e\frac{\Delta r}{\rho(s)}\right\}$$
$$\geqq-\frac{m(s)}{\rho(s)}\{\rho(s-\Delta s)-\rho(s)+e\Delta r\},$$

thus

$$\frac{m(s-\Delta s)-m(s)}{-\Delta s}\leqq-\frac{m(s)}{\rho(s)}\left\{\frac{\rho(s-\Delta s)-\rho(s)}{-\Delta s}-e\frac{\Delta r}{\Delta s}\right\}. \tag{3.1}$$

Clearly $\Delta s\to 0$, $\Delta r/\Delta s\to 1$ as $\Delta t\to+0$. Hence by taking the limit as $\Delta t\to+0$, (3.1) yields the upper bound

$$\frac{dm}{ds}\leqq-\frac{m(s)}{\rho(s)}\left(\frac{d\rho}{ds}-e\right)\quad\text{a. e. in } 0<s\leqq s(z(r_n))\equiv s_n \tag{3.2}$$

for the left upper derivative $\frac{dm}{ds}$, since $\rho(s)$ is absolutely continuous in $0\leqq s\leqq s_n$. If we divide the both sides of (3.2) by $m(s)$, and then integrate between $s=s(z(t))$ and s_n, we obtain

$$\log\frac{m(s_n)}{m(s)}\leqq-\log\frac{\rho(s_n)}{\rho(s)}+e\int_{s(z(t))}^{s_n}\frac{ds}{\rho(s)},\quad\text{i. e.}$$

$$m(s)\geqq\frac{\rho(s_n)}{\rho(s)}m(s_n)\exp\left\{-e\int_{s(z(t))}^{s_n}\frac{ds}{\rho(s)}\right\}.$$

Here we can use [3, Theorem 1, (2.6), p. 74] to show $m(s_n)\geqq(2/\pi)\tan^{-1}(1/\pi)>1/2\pi$. Consequently, we have

$$\omega(z(t))\geqq m(s)>\frac{1}{2\pi}\frac{\rho(s_n)}{\rho(s)}\exp\left\{-e\int_{s(z(t))}^{s_n}\frac{ds}{\rho(s)}\right\}\quad(2t_2<t\leqq r_n). \tag{3.3}$$

4. Let $L: z=te^{i\alpha(t)}$ $(t\geqq t_0)$ be the parametric equation of a B-regular curve. In this section we show that the point $te^{i(\alpha(t)+\gamma)}$ $(0<|\gamma|\leqq\pi)$ is at a distance

$$d\geqq t\cdot l(B,\gamma)\equiv\begin{cases}\dfrac{2t\left|\sin\frac{\gamma}{2}\right|}{B+1} & \left(\left|\sin\frac{\gamma}{2}\right|\geqq\dfrac{2B+1}{2(B+1)}\right)\\[2ex] \dfrac{2t\left|\sin\frac{\gamma}{2}\right|\left\{\sqrt{B^2+\sin^2\frac{\gamma}{2}}-\left|\sin\frac{\gamma}{2}\right|\right\}}{B^2} & \left(\left|\sin\frac{\gamma}{2}\right|<\dfrac{2B+1}{2(B+1)}\right)\end{cases} \tag{4.1}$$

from L.

4.1. Proof of $d \geqq 2t\left|\sin\frac{\gamma}{2}\right| \Big/ (B+1)\ (\equiv \rho)$: If this estimate were not true, it would be possible to find $u\ (\geqq t_0)$ and $t\ (\geqq t_0)$ such that

$$|te^{i(\alpha(t)+\gamma)} - ue^{i\alpha(u)}| < \rho. \tag{4.2}$$

In particular, this implies

$$|t-u| < \rho. \tag{4.3}$$

By the triangle inequality, the definition of B-regular curve, (4.2) and (4.3)

$$\begin{aligned} 2t\left|\sin\frac{\gamma}{2}\right| - \rho &< |te^{i\alpha(t)} - te^{i(\alpha(t)+\gamma)}| - |te^{i(\alpha(t)+\gamma)} - ue^{i\alpha(u)}| \\ &\leqq |te^{i\alpha(t)} - ue^{i\alpha(u)}| \leqq B|t-u| < B\rho. \end{aligned}$$

Hence we have $\rho > 2t\left|\sin\frac{\gamma}{2}\right| \Big/ (B+1) = \rho$, a contradiction.

4.2. Proof of $d \geqq 2t\left|\sin\frac{\gamma}{2}\right|\left\{\sqrt{B^2+\sin^2\frac{\gamma}{2}} - \left|\sin\frac{\gamma}{2}\right|\right\} \Big/ B^2$: In the case $B=1$ this estimate follows from

$$d \geqq \begin{cases} t|\sin\gamma| & (0 < |\gamma| \leqq \pi/2) \\ t & (\pi/2 < |\gamma| \leqq \pi) \end{cases} \geqq 2t\left|\sin\frac{\gamma}{2}\right|\left\{\sqrt{1+\sin^2\frac{\gamma}{2}} - \left|\sin\frac{\gamma}{2}\right|\right\}.$$

Consider next the case $B>1$. Choose $U: ue^{i\alpha(u)}$ so that

$$|te^{i(\alpha(t)+\gamma)} - ue^{i\alpha(u)}| = d. \tag{4.4}$$

If we put $T: te^{i\alpha(t)}$ and $T_\gamma: te^{i(\alpha(t)+\gamma)}$, then we have

$$\begin{aligned} TU^2 &= T_\gamma U^2 + T_\gamma T^2 - 2T_\gamma U \cdot T_\gamma T \cos\angle TT_\gamma U \\ &= d^2 + 4t^2\sin^2\frac{\gamma}{2} - 2\{[u\cos\alpha(u) - t\cos(\alpha(t)+\gamma)] \\ &\qquad \times [t\cos\alpha(t) - t\cos(\alpha(t)+\gamma)] + [u\sin\alpha(u) - t\sin(\alpha(t)+\gamma)] \\ &\qquad \times [t\sin\alpha(t) - t\sin(\alpha(t)+\gamma)]\} \\ &= d^2 + 4t^2\sin^2\frac{\gamma}{2} - 2\left\{t^2(1-\cos\gamma) + 2tu\sin\frac{\gamma}{2}\sin\left(\alpha(t)-\alpha(u)+\frac{\gamma}{2}\right)\right\} \\ &= d^2 - 4tu\sin\frac{\gamma}{2}\sin\left(\alpha(t)-\alpha(u)+\frac{\gamma}{2}\right). \end{aligned} \tag{4.5}$$

By the definition of B-regular curve, (4.4) and (4.5)

$$d^2 - 4tu\sin\frac{\gamma}{2}\sin\left(\alpha(t)-\alpha(u)+\frac{\gamma}{2}\right) \leqq B^2(t-u)^2 = B^2\left\{d^2 - 4tu\sin^2\left(\frac{\alpha(t)+\gamma-\alpha(u)}{2}\right)\right\}.$$

Hence

$$(B^2-1)d^2 \geqq 4tu\left\{B^2\sin^2\left(\frac{\alpha(t)+\gamma-\alpha(u)}{2}\right)-\sin\frac{\gamma}{2}\sin\left(\alpha(t)-\alpha(u)+\frac{\gamma}{2}\right)\right\}$$

$$=4tu\left\{\frac{B^2}{2}-\frac{B^2}{2}\cos\frac{\gamma}{2}\cos\left(\alpha(t)-\alpha(u)+\frac{\gamma}{2}\right)\right.$$

$$\left.+\left(\frac{B^2}{2}-1\right)\sin\frac{\gamma}{2}\sin\left(\alpha(t)-\alpha(u)+\frac{\gamma}{2}\right)\right\}$$

$$\geqq 4tu\left\{\frac{B^2}{2}-\sqrt{\frac{B^4}{4}-(B^2-1)\sin^2\frac{\gamma}{2}}\right\}$$

$$\geqq 4tu\frac{B^2-1}{B^2}\sin^2\frac{\gamma}{2},$$

and thus

$$d^2 \geqq \frac{4tu}{B^2}\sin^2\frac{\gamma}{2} \tag{4.6}$$

since $B>1$. Also from (4.4)

$$u \geqq t-d. \tag{4.7}$$

Combining (4.6) and (4.7), we have

$$d^2 \geqq \frac{4t(t-d)}{B^2}\sin^2\frac{\gamma}{2},$$

and consequently

$$d \geqq \frac{2t\left|\sin\frac{\gamma}{2}\right|\left\{\sqrt{B^2+\sin^2\frac{\gamma}{2}}-\left|\sin\frac{\gamma}{2}\right|\right\}}{B^2}.$$

5. From (4.1) we easily deduce that for $t\in[K_1t_2, r_n]$ $(K_1=K_1(B, c, \delta)$, $n\geqq n_4$ $(\geqq n_3))$

$$\rho(s(z(t)))\geqq t\cdot l\left(B, \frac{c}{2}-2\delta\right). \tag{5.1}$$

From (5.1) and the fact that $\rho(s(z(t)))\leqq t$ we have

$$\frac{\rho(s_n)}{\rho(s(z(u)))}\geqq l\left(B, \frac{c}{2}-2\delta\right)\left(\frac{r_n}{u}\right) \quad (u\in[K_1t_2, r_n],\ n\geqq n_4). \tag{5.2}$$

Also (5.1) and the fact that $\mathcal{L}_n$ is B-regular imply

$$\int_{s(z(u))}^{s_n}\frac{ds}{\rho(s)}\leqq\int_u^{r_n}\frac{B}{\rho(s(z(t)))}dt\leqq\frac{B}{l\left(B, \frac{c}{2}-2\delta\right)}\log\left(\frac{r_n}{u}\right) \tag{5.3}$$

$$(u\in[K_1t_2. r_n],\ n\geqq n_4).$$

Substituting (5.2) and (5.3) into (3.3), we obtain

$$\omega(z(u)) > \frac{l\left(B, \frac{c}{2}-2\delta\right)}{2\pi}\left(\frac{u}{r_n}\right)^{\frac{eB}{l\left(B, \frac{c}{2}-2\delta\right)}-1} \quad (u \in [K_1 t_2, r_n],\ n \geqq n_4). \tag{5.4}$$

Hence from (2.4), (5.4) and (1.4) we deduce that for $u \in [K_1 t_2, r_n]$ $(n \geqq n_4)$

$$\log\left|\frac{f'(z(u))}{f(z(u))}\right| < h \log u - \frac{K \cdot l\left(B, \frac{c}{2}-2\delta\right)}{2\pi} T(r_n, f)\left(\frac{u}{r_n}\right)^{\frac{eB}{l\left(B, \frac{c}{2}-2\delta\right)}-1} \tag{5.5}$$

$$< h \log r_n - \frac{K \cdot l\left(B, \frac{c}{2}-2\delta\right)}{2\pi} u^{\frac{eB}{l\left(B, \frac{c}{2}-2\delta\right)}-1} r_n^{\lambda-\varepsilon-\left(\frac{eB}{l\left(B, \frac{c}{2}-2\delta\right)}-1\right)}.$$

As $n \to \infty$, the right hand side of (5.5) tends to $-\infty$, by (2.1) or (2.1)′. Therefore

$$\frac{f'(z(u))}{f(z(u))} = 0$$

for every $z(u) = ue^{i\left(\alpha_1(u)+\frac{c}{2}\right)}$ $(u > K_1 t_2)$. Thus $f'(z)/f(z)$ vanishes identically, which is only possible if $f(z)$ is a constant. This contradicts (1.1) and hence completes the proof of Theorem.

References

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