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DUPIN HYPERSURFACES AND A LIE INVARIANT

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Dedicated to Professor Tominosuke Otsuki on his 70th birthday

§1. Introduction.

In [9], Pinkall developes a Lie's sphere geometry on hypersurfaces in a space form and successfully applies the theory to a class of hypersurfaces called "Dupin". A Dupin hypersurface is a hypersurface each of which principal curvatures has a constant multiplicity with a vanishing derivative in the corresponding curvature direction. One of his results is the local Lie equivalence of cyclides of Dupin with isoparametric hypersurfaces, where a cyclide of Dupin is a Dupin hypersurface with exactly two principal curvatures. This is essentially used in [4] to find a solution to a simple progressing wave equation.

For any integer g, we can construct a Dupin hypersurface with g principal curvatures of arbitrary multiplicities. Isoparametric hypersurfaces, however, have $g \in \{1, 2, 3, 4, 6\}$ principal curvatures with non-arbitrary multiplicities if $g \geqq 3$. Thus the equivalence problem between Dupin hypersurfaces and isoparametric hypersurfaces for $g \geqq 3$ requires some more conditions on Dupin hypersurfaces.

In [11], Thorbergsson guarantees coincidence of *compact embedded* Dupin hypersurfaces with isoparametric hypersurfaces in cohomology level. This motivates Cecil and Ryan a conjecture [3]: A compact embedded Dupin hypersurface is Lie equivalent to an isoparametric hypersurface. Besides the trivial case $g=1$, this is already known true when $g=2$ [2]. For $g=3$, the author gives a positive answer in [4]. In this paper, we find a certain Lie invariant by which we get a non-trivial necessary condition for the equivalence when $g=4$ and 6. A sufficient condition for $g=4$ is obtained as well, and in the forthcoming paper, we give it for $g=6$.

After this paper was finished, Pinkall and Thorbergsson construct counterexamples to the conjecture for $g=4$ [10]. Independently, Ozawa and the author get counterexamples for $g=4$ and 6 in [6], using a new method producing taut embeddings. Both examples are shown to be not Lie equivalent to isoparametric hypersurfaces by using the Lie invariant obtained in the present paper.

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Now our results are:

THEOREM I. *Let λ, μ, ν, τ be distinct principal curvatures of a hypersurface M in a space form $\overline{M}(c)$. Then the function*

$$\Psi=\frac{(\lambda-\mu)(\tau-\nu)}{(\lambda-\nu)(\tau-\mu)}$$

is invariant under Lie transformations.

An immediate consequence of this theorem is:

COROLLALY. *For a Dupin hypersurface M in $\overline{M}(c)$ with four or six principal curvatures to be Lie equivalent to an isoparametric hypersurface, the function Ψ must be constant on M for any distinct four principal curvatures λ, μ, ν, τ of M.*

As for a sufficient condition, we have:

THEOREM II. *Let M be a compact embedded Dupin hyperface with four principal curvatures $\lambda>\mu>\nu>\tau$ in $\overline{M}(c)$. If Ψ is constant on M and if for each λ-leaf L^{λ}(τ-leaf L^{τ}, resp.), there exists a ν-leaf L^{ν}_{λ}(μ-leaf L^{μ}_{τ}, resp.) such that $L^{\mu}_{q}\cap L^{\nu}_{\lambda}\neq\emptyset$ for all $q\in L^{\lambda}$($L^{\nu}_{q}\cap L^{\mu}_{\tau}\neq\emptyset$ for all $q\in L^{\tau}$, resp.), then M is Lie equivalent to an isoparametric hypersurface in a sphere.*

In §2-3, we give a summery of Lie's sphere geometry, and prove Theorem I in §4. In §5-7, we investigate compact embedded Dupin hypersurfaces with four principal curvatures. We use the constantness of Ψ in §8-9. The assump tion on leaves is used in the last section to prove Theorem II.

§2. Definitions.

Let $\overline{M}(c)$ be an n-dimensional complete simply-connected space form of constant curvature c. An immersed hypersurface M in $\overline{M}(c)$ is called *Dupin* if it satisfies:

(i) The multiplicity of each principal curvature is constant on M.

(ii) Each principal curvature is constant along its leaf of the corresponding curvature distribution.

By an *isoparametric hypersurface*, we mean an immersed hypersurface in $\overline{M}(c)$ with constant principal curvatures. Obviously, isoparametric hypersurfaces are Dupin, but a (non-isometric) conformal image of an isoparametric hypersurface and its parallel hypersurfaces are non-isoparametric Dupin hypersurfaces.

By conformal invariance of Dupin hypersurfaces and the function Ψ, we discuss on the objects in $S^n=\overline{M}(1)$ in this paper.

A Lie transformation is a transformation on a space of all oriented hypershperes in S^n that preserves oriented contact (see § 3).

An immersed submanifold X in S^n is said to be *taut*, if for all $x \in S^n$ such that $d_x := d(x, \ )^2$(square of the spherical distance) is a Morse function on M, the number of critical points of d_x is equal to $\beta(M; \boldsymbol{Z}_2)$, the sum of $\boldsymbol{Z}_2$-Betti numbers of M.

Lie invariance of tautness is first mentioned by Pinkall [8], and is proved by Cecil and Chern [1]. Now the important is:

THEOREM (Thorbergsson [11]). *A complete embedded Dupin hypersurface M in $\overline{M}(c)$ is taut. If M is compact, then g is* 1, 2, 3, 4, 6. *Moreover, $H^*(M; \boldsymbol{Z}_2)$ is obtained by Münzner's method.*

This theorem motivates Cecil and Ryan the conjecture, and is essentially used in this paper.

§ 3. Basic facts.

In this section, we review Lie geometric description of hypersurfaces in S^n, and give basic facts on Lie transformations. For details, see [9].

Let S^n be the unit shpere in $\boldsymbol{R}^{n+1}$. For $k = {}^t(z, x, y) \in V_{n+3} := \boldsymbol{R}^{n+1} \times \boldsymbol{R} \times \boldsymbol{R}$, define the symmetric bilinear form $\langle \ , \ \rangle$ on V_{n+3} by

$$\langle k, k \rangle = \|z\|^2 - x^2 - y^2. \tag{3.1}$$

Let P^{n+2} be the real projective space corresponding to V_{n+3} and denote by $[k]$ the projective point spanned by $k \in V_{n+3}$. Then the quadric

$$Q = \{[k] \in P^{n+2} \mid \langle k, k \rangle = 0\}$$

is identified with the set of all oriented hyperspheres in S^n. Two oriented hyperspheres $[k_1]$ and $[k_2]$ are in oriented contact iff $\langle k_1, k_2 \rangle = 0$, which is equivalent to that the projective line in P^{n+2} spanned by $[k_1]$ and $[k_2]$ is entirely contained in Q. A Lie transformation of S^n is interpreted as a line preserving diffeo $\varphi: Q \to Q$. This is known to be the restriction to Q of an projective transformation $\alpha: P^{n+2} \to P^{n+2}$ leaving Q fixed [9, Lemma 4]. Therefore, a Lie transformation is represented by $A = (a_1, a_2, \cdots, a_{n+3}) \in O(n+1, 2)$, where $\{a_1, a_2, \cdots, a_{n+3}\}$ is the so-called Lie frame, i.e. an ordered set of vectors in V_{n+3} satisfying

$$(\langle a_i, a_j \rangle)_{1 \le i, j \le n+3} = \begin{pmatrix} I_{n+1} & 0 \\ 0 & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \end{pmatrix}.$$

LEMMA 3.1. *A is in $O(n+1, 2)$ iff ${}^tA \in O(n+1, 2)$. In particular, both ordered sets of colume vectors and line vectors of A are Lie frames of V_{n+3}.*

Proof. Put

$$A=\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where $A_1 \in M_{n+1}(\boldsymbol{R})$, A_2, ${}^tA_3 \in M_{n+1,2}(\boldsymbol{R})$, $A_4 \in M_2(\boldsymbol{R})$. Define

$$\tilde{A}=\begin{pmatrix} A_1 & \sqrt{-1}A_2 \\ \sqrt{-1}A_3 & -A_4 \end{pmatrix}.$$

Then it is easy to see that $A \in O(n+1, 2)$ iff ${}^t\tilde{A}\tilde{A}=I$. Since ${}^t\tilde{A}\tilde{A}=I$ iff $\tilde{A}{}^t\tilde{A}=I$, we get ${}^tA \in O(n+1, 2)$ iff $A=O(n+1, 2)$. q.e.d.

Remark 3.2. We obtain Lie frames of V_{n+3} as follows: Choose two linearly independent timelike vectors in V_{n+3} so that they span a timelike plane T. Let S be the $(n+1)$-dimensional spacelike subspace orthogonal to T with respect to $\langle\ ,\ \rangle$. Since $\langle\ ,\ \rangle$ is definite on S and T, we can choose $a_1, a_2, \cdots, a_{n+1} \in S$ and $a_{n+2}, a_{n+3} \in T$ so that

$$(\langle a_i, a_j \rangle)_{1 \leq i, j \leq n+3}=\begin{pmatrix} I_{n+1} & 0 \\ 0 & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \end{pmatrix},$$

i.e. $\{a_1, a_2, \cdots, a_{n+3}\}$ is a Lie frame of V_{n+3}. In particular, $\det A \neq 0$ since $a_1, \cdots, a_{n+3}$ are linearly independent.

Now, for an $(n-1)$-dimensional manifold M and a pair of mappings k_1, k_2: $M \to V_{n+3}$, (M, k_1, k_2) is called *a Lie geometric hypersurface* if it satisfies:

a) For all $p \in M$, the vectors $k_1(p)$ and $k_2(p)$ are linearly independent and we have

$$\langle k_i, k_j \rangle=0, \qquad i, j=1, 2.$$

b) There is no $p \in M$, $X \in T_pM$ such that simultaneously $dk_1(X)$ and $dk_2(X)$ are in span $(k_1(p), k_2(p))$.

c) $\langle dk_1, k_2 \rangle=0$.

Remark 3.3. Obviously, if (M, k_1, k_2) is a Lie geometric hypersurface in S^n, so is (M, Ak_1, Ak_2) for any Lie transformation A.

Remark 3.4. The properties a), b), c) are preserved by

$$\tilde{k}_1=\alpha k_1+\beta k_2, \qquad \tilde{k}_2=\gamma k_1+\delta k_2 \tag{3.2}$$

where $\alpha, \beta, \gamma, \delta$: $M \to \boldsymbol{R}$ are functions such that $\alpha\delta-\beta\gamma$ is everywhere different from zero on M.

Example 3.5. Let $\boldsymbol{p}$: $M \to S^n$ be an oriented hypersurface and let $\boldsymbol{n}$: $M \to S^n$ be defined by $\boldsymbol{n}(p)$=the unit normal vector at $p \in M$. Then for

$$k_1 = {}^t(\boldsymbol{p}, 1, 0), \qquad k_2 = {}^t(\boldsymbol{n}, 0, 1) \tag{3.3}$$

(M, k_1, k_2) is a Lie geometric hypersurface of S^n.

LEMMA 3.6. *If (M, k_1, k_2) is a Lie geometric hypersurface of S^n, then there are functions $\alpha, \beta, \gamma, \delta$: $M \to \boldsymbol{R}$ and maps $\boldsymbol{p}, \boldsymbol{n}$: $M \to S^n$ such that*

$$\alpha k_1 + \beta k_2 = {}^t(\boldsymbol{p}, 1, 0), \qquad \gamma k_1 + \delta k_2 = {}^t(\boldsymbol{n}, 0, 1). \tag{3.4}$$

Proof. Let $k_1 = {}^t(\boldsymbol{h}_1, a, b)$ and $k_2 = {}^t(\boldsymbol{h}_2, c, d)$. Then we have

$$\begin{aligned}(ad-bc)^2 &= (a^2+b^2)(c^2+d^2) - (ac+bd)^2 \\ &= \|\boldsymbol{h}_1\|^2 \|\boldsymbol{h}_2\|^2 - \langle \boldsymbol{h}_1, \boldsymbol{h}_2 \rangle^2 \\ &> 0\end{aligned}$$

by a). Thus there exists the inverse matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which we denote by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. q.e.d.

For $\boldsymbol{p}, \boldsymbol{n}$: $M \to S^n$ in Example 3.5, *a parallel hypersurface* is defined by

$$\boldsymbol{p}_\alpha = \cos \alpha \boldsymbol{p} + \sin \alpha \boldsymbol{n}, \qquad \alpha \in [\pi, \pi).$$

of which the unit normal vector field $\boldsymbol{n}_\alpha$ is given by

$$\boldsymbol{n}_\alpha = -\sin \alpha \boldsymbol{p} + \cos \alpha \boldsymbol{n}.$$

Remark 3.7. $\boldsymbol{p}$: $M \to S^n$ in Lemma 3.6 is not always an immersion. But singularlities can be removed by passing to a parallel hypersurface of $\boldsymbol{p}$ [9, Theorem 1].

Remark 3.8. Define $A \in O(n+1, 2)$ by

$$A = \begin{pmatrix} I_{n+1} & 0 & \\ 0 & \begin{matrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{matrix} \end{pmatrix}. \tag{3.5}$$

Then we have for k_1, k_2 in (3.3),

$$Ak_1 = {}^t(\boldsymbol{p}, \cos\alpha, -\sin\alpha), \qquad Ak_2 = {}^t(\boldsymbol{n}, \sin\alpha, \cos\alpha).$$

Using Lemma 3.6, we obtain

$$\begin{aligned}\tilde{k}_1 &= \cos\alpha A k_1 + \sin\alpha A k_2 = {}^t(\boldsymbol{p}_\alpha, 1, 0) \\ \tilde{k}_2 &= -\sin\alpha A k_1 + \cos\alpha A k_2 = {}^t(\boldsymbol{n}_\alpha, 0, 1).\end{aligned}$$

Thus parallel hypersurfaces are obtained by a Lie transformation of type (3.5).

DEFINITION. Let (M, k_1, k_2) be a Lie geometric hypersurface, $p\in M$, λ, $\mu\in \boldsymbol{R}$, $(\lambda, \mu)\neq(0, 0)$. Then the hypersphere $k=\lambda k_1(p)+\mu k_2(p)$ is called *a curvature sphere* at p if there is a tangent vector $X\in T_pM$, $X\neq 0$ such that

$$\lambda dk_1(X)+\mu dk_2(X)\in \mathrm{span}(k_1(p),\ k_2(p)).$$

X is called *a direction of curvature* corresponding to k.

Remark 3.9. The notion of curvature sphere is invariant under Lie transformation. Especially, the direction of a curvature is preserved by Lie transformations. When (M, k_1, k_2) is given by (3.3), k is a curvature sphere iff $\tilde{\lambda}=\lambda/\mu$ is a principal curvature of the hypersurface at p. The direction of curvature is a principal vector with respect to $\tilde{\lambda}$.

LEMMA 3.10. *Let (M, k_1, k_2) be a Lie geometric hypersurface of S^n aud let $(M, \tilde{k}_1, \tilde{k}_2)$ be its another description by* (3.2). *Then $\lambda k_1(p)+\mu k_2(p)$ is a curvature sphere at $p\in M$ iff $\sigma\tilde{k}_1(p)+\tau\tilde{k}_2(p)$ is a curvature sphere at p, where*

$$\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}\begin{pmatrix}\sigma\\ \tau\end{pmatrix}=r\begin{pmatrix}\lambda\\ \mu\end{pmatrix},\qquad r\in \boldsymbol{R},\ r\neq 0.$$

Proof. Put $L=\mathrm{span}(k_1(p),\ k_2(p))=\mathrm{span}(\tilde{k}_1(p),\ \tilde{k}_2(p))$. Then

$$\begin{aligned}\sigma d\tilde{k}_1+\tau d\tilde{k}_2&\equiv\sigma(\alpha dk_1+\beta dk_2)+\tau(\gamma dk_1+\delta dk_2)\pmod{L}\\ &=(\alpha\sigma+\gamma\tau)dk_1+(\beta\sigma+\delta\tau)dk_2.\end{aligned}$$

q. e. d.

§ 4. Principal curvatures of a Lie image.

In this section, we compute principal curvatures of a Lie image of a hypersurface M in S^n. To define "a Lie image" precisely, we review the Legendre map from M to the space of lines in Q which we denote by Λ^{2n-1}. Then we study about spherical projections from Λ^{2n-1} to a sphere, and show that it is sufficient to take a special spherical projection when we study Lie images of M.

DEFINITION 4.1. For a hypersurface M in S^n, let (M, k_1, k_2) be the corresponding Lie geometric hypersurface given in (3.3). Then *the Legendre map* $\mathcal{L}: M\rightarrow\Lambda^{2n-1}$ is defined for $p\in M$ by $\mathcal{L}(p)=$ the line in Q spanned by $k_1(p)$ and $k_2(p)$.

DEFINITION 4.2. For an ordered pair (u, v) of unit timelike vectors satisfying $\langle u, v\rangle=0$, *the spherical projection*

$$\pi_{u,v}\colon \Lambda^{2n-1}\longrightarrow S^n_{u,v}$$

is defined as follows: For $l\in\Lambda^{2n-1}$, let

$$l_1=l\cap v^{\perp}\qquad(\in Q)$$

which can be defined since $\dim\{\operatorname{span} l\}=2$ and $\dim v^{\perp}=n+2$ in V_{n+3}. Then represent l_1 by the (unique) vector $\tilde{l}_1$ in V_{n+3} satisfying $\langle\tilde{l}_1, u\rangle=-1$, which is possible since $l_1\in Q$ and $\langle l_1, v\rangle=0$. Then $\pi_{u,v}(l)$ is defined to be the vector $\tilde{l}_1-u$. Obviously, $\langle\pi_{u,v}(l), \pi_{u,v}(l)\rangle=1$, and so it lies in the unit sphere $S^n_{u,v}$ of $u^{\perp}\cap v^{\perp}$ in V_{n+3}.

Now let $u_0=(\mathbf{0}, 1, 0)$ and $v_0=(\mathbf{0}, 0, 1)$ in our standard coordinate system of V_{n+3}. Note that $S^n=S^n_{u_0,v_0}$, and write $\pi_0=\pi_{u_0,v_0}$. For the pair (u, v) above, there is a transformation $A_0\in O(n+1, 2)$ such that $A_0u=u_0$ and $A_0v=v_0$. Let $\tilde{A}_0$ be the Lie transformation represented by A_0. Since $\tilde{A}_0$ induces a transformation on Λ^{2n-1}, it induces a map $f_{A_0}\colon S^n_{u,v}\to S^n$, so that the following diagram is commutative:

$$
\begin{tikzcd}
\Lambda^{2n-1} \arrow[r, "\tilde{A}_0"] \arrow[d, "\pi_{u,v}"] & \Lambda^{2n-1} \arrow[d, "\pi_0"] \\
S^n_{u,v} \arrow[r, "f_{A_0}"'] & S^n
\end{tikzcd}
$$

In fact, for $l\in\Lambda^{2n-1}$ with $\pi_{u,v}(l)=\tilde{l}_1-u$, we have

$$0=\langle\tilde{l}_1, v\rangle=\langle A_0\tilde{l}_1, A_0v\rangle=\langle A_0\tilde{l}_1, v_0\rangle$$

and

$$-1=\langle\tilde{l}_1, u\rangle=\langle A_0\tilde{l}_1, A_0u\rangle=\langle A_0\tilde{l}_1, u_0\rangle$$

where $\tilde{l}_1\in l$, i.e. $\langle\tilde{l}_1, l\rangle=0=\langle A_0\tilde{l}_1, \tilde{A}_0l\rangle$. So we get

$$\pi_0(\tilde{A}_0l)=A_0\tilde{l}_1-u_0=A_0(\tilde{l}_1-u)=A_0(\pi_{u,v}(l)).$$

This means that $\pi_{u,v}(l)=\pi_{u,v}(l')$ leads to $\pi_0(\tilde{A}_0l)=\pi_0(\tilde{A}_0l')$, that is, f_{A_0} is well-defined and $f_{A_0}(p)=A_0p$ for $p\in S^n_{u,v}$. From this follows immediately:

PROPOSITION 4.3. *$f_{A_0}\colon S^n_{u,v}\to S^n$ is an isometry.*

DEFINITION 4.4. For a hypersurface M in S^n, a Lie image of M is a spherically projected image of $\tilde{A}\mathcal{L}$ where $\mathcal{L}$ is the Legendre map of M and $\tilde{A}$ is some Lie transformation.

From Proposition 4.3, we get:

COROLLARY 4.5. *Lie images of M are obtained, up to isometry, by $\pi_0(\tilde{A}\mathcal{L})$ for some $A\in O(n+1, 2)$.*

Now, we compute principal curvatures of $\pi_0(\tilde{A}\mathcal{L})$ where $\tilde{A}$ is the Lie transformation represented by $A\in O(n+1, 2)$. For (M, k_1, k_2) given by (3.3), we may denote

$$Ak_1={}^t(\boldsymbol{h}_1, a, b), \qquad Ak_2={}^t(\boldsymbol{h}_2, c, d)$$

where $\boldsymbol{h}_1, \boldsymbol{h}_2: M \to V_{n+3}$ and $a, b, c, d: M \to \boldsymbol{R}$. Then we have

(4.1) $$\tilde{k}_1 = \alpha A k_1 + \beta A k_2 = {}^t(\boldsymbol{q}, 1, 0), \quad \tilde{k}_2 = \gamma A k_1 + \delta A k_2 = {}^t(\boldsymbol{m}, 0, 1)$$

where $\boldsymbol{q}, \boldsymbol{m}: M \to S^n$ and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1},$$

in fact, $\pi_0(\tilde{A}\mathcal{L})$ is nothing but $\boldsymbol{q}$.

Now let λ be a principal curvature of the immersion $\boldsymbol{p}$ at $p \in M$, and assume that $\boldsymbol{q}: M \to S^n$ is an immersion around p. Then $\lambda k_1(p) + k_2(p)$ is the curvature sphere and so is $\lambda A k_1(p) + A k_2(p)$ (Remark 3.9). Thus by Lemma 3.10, the corresponding principal curvature $\tilde{\lambda}$ of the immersion $\boldsymbol{q}: M \to S^n$ at $\boldsymbol{q}(p)$ is given by $\tilde{\lambda} = \sigma/\tau$, where

$$\begin{pmatrix} \sigma \\ \tau \end{pmatrix} = r \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}^{-1} \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = r \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = r \begin{pmatrix} a\lambda + c \\ b\lambda + d \end{pmatrix}, \qquad r \in \boldsymbol{R},\ r \neq 0.$$

This implies

(4.2) $$\tilde{\lambda} = \frac{a\lambda + c}{b\lambda + d}.$$

Therefore, we get:

PROPOSITION 4.6. *Between any principal curvature λ of a hypersurface M in S^n and the corresponding principal curvature $\tilde{\lambda}$ of a Lie image of M, there is a relation* (4.2) *where a, b, c, d are functions on M determined by the Lie transformation.*

Thus we have:

THEOREM I. *Let λ, μ, ν, τ be distinct principal curvatures of a hypersurface M in $\overline{M}(c)$. Then the function*

$$\Psi = \frac{(\lambda - \mu)(\tau - \nu)}{(\lambda - \nu)(\tau - \mu)}$$

is invariant under Lie transformations.

Remark 4.7. Note that $\dfrac{(\lambda - \mu)}{(\lambda - \nu)}$ and $\dfrac{(\tau - \nu)}{(\tau - \mu)}$ are invariant under conformal transformation and so the result holds for M in $\overline{M}(c)$.

Remark 4.8. It is easily seen that Ψ is the cross ratio among four projective points $k^\lambda, k^\mu, k^\nu, k^\tau$ on the line in Q determined by k_1 and k_2 given by (3.3), where k^* is the curvature sphere with respect to $*$. This is another proof of Theorem I. In the following, we denote the cross ratio Ψ by $[\lambda, \tau; \mu, \nu]$.

COROLLARY. *For a Dupin hypersurface in a sphere with four or six principal curvatures to be Lie equivalent to an isoparametric hypersurface, the function* Ψ *must be constant on* M *for any distinct four principal curvatures* λ, μ, ν, τ *of* M.

Next, for later use, we compute derivatives of principal curvatures (4.2) of the spherical projection (4.1). Let $X\in T_pM$ be a direction of curvature corresponding to the curvature sphere $\lambda k_1(p)+k_2(p)$, or equivalently to $\lambda Ak_1(p)+Ak_2(p)$. From (4.1), we get

$$d\tilde{k}_1(X)=X(\alpha)Ak_1+X(\beta)Ak_2+A(\alpha dk_1(X)+\beta dk_2(X))$$
$$\equiv(\alpha-\beta\lambda)Adk_1(X) \qquad (\mathrm{mod\,span}(Ak_1(p),\ Ak_2(p))).$$

So if $Y\in T_pM$ is also a direction of curvature corresponding to $\lambda k_1(p)+k_2(p)$, noting a) and c) in §3, we have

$$\langle d\boldsymbol{q}(X),\ d\boldsymbol{q}(Y)\rangle=\langle d\tilde{k}_1(X),\ d\tilde{k}_1(Y)\rangle \tag{4.3}$$
$$=(\alpha-\beta\lambda)^2\langle dk_1(X),\ dk_2(Y)\rangle$$
$$=(\alpha-\beta\lambda)^2\langle d\boldsymbol{p}(X),\ d\boldsymbol{p}(Y)\rangle.$$

Since it is easy to see that $\alpha-\beta\lambda=\dfrac{b\lambda+d}{ad-bc}$, we obtain:

LEMMA 4.9. *If* $X\in T_pM$ *is a direction of the curvature sphere* $\lambda k_1(p)+k_2(p)$ *such that* $\|dp(X)\|=1$, *then* $\dfrac{ad-bc}{b\lambda+d}d\boldsymbol{q}(X)$ *is a unit principal vector corresponding to the principal curvature* $\tilde{\lambda}=\dfrac{a\lambda+c}{b\lambda+d}$.

Now, let U be a neighbourhood of M on which principal curvatures satisfy $\lambda_1>\lambda_2>\cdots>\lambda_g$. Then the tangent bundle TU is decomposed into $TU=T_1+\cdots+T_g$ where T_i is the curvature distribution corresponding to λ_i and $\tilde{\lambda}_i$. We can choose a local frame $X_1, \cdots X_{n-1}$ of TU so that $X_{m_1+\cdots+m_{i-1}+1}, \cdots, X_{m_i}\in T_i$, where m_i is the multiplicitiy of λ_i, and that $e_i=d\boldsymbol{p}(X_i)$, $i=1, \cdots, n-1$ form an orthonormal frame with respect to $\boldsymbol{p}$: $M\to S^n$. Denoting the connection of S^n by ∇, we put

$$\nabla_{e_i}e_j=\sum_{k=1}^{n-1}\Lambda_{ij}^k e_k+\lambda_i\delta_{ij}\boldsymbol{n}.$$

Then we have [5]:

$$e_i(\lambda_j)=\Lambda_{jj}^i(\lambda_j-\lambda_i), \qquad i\neq j. \tag{4.4}$$

Remark 4.10. Put $[i]:=\{j\mid\lambda_i=\lambda_j\}$. Then a Dupin hypersurface $\boldsymbol{p}$: $M\to S^n$ is isoparametric iff $\Lambda_{jj}^i\equiv0$ for all i, j such that $i\notin[j]$.

Since

$$(4.5)\qquad f_i=\frac{ad-bc}{b\lambda_i+d}\,d\boldsymbol{q}(X_i),$$

$i=1, \cdots, n-1$, form an orthnormal frame with respect to $\boldsymbol{q}: M\to S^n$ by (4.3), putting

$$\nabla_{f_i}f_j=\sum_{k=1}^{n-1}\tilde{\Lambda}_{ij}^k f_k+\tilde{\lambda}\delta_{ij}\boldsymbol{m},$$

we get

$$(4.6)\qquad f_i(\tilde{\lambda}_j)=\tilde{\Lambda}_{jj}^i(\tilde{\lambda}_j-\tilde{\lambda}_i),\qquad i\neq j.$$

For convenience, put $A=\begin{pmatrix} {}^tb_1\\ \vdots\\ {}^tb_{n+3}\end{pmatrix}=\begin{pmatrix} {}^tc_1 & & \\ \vdots & a_{n+2} & a_{n+3}\\ {}^tc_{n+3} & & \end{pmatrix}$, where ${}^tb_\alpha=(a_{\alpha\beta})_{1\leq\beta\leq n+3}$, ${}^tc_\alpha=(a_{\alpha i})_{1\leq i\leq n+1}$ for $\alpha=1, \cdots, n+3$, $a_{n+2}=(a_{\alpha\, n+2})_{1\leq\alpha\leq n+3}$ and $a_{n+3}=(a_{\alpha\, n+3})_{1\leq\alpha\leq n+3}$.

LEMMA 4.11. *We have*

$$\tilde{\Lambda}_{jj}^i=0\qquad \textit{if}\quad j\neq i\in[j],$$

and for $i\notin[j]$,

$$(4.7)\qquad \tilde{\Lambda}_{jj}^i=\frac{1}{b\lambda_j+d}\{(b\lambda_j+d)x_i-(a\lambda_j+c)y_i+(ad-bc)\Lambda_{jj}^i\}$$

where $x_i=\langle c_{n+2}, e_i\rangle$ *and* $y_i=\langle c_{n+3}, e_i\rangle$, *using the inner product* $\langle\ ,\ \rangle$ *of* $\boldsymbol{R}^{n+1}$.

Proof. Since $a=\langle c_{n+2}, \boldsymbol{p}\rangle+a_{n+2\, n+2}$, $b=\langle c_{n+3}, \boldsymbol{p}\rangle+a_{n+3\, n+2}$, $c=\langle c_{n+2}, \boldsymbol{n}\rangle+a_{n+2\, n+3}$ and $d=\langle c_{n+3}, \boldsymbol{n}\rangle+a_{n+3\, n+3}$, it is easy to see that:

1) $e_i(a\lambda_j+c)=e_i(a)\lambda_j+e_i(c)=\langle c_{n+2}, \lambda_je_i+\nabla_{e_i}\boldsymbol{n}\rangle=0,\qquad$ if $j\neq i\in[j]$

2) $e_i(b\lambda_j+d)=\langle c_{n+3}, \lambda_je_i+\nabla_{e_i}\boldsymbol{n}\rangle=0,\qquad$ if $j\neq i\in[j]$

3) $e_i(a\lambda_j+c)=\langle c_{n+2}, \lambda_je_i+\nabla_{e_i}\boldsymbol{n}\rangle+ae_i(\lambda_j)$
$\qquad=(\lambda_j-\lambda_i)(\langle c_{n+2}, e_i\rangle+a\Lambda_{jj}^i),\qquad$ if $i\notin[j]$

4) $e_i(b\lambda_j+d)=(\lambda_j-\lambda_i)(\langle c_{n+3}, e_i\rangle+b\Lambda_{jj}^i),\qquad$ if $i\notin[j]$.

So we have

$$e_i(\tilde{\lambda}_j)=0,\qquad \text{if}\quad j\neq i\in[j]$$

and for $i\notin[j]$,

$$(4.8)\qquad e_i(\tilde{\lambda}_j)=\frac{(\lambda_j-\lambda_i)}{(b\lambda_j+d)^2}\{(b\lambda_j+d)x_i-(a\lambda_j+c)y_i+(ad-bc)\Lambda_{jj}^i\}.$$

Here we note that

$$e_i(\lambda_j)=d\boldsymbol{p}(X_i)(\lambda_j)=X_i(\lambda_j)$$

and

$$e_i(\tilde{\lambda}_j)=X_i(\tilde{\lambda}_j)$$

considering $\tilde{\lambda}_j$ as a function on M.

On the other hand, using (4.5), we get

$$f_i(\tilde{\lambda}_j)=\frac{ad-bc}{b\lambda_i+d}d\boldsymbol{q}(X_i)(\tilde{\lambda}_j)=\frac{ad-bc}{b\lambda_i+d}X_i(\tilde{\lambda}_j)=\frac{ad-bc}{b\lambda_i+d}e_i(\tilde{\lambda}_j).$$

Thus we have

$$e_i(\tilde{\lambda}_j)=\frac{b\lambda_i+d}{ad-bc}\tilde{\Lambda}^i_{jj}(\tilde{\lambda}_j-\tilde{\lambda}_i)=\tilde{\Lambda}^i_{jj}\frac{(\lambda_j-\lambda_i)}{(b\lambda_j+d)},$$

from which with (4.8) follows the lemma. q.e.d.

§5. Compact Dupin hypersurfaces with four principal curvatures.

From now on, let $\boldsymbol{p}\colon M\to S^n$ be a compact connected embedded Dupin hypersurface with four principal curvatures $\lambda>\mu>\nu>\tau$. Then we know that $m_1=m_3$ and $m_2=m_4$, where m_1, m_2, m_3, m_4 are multiplicities of λ, μ, ν, τ, respectively [7]. We use the index convention: $1\leqq a, b, c\leqq m_1<i, j, k\leqq m_1+m_2<r, s, t\leqq 2m_1+m_2<u, v, w\leqq n-1$. A local orthonormal frame (e_a, e_i, e_r, e_u) is chosen as in §4.

The purpose is now to find a condition under which there exists $A\in O(n+1, 2)$ such that $\tilde{\Lambda}^\alpha_{\beta\beta}\equiv 0$ for all α, β such that $\alpha\notin[\beta]$. But we come against difficulty soon if we investigate equations

$$\tilde{\Lambda}^a_{ii}=0,\quad \tilde{\Lambda}^a_{rr}=0,\quad \tilde{\Lambda}^a_{uu}=0,$$

even at a point, with (x_a, y_a), defined in Lemma 4.11, as unknowns. At a critical point of Ψ, however, this can be done.

PROPOSITION 5.1. *Let p be a critical point of Ψ. Then we obtain $A\in O(n+1, 2)$ such that*

$$\tilde{\Lambda}^\alpha_{\beta\beta}(p)=0,\quad \textit{for all } \alpha, \beta \textit{ such that } \alpha\notin[\beta]. \tag{5.1}$$

Remark 5.2. At p, the image hypersurface has the "common" normal geodesic defined in [5].

For the proof, we need:

LEMMA 5.3. *At a critical point $p\in M$ of Ψ, we have:*

(i) $$\frac{\Lambda^a_{ii}-\Lambda^a_{rr}}{\mu-\nu}=\frac{\Lambda^a_{ii}-\Lambda^a_{uu}}{\mu-\tau}=\frac{\Lambda^a_{rr}-\Lambda^a_{uu}}{\nu-\tau}(=:R_a)$$

(ii) $$\frac{\Lambda^i_{aa}-\Lambda^i_{rr}}{\lambda-\nu}=\frac{\Lambda^i_{aa}-\Lambda^i_{uu}}{\lambda-\tau}=\frac{\Lambda^i_{rr}-\Lambda^i_{uu}}{\nu-\tau}(=:R_i)$$

(iii) $$\frac{\Lambda^r_{aa}-\Lambda^r_{ii}}{\lambda-\mu}=\frac{\Lambda^r_{aa}-\Lambda^r_{uu}}{\lambda-\tau}=\frac{\Lambda^r_{ii}-\Lambda^r_{uu}}{\mu-\tau}(=:R_r)$$

(iv) $\dfrac{\Lambda^u_{aa}-\Lambda^u_{ii}}{\lambda-\mu}=\dfrac{\Lambda^u_{aa}-\Lambda^u_{rr}}{\lambda-\nu}=\dfrac{\Lambda^u_{ii}-\Lambda^u_{rr}}{\mu-\nu}(=: R_u)$.

Proof. Let $\Phi=\log\Psi$. Then p is a critical point of Ψ iff p is a critical point of Φ. For instance, we have, using (4.4),

$$
\begin{aligned}
0=e_a(\Phi)&=\frac{e_a(\lambda-\mu)}{\lambda-\mu}-\frac{e_a(\lambda-\nu)}{\lambda-\nu}+\frac{e_a(\tau-\nu)}{\tau-\nu}-\frac{e_a(\tau-\mu)}{\tau-\mu}\\
&=\Lambda^a_{ii}-\Lambda^a_{rr}+\frac{\Lambda^a_{uu}(\tau-\lambda)-\Lambda^a_{rr}(\nu-\lambda)}{\tau-\nu}-\frac{\Lambda^a_{uu}(\tau-\lambda)-\Lambda^a_{ii}(\mu-\lambda)}{\tau-\mu}\\
&=(\lambda-\tau)\left\{\frac{\Lambda^a_{ii}-\Lambda^a_{uu}}{\mu-\tau}-\frac{\Lambda^a_{rr}-\Lambda^a_{uu}}{\nu-\tau}\right\}.
\end{aligned}
$$

The first equality in (i), then follows automatically. (ii) and (iii) are similarly proved. q. e. d.

To prove Proposition 5.1, it is sufficient to show that there is a Lie transformation $A_1\in O(n+1, 2)$ such that the image hypersurface has the "common" normal *circle* at p [5], i. e.:

LEMMA 5.4. *Let p be a critical point of Ψ. Then we obtain $A_1\in O(n+1, 2)$ such that*

(5.2) $\tilde{\Lambda}^\alpha_{\beta\beta}(p)=\tilde{\Lambda}^\alpha_{\gamma\gamma}(p)$ *for all α, β, γ such that $\alpha\notin[\beta]\cup[\gamma]$, $\beta\notin[\gamma]$.*

Proof. Choose the coordinate of $\boldsymbol{R}^{n+1}$ so that $e_\alpha(p)={}^t(0, \cdots \overset{\alpha}{1} \cdots 0)$, $\alpha=1, \cdots, n-1$, $\boldsymbol{p}(p)={}^t(0, \cdots 1, 0)$, and $\boldsymbol{n}(p)={}^t(0, \cdots, 1)$. Consider simultaneous equations:

$$\tilde{\Lambda}^a_{ii}(p)=\tilde{\Lambda}^a_{rr}(p)=\tilde{\Lambda}^a_{uu}(p) \tag{5.3}$$

with unknowns (x_a, y_a) for a fixed a, where we use notations in Lemma 4.11. For simplicity, try to solve (5.3) by putting $b=0$. Then from (4.7), (5.3) becomes

$$a(\mu-\nu)y_a-ad(\Lambda^a_{ii}-\Lambda^a_{rr})=0$$

$$a(\mu-\tau)y_a-ad(\Lambda^a_{ii}-\Lambda^a_{uu})=0$$

By (i) of Lemma 5.3, at p, a solution is, for instance,

$$(x_a, y_a)=(0, dR_a), \qquad a=1, \cdots, m_1.$$

Just in the same way, solutions to

$$\tilde{\Lambda}^i_{aa}(p)=\tilde{\Lambda}^i_{rr}(p)=\tilde{\Lambda}^i_{uu}(p),$$

$$\tilde{\Lambda}^r_{aa}(p)=\tilde{\Lambda}^r_{ii}(p)=\tilde{\Lambda}^r_{uu}(p),$$

$$\tilde{\Lambda}^u_{aa}(p)=\tilde{\Lambda}^u_{ii}(p)=\tilde{\Lambda}^u_{rr}(p),$$

are, respectively,

$$(x_\iota, y_\iota)=(0, dR_\iota) \qquad m_1<i\leqq m_1+m_2,$$

$$(x_r, y_r)=(0, dR_r) \qquad m_1+m_2<r\leqq 2m_1+m_2,$$

$$(x_u, y_u)=(0, dR_u) \qquad 2m_1+m_2<u\leqq n-1.$$

Now, we have to find a Lie frame $b_1, \cdots, b_{n+3}$ such that

$$b_{n+2}={}^t(x, x_n, x_{n+1}, a-x_n, c-x_{n+1})$$

$$b_{n+3}={}^t(y, y_n, y_{n+1}, -y_n, d-y_{n+1}),$$

where $x=(x_a, x_\iota, x_r, x_u)$ and $y=(y_a, y_\iota, y_r, y_u)$ are given above, $x_n=\langle c_{n+2}, \boldsymbol{p}\rangle$, $x_{n+1}=\langle c_{n+2}, \boldsymbol{n}\rangle$, $y_n=\langle c_{n+3}, \boldsymbol{p}\rangle$ and $y_{n+1}=\langle c_{n+3}, \boldsymbol{n}\rangle$. Put $x_n=x_{n+1}=y_n=0$, $a=1$, $c=0$ and $y_{n+1}=-Nd$. Then obviously, we have $\langle b_{n+2}, b_{n+2}\rangle=-1$ and $\langle b_{n+2}, b_{n+3}\rangle=0$. On the other hand, since

$$\langle b_{n+3}, b_{n+3}\rangle=d^2\left\{\sum_{\alpha=1}^{n-1} R_\alpha^2+N^2-(N+1)^2\right\}=d^2\left\{\sum_{\alpha=1}^{n-1} R_\alpha^2-1-2N\right\},$$

we have $\langle b_{n+3}, b_{n+3}\rangle<0$ for a sufficiently large N, and then putting $d=\sqrt{\left|\sum_{\alpha=1}^{n-1} R_\alpha^2-1-2N\right|}^{-1}$, we get $\langle b_{n+3}, b_{n+3}\rangle=-1$. Therefore according to Remark 3.2, we can extend b_{n+2}, b_{n+3} to a Lie frame $b_1, \cdots, b_{n+3}$ so that $A_1=\begin{pmatrix} {}^tb_1 \\ \vdots \\ {}^tb_{n+3}\end{pmatrix}\in O(n+1, 2)$. q.e.d.

Remark 5.5. The desired Lie transformation A in Proposition is obtained now by $A=A_2\cdot A_1$, where A_2 is the conformal transformation of S^n given by $A_2=\begin{pmatrix} {}^tb_1' \\ \vdots \\ {}^tb_{n+3}'\end{pmatrix}\in O(n+1, 2)$; $b_1', \cdots, b_{n+3}'$ is a Lie frame extended from

$$b_{n+2}'={}^t(x', -aK, 0, a(K+1), 0),$$

$$b_{n+3}'={}^t(0, 0, 0, 0, 1)$$

where $x'=(-a\tilde{\Lambda}_{ii}^a(p), -a\tilde{\Lambda}_{aa}^\iota(p), -a\tilde{\Lambda}_{aa}^r(p), -a\tilde{\Lambda}_{aa}^u(p))\in\boldsymbol{R}^{n-1}$, K and a are chosen so that $\langle b_{n+2}', b_{n+2}'\rangle=-1$.

§ 6. Using tautness.

We can start now from a compact embedded Dupin hypersurface $\boldsymbol{q}: M\to S^n$ which has a point $p\in M$, at which the normal geodesic is "common". The purpose of this section is to show:

PROPOSITION 6.1. *The normal geodesic γ at p cuts M at eight points $p_1=$*

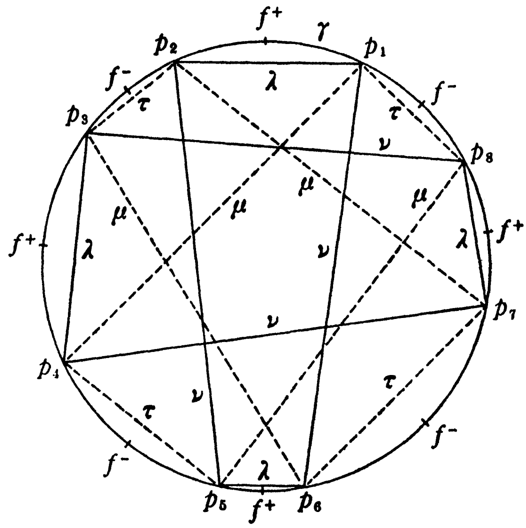

Fig. 6.1.

$p, p_2, \cdots, p_8$. γ is the common normal geodesic at every point $p_i (i=1, \cdots, 8)$. Moreover, all leaves at p_i's are connected as in Fig. 6.1.

Remark 6.2. The leaf L_p^λ of a Dupin hypersurface M corresponding to the principal curvature λ at p is an m_1-sphere centered at

$$c=p+\frac{H_a}{\|H_a\|^2},$$

where

$$H_a=\sum_{\alpha\notin[a]} A_{aa}^{\alpha} e_\alpha+\lambda \boldsymbol{n}-\boldsymbol{p}$$

[5]. Obviously, L_p^λ lies on the hypersphere S_p^λ with center at

$$f_p^\lambda=\cos\theta^\lambda \boldsymbol{p}+\sin\theta^\lambda \boldsymbol{n},$$

where $\theta^\lambda=\cot^{-1}\lambda\left(-\frac{\pi}{2}<\theta^\lambda\leqq\frac{\pi}{2}\right)$ is its radius. With the normal geodesic γ at p, L_p^λ has another intersection $p'\neq p$, iff $A_{aa}^{\alpha}(p)=0$ for all $\alpha\notin[a]$, that is if L_p^λ is totally geodesic in S_p^λ. In this case, γ is the normal geodesic at p', too [5, Remark 2.3].

Remark 6.3. By Thorbergsson [11], $H_i(M; \boldsymbol{Z}_2)\neq 0$ only when $i=0, m_1, m_2, m_1+m_2, 2m_1+m_2, m_1+2m_2, n-1$ and $\beta(M; \boldsymbol{Z}_2)=8$. Non-zero cycles of $H_{m_1}(M)$ and $H_{m_2}(M)$ are represented by L^λ and L^τ, respectively. We denote other non-trivial cycles by

$$[c^{\mu\lambda}]\in H_{m_1+m_2}(M),\ [c^{\nu\tau}]\in H_{m_1+m_2}(M),$$

$$[c^{\nu\mu\lambda}]\in H_{2m_1+m_2}(M),\ [c^{\mu\nu\tau}]\in H_{m_1+2m_2}(M),$$

which are image cycles of the fundamental cycles of N_i's by maps h_i's defined in [11]. Note that the intersection number $S(\ ,\ )$ satisfies

$$S(c^{\mu\lambda}, c^{\nu\tau})=1,\ S(L^{\tau}, c^{\nu\mu\lambda})=1,\ S(L^{\lambda}, c^{\mu\nu\tau})=1. \tag{6.1}$$

We denote by $B_p^{\lambda\pm}$, the ball such that $\partial B_p^{\lambda\pm}=S_p^{\lambda}$ where $\boldsymbol{n}(p)$ is the inner (outer, resp.) normal to $B_p^{\lambda+}(B_p^{\lambda-}$, resp.). $B_p^{\mu\pm}$, $B_p^{\nu\pm}$ and $B_p^{\tau\pm}$ are similarly defined.

LEMMA 6.4. *For any $p\in M$ and any $L^{\lambda}(L^{\tau}$, resp.), $B_p^{\mu-}\cap L^{\lambda}\neq\emptyset(B_p^{\nu+}\cap L^{\tau}\neq\emptyset$, resp.).*

Proof. $B_p^{\mu-}$ contains the homology cycle $c^{\mu\nu\tau}$ with which any m_1-cycle L^{λ} has the intersection number $\neq 0$. q.e.d.

Proof of Proposition 6.1. At p, by (5.1) and by Remark 6.2, we have $\gamma\cap L_p^{\lambda}=\{p, p_2\}$, $\gamma\cap L_p^{\mu}=\{p, p_4\}$, $\gamma\cap L_p^{\nu}=\{p, p_6\}$, $\gamma\cap L_p^{\tau}=\{p, p_8\}$. We have $f_{p_2}^{\tau}$ $(f_{p_4}^{\lambda}$, resp.) on γ in the direction from p_2 to $p_4(p_4$ to p_2, resp.), since along L_p^{λ} and L_p^{μ}, the positive normal direction is preserved. Moreover by tautness, we have $(B_{p_2}^{\tau})^{\circ}\cap M=\emptyset$ and $(B_{p_4}^{\lambda})^{0}\cap M=\emptyset$. Therefore $f_{p_2}^{\tau}$ and $f_{p_4}^{\lambda}$ are situated in this order along $\widehat{p_2p_4}$ from p_2 to p_4. On the other hand, by Thorbergsson [11], S^n is decomposed into two disk bundles over two focal submanifolds $M_{\pm}:=\{f_p^{\pm}\mid p\in M\}$, $f_p^{+}=f_p^{\lambda}$, $f_p^{-}=f_p^{\tau}$, both of which have M as their boundaries. Thus there exists a point $p_3\in M$ between $f_{p_2}^{\tau}$ and $f_{p_4}^{\lambda}$ on $\widehat{p_2p_4}$. Similarly, there exist p_5, $p_7\in M$ on the arcs $\widehat{p_4p_6}$, $\widehat{p_6p_8}$, respectively.

Remark 6.5. At this stage, we say nothing about the relation among p_2, p_3, p_4. Neither do we know whether M cuts γ orthogonally at p_3, p_5, p_7, or not. The proof is completed by Lemma 6.6 and 6.7.

In the following (and in §9 as well), we denote L_i^*, S_i^*, $B_i^{*\pm}$ for $L_{p_i}^*$, $S_{p_i}^*$, $B_{p_i}^{*\pm}$, respectively, for $i=1, \cdots, 8$ and $*=\lambda, \mu, \nu, \tau$.

LEMMA 6.6. *$L_i^{\tau}\cap L_{i+2}^{\lambda}=p_{i+1}$, for $i=2$, 4, 6.*

Proof. [$i=2$]. By tautness, we have $L_2^{\tau}=B_2^{\tau}\cap M\subset S_2^{\tau}$ and $L_4^{\lambda}=B_4^{\lambda}\cap M\subset S_4^{\lambda}$ [5, Lemma 3.5]. Then denoting here $\theta_i^*=|\theta_i^*|$, we have

$$2\theta_2^{\tau}\leqq d(p_2, p_3) \quad \text{and} \quad 2\theta_4^{\lambda}\leqq d(p_4, p_3)$$

where equalities imply $p_3\in L_2^{\tau}$ and $p_3\in L_4^{\lambda}$, respectively. Hence $L_2^{\tau}\cap L_4^{\lambda}\neq\emptyset$ implies $L_2^{\tau}\cap L_4^{\lambda}=\{p_3\}$. Now, suppose $L_2^{\tau}\cap L_4^{\lambda}=\emptyset$. Then

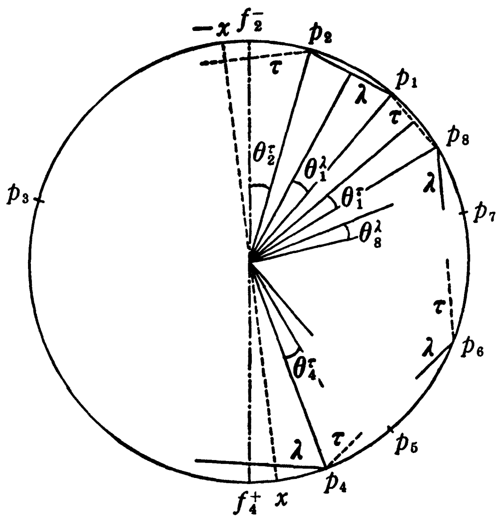

Fig. 6.2.

$$\theta_2^\tau < d(p_3, f_2^-), \qquad \theta_4^\lambda < d(p_3, f_4^+).$$

Therefore we can transform S^n conformally preserving γ so that S_2^τ and S_4^λ are centered at mutually antipodal points (Without confusion, we denote the image by the same letter). Let $x \in \widehat{p_4 f_4^+}$ be sufficiently near f_4^+ so that d_x is a Morse function on M. Let $B(x, r)$ be the ball centered at x with radius r.

In $B(-x, d(-x, p_2))$, d_{-x} has critical points of indicies 0 and m_2, the latter is nothing but p_2. Obviously as a critical point of d_{-x}, p_1 is of index m_1. Since

$$d(p_8, f_8^\nu) \leqq \theta_1^\tau + \theta_1^\lambda + \theta_2^\tau$$

by Lemma 6.4, the index of p_8 as a critical point of $d_{-x} \geqq m_1 + m_2$. In the same way, the index of p_8 as a critical point of $d_x \geqq m_1 + m_2$. Thus the index of p_8 with respect to d_{-x} is $m_1 + m_2$, corresponding to the cycle $c^{\nu\tau}$. On the other hand, since $B_2^{\mu+}$ contains $c^{\mu\lambda}$ which is not homotopic to $c^{\nu\tau}$, and since $d(f_2^\mu, p_2) \leqq \theta_1^\lambda + \theta_1^\tau + \theta_8^\lambda$ by Lemma 6.4, d_{-x} should have another critical point of index $m_1 + m_2$ in $B := B(-x, d(-x, p_8) + 2\theta_8^\lambda)$. Now, in $B(x, d(x, p_4) + 2\theta_4^\tau)$, d_x has three critical points corresponding to a point, L_4^λ and L_4^τ. (Note that we may assume $d(x, f_4^+) < \theta_4^\tau$.) Thus from $p_6 \in B^c$, d_x has four critical points in B^c. Finally, d_{-x} should have nine critical points on M, which contradicts tautness.

[$i=6$]. Similar to [$i=2$].

[$i=4$]. Easier than above two cases. q. e. d.

LEMMA 6.7. $L_2^\mu = L_7^\mu$, $L_3^\mu = L_6^\mu$, $L_5^\mu = L_8^\mu$,

$$L_2^\nu = L_5^\nu,\ L_3^\nu = L_8^\nu,\ L_4^\nu = L_7^\nu.$$

Proof. $[L_2^\mu = L_7^\mu]$. Consider the cycle $c_2^{\mu\lambda} \subset B_2^{\mu+}$ at p_2 and $c_7^{\nu\tau} \subset B_7^{\nu-}$ at p_7. Then $S(c_2^{\mu\lambda}, c_7^{\nu\tau})=1$ and $S(c_2^{\mu\nu\tau}, L_7^\lambda)=1$ imply $B_2^{\mu+} \cap B_7^{\nu-} = \{p_7\}$ and $p_7 \in c_2^{\mu\lambda}$. Since $c_2^{\mu\lambda} \cap S_2^{\mu\lambda} = L_2^\mu$, we get $p_7 \in L_2^\mu$, that is $L_2^\mu = L_7^\mu$. The other cases follow similarly. q. e. d.

LEMMA 6.8. *Let z_i be the complex number corresponding to $p_i \in S^1 \subset \boldsymbol{R}^2 \cong \boldsymbol{C}$, $i=1, \cdots, 8$. Then we have*

$$\Psi(p_1) = [z_2, z_8 ; z_4, z_6],$$

where $[z_2, z_8 ; z_4, z_6] = \dfrac{z_2 - z_4}{z_2 - z_6} \Big/ \dfrac{z_8 - z_4}{z_8 - z_6}$ *is the cross ratio.* $\Psi(p_i)$, $i=2, \cdots, 8$ *are similarly obtained.*

Proof. As is well-known,

$$[z_1, -z_1 ; z_2, z_4] = \tan\frac{1}{2}\{\arg(z_2/z_1)\} / \tan\frac{1}{2}\{\arg(z_4/z_1)\}$$

$$= \frac{\mu(p_1)}{\lambda(p_1)}.$$

Therefore

$$\Psi(p_1) = \frac{1-\mu/\lambda}{1-\nu/\lambda} \Big/ \frac{1-\mu/\tau}{1-\nu/\tau} = \frac{1-[z_1, -z_1 ; z_2, z_4]}{1-[z_1, -z_1 ; z_2, z_6]} \Big/ \frac{1-[z_1, -z_1 ; z_8, z_4]}{1-[z_1, -z_1 ; z_8, z_6]}$$

$$= [z_2, z_8 ; z_4, z_6].$$

q. e. d.

§ 7. Lie transformation of S^1.

In this section, we find a Lie transformation $A_3 \in O(2, 2) \subset O(n+1, 2)$, which maps $\gamma \cap M$ from the position in Fig. 6.1 to the position in Fig. 7.2. We follow the argument in [5]. That is, we restrict our argument to $\gamma = S^1$, on which, any pair of points connected by some leaf is considered as an oriented hypersphere of S^1. We give positive orientation to a leaf L_i^* if $\theta_i^* = \cot^{-1} *_i > 0$, $* = \lambda, \mu, \nu, \tau$. (Recall that θ_i^* is chosen so that $-\pi/2 < \theta_i^* \leqq \pi/2$).

First of all, since $L_1^\lambda \cap L_5^\lambda = \{\phi\}$, we can transform S^1 conformally so that the images of L_1^λ and L_5^λ are centered at mutually antipodal points. Then by a conformal transformation fixing these centers, we may assume that $\lambda(p_1) = \cot\theta = \lambda(p_5)$ for some $0 < \theta < \pi/2$. Note that in § 9, we denote the combination of these conformal transformations by C.

Now, put $\varphi = \cot^{-1}(-\tau(p_2))$, $\phi = \cot^{-1}(-\tau(p_1))$, $\varphi' = \cot^{-1}(-\tau(p_4))$ and $\phi' = \cot^{-1}(-\tau(p_6))$, where $0 < \varphi, \phi, \varphi', \phi' < \pi/2$ (see Fig. 7.1). Then since $(\theta+\varphi+\phi)+(\theta+\varphi'+\phi')$ is less than π, we may assume

$$\theta + \varphi + \phi < \frac{\pi}{2}, \tag{7.1}$$

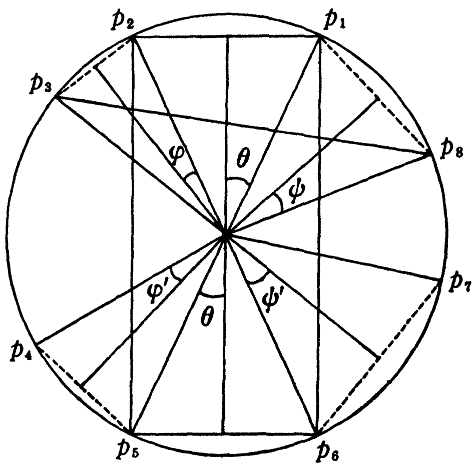

Fig. 7.1.

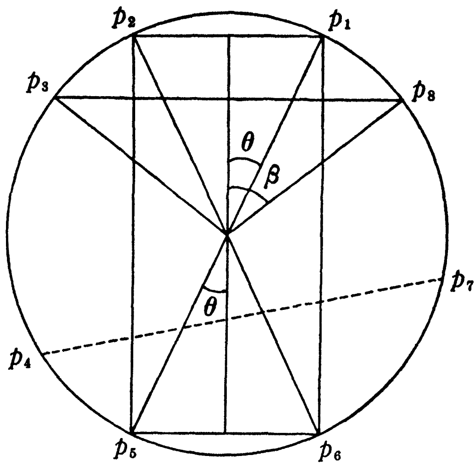

Fig. 7.2

without loss of generality. Let

$$k_1=\{p_1,\ p_2\}={}^t\!\left(\frac{1}{\cos\theta},\ 0,\ 1,\ \tan\theta\right),$$

$$k_2=\{p_5,\ p_6\}={}^t\!\left(-\frac{1}{\cos\theta},\ 0,\ 1,\ \tan\theta\right)$$

$$k_3=\{p_3,\ p_8\}={}^t\!\left(\frac{\cos\omega}{\cos\sigma},\ \frac{\sin\omega}{\cos\sigma},\ 1,\ -\tan\sigma\right),$$

where $\sigma=\theta+\varphi+\psi$, $\omega=\psi-\varphi$, and

$$k_4={}^t(0, 0, 0, 1).$$

Note that $k_1, k_2, k_3 \in Q$ but $k_4 \notin Q$. Moreover, let

$$h_1=k_1,$$

$$h_2=uk_2, \qquad u\in\boldsymbol{R}-\{0\},$$

$$h_3=v\,{}^t\!\left(\frac{1}{\cos\beta}, 0, 1, -\tan\beta\right), \quad v\in\boldsymbol{R}-\{0\},$$

$$h_4={}^t(x, y, z, w).$$

Note that $h_1, h_2, h_3 \in Q$. For $K=(k_1, k_2, k_3, k_4)$ and $H=(h_1, h_2, h_3, h_4)$, find $A_3\in O(2, 2)$ such that $A_3K=H$. Since

$$K^{-1}=\begin{pmatrix} \dfrac{\cos\theta}{2} & -\dfrac{\cos\sigma}{2\sin\omega}\left(1+\dfrac{\cos\theta\cos\omega}{\cos\sigma}\right) & \dfrac{1}{2} & 0 \\ -\dfrac{\cos\theta}{2} & -\dfrac{\cos\sigma}{2\sin\omega}\left(1-\dfrac{\cos\theta\cos\omega}{\cos\sigma}\right) & \dfrac{1}{2} & 0 \\ 0 & \dfrac{\cos\sigma}{\sin\omega} & 0 & 0 \\ 0 & \dfrac{\cos\sigma}{\sin\omega}(\tan\sigma+\tan\theta) & -\tan\theta & 1 \end{pmatrix},$$

$A_3=HK^{-1}=(a_1, a_2, a_3, a_4)$ is given by

$$a_1=\frac{\cos\theta}{2}(h_1-h_2),$$

$$a_2=\frac{\cos\sigma}{\sin\omega}\left(-\frac{\cos\omega}{\cos\sigma}a_1-a_3+h_3+\tan\sigma h_4\right),$$

$$a_3=\frac{1}{2}(h_1+h_2)-\tan\theta h_4,$$

$$a_4=h_4.$$

By writing down the condition $A_3\in 0(2, 2)$, i.e.

$$\langle a_1, a_1\rangle=\langle a_2, a_2\rangle=-\langle a_3, a_3\rangle=-\langle a_4, a_4\rangle=1,$$

$$\langle a_i, a_j\rangle=0 \qquad \text{for} \quad i\neq j,$$

as equations with unknowns $(u, v, \beta, x, y, z, w)$, we have

$$u=1 \tag{7.2}$$

$$v=\frac{\cos\omega\cos\beta}{\cos\sigma} \tag{7.3}$$

$$\cos(\theta+\beta)=\frac{\cos(\theta+\sigma)}{\cos\omega} \tag{7.4}$$

$$x=0 \tag{7.5}$$

$$z+w\tan\theta=\tan\theta \tag{7.6}$$

$$-z+w\tan\beta=\frac{\sin\sigma}{\cos\omega\cos\beta} \tag{7.7}$$

$$y^2-z^2-w^2=-1. \tag{7.8}$$

In fact,

(1) $1=\langle a_1, a_1\rangle$ implies (7.2) and $a_1={}^t(1, 0, 0, 0)$, $1/2(h_1+h_2)={}^t(0, 0, 1, \tan\theta)$.

(2) $0=\langle a_1, a_4\rangle$ implies (7.5).

(3) $\langle a_1, a_3\rangle=0$ follows from (1) and (2).

(4) $0=\langle a_1, a_2\rangle=-\dfrac{\cos\omega}{\sin\omega}+\dfrac{\cos\sigma}{\sin\omega}\langle a_1, h_3\rangle$ implies $\langle a_1, h_3\rangle=\dfrac{\cos\omega}{\cos\sigma}$ and (7.3).

(5) $0=\langle a_3, a_4\rangle$ implies (7.6).

(6) $\langle a_3, a_3\rangle=-1-\tan^2\theta-\tan^2\theta-2\tan\theta(-z-w\tan\theta)=-1$ follows from (5).

(7) $0=\langle a_2, a_4\rangle=\dfrac{\cos\sigma}{\sin\omega}(\langle h_3, h_4\rangle-\tan\sigma)$ implies $\langle h_3, h_4\rangle=\tan\sigma$: (7.7).

(8) $0=\langle a_2, a_3\rangle=\dfrac{\cos\sigma}{\sin\omega}(1+\langle h_3, a_3\rangle)$ implies $\langle a_3, h_3\rangle=-1$ so that $v(-1+\tan\theta\tan\beta)-\tan\theta\tan\sigma=-1$, i.e. (7.4).

(9)
$$\begin{aligned}\langle a_2, a_2\rangle&=\frac{\cos^2\sigma}{\sin^2\omega}\Big(\frac{\cos^2\omega}{\cos^2\sigma}-1-\tan^2\sigma-2\frac{\cos\omega}{\cos\sigma}\langle a_1, h_3\rangle\\&\quad-2\langle a_3, h_3\rangle+2\tan\sigma\langle h_3, h_4\rangle\Big)\\&=1\end{aligned}$$

follows from (4), (7) and (8).

(10) $-1=\langle a_4, a_4\rangle$ implies (7.8).

Case 1. If $\theta+\sigma=2\theta+\varphi+\psi\leqq\pi/2$ is satisfied, then since

$$0<\varphi\sim\psi<\varphi+\psi<2\theta+\varphi+\psi=\theta+\sigma\leqq\frac{\pi}{2},$$

we get

$$\cos\omega=\cos(\psi-\varphi)>\cos(\theta+\sigma)\geqq 0,$$

and so

$$0\leqq\frac{\cos(\theta+\sigma)}{\cos\omega}<1.$$

Thus we can find β satisfying (7.4). Then, v is obtained by (7.3). On the other hand, we get

$$w=\frac{1}{\sin(\theta+\beta)}\left\{\sin\theta\cos\beta+\frac{\sin\sigma}{\cos\omega}\cos\theta\right\}$$

from (7.6) and (7.7), so it follows

$$w-1=\frac{\cos\theta}{\sin(\theta+\beta)}\left\{\frac{\sin\sigma}{\cos\omega}-\sin\beta\right\}.$$

Now,

$$\cos(\theta+\beta)=\frac{\cos(\theta+\sigma)}{\cos\omega}\geqq\cos(\theta+\sigma)$$

implies $\beta\leqq\sigma$ because of $\theta+\sigma\leqq\pi/2$, and hence we have

$$\sin\beta\leqq\sin\sigma\leqq\frac{\sin\sigma}{\cos\omega},$$

i.e. $w\geqq 1$. This guarantees $y^2\geqq 0$ in (7.8), and finally, the whole solution is given by

$$\text{(7.9)}\quad\begin{cases} u=1,\ v=\dfrac{\cos\omega\cos\beta}{\cos\sigma} \quad \text{where} \quad \beta=\cos^{-1}\left\{\dfrac{\cos(\theta+\sigma)}{\cos\omega}\right\}-\theta, \\ x=0,\ y=\sqrt{z^2+w^2-1}, \quad \text{where} \quad z=\dfrac{\sin\theta}{\sin(\theta+\beta)}\left\{\sin\beta-\dfrac{\sin\sigma}{\cos\omega}\right\}, \\ w=\dfrac{1}{\sin(\theta+\beta)}\left\{\sin\theta\cos\beta+\dfrac{\sin\sigma}{\cos\omega}\cos\theta\right\}. \end{cases}$$

Case 2. When $\theta+\sigma=2\theta+\varphi+\psi>\pi/2$, we have

$$0<\varphi\sim\psi<\pi-2\theta-\varphi-\psi=\pi-\theta-\sigma<\frac{\pi}{2}$$

since $2(\theta+\varphi)<\pi$ and $2(\theta+\psi)<\pi$ by (7.1). This implies

$$\cos\omega>\cos(\pi-\theta-\sigma)=-\cos(\theta+\sigma)>0,$$

and we get β satisfying (7.4), $0<\beta<\pi$. On the other hand, from

$$\cos(\theta+\beta)=\frac{\cos(\theta+\sigma)}{\cos\omega}\leqq\cos(\theta+\sigma)<0,$$

noting that $\theta+\sigma<\pi$, we have $\beta\geqq\sigma$. Thus we obtain, using (7.1),

$$\cos\beta\leqq\cos\sigma\leqq\frac{\cos\sigma}{\cos\omega}.$$

Now, since (7.4) is equal to

$$\cos\theta\cos\beta-\sin\theta\sin\beta=\frac{\cos\theta\cos\sigma-\sin\theta\sin\sigma}{\cos\omega},$$

we get

$$\sin\beta \leqq \frac{\sin\sigma}{\cos\omega},$$

equivalently $w \geqq 1$. Thus, in this case again, solutions are given by (7.9).

§ 8. Under the assumption that Ψ is constant on M.

Here, for the first time, we assume that Ψ is constant on M.

PROPOSITION 8.1. *If Ψ is constant on M, then there is a Lie transformation $A_4 \in O(2, 2)$ which maps $\gamma \cap M$ in Fig. 7.2 to the position in Fig. 8.1.*

Proof. Consider the image hypersurface shown in Fig. 7.2. Note that $\nu(p_1)=\nu(p_2)$ and $\tau(p_1)=\tau(p_2)$ imply $\mu(p_1)=\mu(p_2)$. Now, let $\eta=\cot^{-1}\lambda(p_3)$, $\varphi=\cot^{-1}(-\tau(p_2))$ and $\varphi'=\cot^{-1}(-\tau(p_4))$, where $0<\eta, \varphi, \varphi'<\pi/2$. Without loss of generality, assume that $\varphi \leqq \varphi'$. Then we have

$$\theta+2\varphi+\eta \leqq \frac{\pi}{2}. \tag{8.1}$$

To imitate the previous calculation, rotating the coordinate axes by $\pi/2$, we put

$$k_1=\{p_2, p_5\}={}^t\Big(\frac{1}{\sin\theta}, 0, 1, -\cot\theta\Big),$$

$$k_2=\{p_1, p_6\}={}^t\Big(-\frac{1}{\sin\theta}, 0, 1, -\cot\theta\Big),$$

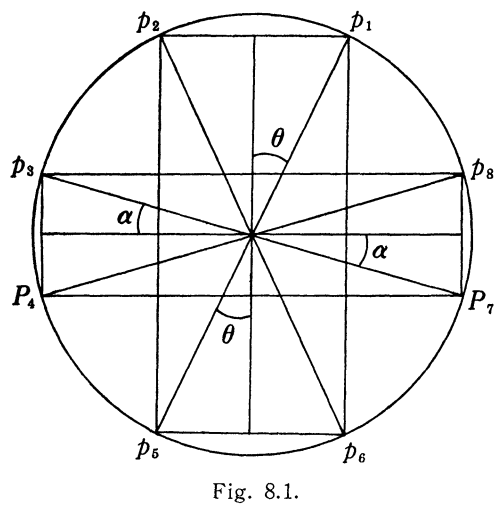

Fig. 8.1.

$$k_3=\{p_3,\ p_4\}={}^t\Big(\frac{\sin\zeta}{\cos\eta},\ \frac{\cos\zeta}{\cos\eta},\ 1,\ \tan\eta\Big),\quad \zeta=\theta+2\varphi+\eta,$$

$$k_4={}^t(0,\ 0,\ 0,\ 1),$$

$$h_1=k_1,$$

$$h_2=uk_2,\quad u\in\boldsymbol{R}-\{0\},$$

$$h_3=v\,{}^t\Big(\frac{1}{\cos\alpha},\ 0,\ 1,\ \tan\alpha\Big),\qquad v\in\boldsymbol{R}-\{0\},$$

$$h_4={}^t(x,\ y,\ z,\ w).$$

Now, find $A_4\in O(2, 2)$ such that $A_4K=H$, where $K=(k_1, k_2, k_3, k_4)$ and $H=(h_1, h_2, h_3, h_4)$. As before, since we have

$$K^{-1}=\begin{pmatrix} \dfrac{\sin\theta}{2} & -\dfrac{\cos\eta}{2\cos\zeta}\Big(1+\dfrac{\sin\zeta\sin\theta}{\cos\eta}\Big) & \dfrac{1}{2} & 0 \\ -\dfrac{\sin\theta}{2} & -\dfrac{\cos\eta}{2\cos\zeta}\Big(1-\dfrac{\sin\zeta\sin\theta}{\cos\eta}\Big) & \dfrac{1}{2} & 0 \\ 0 & \dfrac{\cos\eta}{\cos\zeta} & 0 & 0 \\ 0 & -\dfrac{\cos\eta}{\cos\zeta}\Big(\tan\eta+\tan\theta\Big) & \cot\theta & 1 \end{pmatrix},$$

A_4 is given by

$$a_1=\frac{\sin\theta}{2}(h_1-h_2),$$

$$a_2=\frac{\cos\eta}{\cos\zeta}\Big\{-\frac{\sin\zeta}{\cos\eta}a_1-a_3+h_3-\tan\eta\, h_4\Big\},$$

$$a_3=\frac{1}{2}(h_1+h_2)+\cot\theta\, h_4,$$

$$a_4=h_4.$$

The condition $A_4\in O(2, 2)$ are, this time:

$$u=1 \tag{8.2}$$

$$\sin(\theta-\alpha)=\frac{\sin(\theta-\eta)}{\sin\zeta} \tag{8.3}$$

$$v=\frac{\cos\alpha\sin\zeta}{\cos\eta} \tag{8.4}$$

$$x=0 \tag{8.5}$$

$$-z+w\cot\theta=\cot\theta \tag{8.6}$$

$$z+w\tan\alpha=\frac{\sin\eta}{\cos\alpha\sin\zeta} \tag{8.7}$$

$$y^2-z^2-w^2=-1. \tag{8.8}$$

As before, from (8.1) and

$$\theta\sim\eta<\theta+\eta<\theta+2\varphi+\eta=\zeta,$$

there exists α satisfying (8.3). From (8.6) and (8.7), we get

$$w=\frac{1}{\cos(\theta-\alpha)}\left(\cos\theta\cos\alpha+\frac{\sin\theta\sin\eta}{\sin\zeta}\right)$$

and

$$w-1=\frac{\sin\theta}{\cos(\theta-\alpha)}\left(\frac{\sin\eta}{\sin\zeta}-\sin\alpha\right).$$

When $\theta\geqq\eta$, it follows $\alpha\leqq\eta$ from (8.3) so that

$$\sin\alpha\leqq\sin\eta\leqq\frac{\sin\eta}{\sin\zeta}$$

implies $w\geqq 1$, and hence $y^2\geqq 0$.

When $\theta<\eta$, though $\alpha>\eta$, we have from (8.3),

$$\begin{aligned}\frac{\sin\theta\cos\eta-\cos\theta\sin\eta}{\sin\zeta}&=\sin\theta\cos\alpha-\cos\theta\sin\alpha\\&<\sin\theta\cos\eta-\cos\theta\sin\alpha\\&\leqq\sin\theta\frac{\cos\eta}{\sin\zeta}-\cos\theta\sin\alpha\end{aligned}$$

so that

$$\frac{\sin\eta}{\sin\zeta}>\sin\alpha.$$

Therefore in both cases, solutions are given by

$$\begin{cases} u=1, \quad \alpha=\theta-\sin^{-1}\left\{\dfrac{\sin(\theta-\eta)}{\sin\zeta}\right\}, \quad v=\dfrac{\cos\alpha\sin\zeta}{\cos\eta}, \\ x=0, \quad y=\sqrt{z^2+w^2-1} \quad \text{where} \quad z=\dfrac{\cos\theta}{\cos(\theta-\alpha)}\left(\dfrac{\sin\eta}{\sin\zeta}-\sin\alpha\right), \\ w=\dfrac{1}{\cos(\theta-\alpha)}\left(\cos\theta\,\cos\alpha+\dfrac{\sin\theta\sin\eta}{\sin\zeta}\right). \end{cases}$$

Finally, by using Ψ is constant, we get $\tau(p_2)=\tau(p_4)\overset{(*)}{=}\tau(p_6)=\tau(p_8)$. Put $\rho=$

(*) By an easy calculation, we have for $k_5=\{p_5, p_7\}$, $Ak_5={}^t\left(-\dfrac{1}{\cos\alpha}, 0, 1, \tan\alpha\right)$, and the equality holds.

$\cot^{-1}(-\tau(p_2))$.

PROPOSITION 8.2. *If Ψ is constant on M, then we have $\theta+\rho=\pi/4$ and $\theta=\alpha$.*

Proof. By Lemma 6.8, we have

$$\Psi(p_1)=\frac{z_2-z_4}{z_2-z_6}\Big/\frac{z_8-z_4}{z_8-z_6}=\frac{e^{2\iota(\alpha+\rho)}-1}{-2}\cdot\frac{e^{-2\iota(\alpha+\rho)}-1}{-2}$$

$$\Psi(p_4)=\frac{z_3-z_1}{z_3-z_7}\Big/\frac{z_5-z_1}{z_5-z_7}=\frac{e^{-2\iota(\alpha+\rho)}-1}{-2}\cdot\frac{e^{2\iota(\theta+\rho)}-1}{-2}.$$

From $(\alpha+\rho)+(\theta+\rho)=\pi/2$, it follows

$$e^{2\iota(\theta+\rho)}=e^{\iota(\pi-2(\alpha+\rho))}=-e^{-2\iota(\alpha+\rho)}.$$

So if we put $z=e^{2\iota(\theta+\rho)}$, $\Psi(p_1)=\Psi(p_4)$ is written as

$$(z-1)(\bar{z}-1)=(z+1)(\bar{z}+1).$$

Thus we get

$$\theta+\rho=\alpha+\rho=\frac{\pi}{4}. \qquad \text{q.e.d.}$$

COROLLARY 8.3. *If Ψ is constant on M, then by taking $A_5\in O(2,2)$ of type* (3.5), *we can transform Fig.* 8.1 *so that $\theta=\alpha=\rho=\pi/8$. Therefore, in fact, $\Psi=1/2$.*

§ 9. Proof of Theorem II.

From the results in § 5–8, when Ψ is constant on M, we see that at each point p of M, there exists Lie transformation $A_p=A_5\cdot A_4\cdot A_3\cdot C\cdot A_2\cdot A_1$ of which image satisfies the relation in Fig. 9.1 where we denote the image point of p by p_1 and the normal geodesic at p_1 by γ. Note that this result is obtained not only by a local calculation but also by using a global property "tautness" of M.

Now we show:

PROPOSITION 9.1. *Let N be a Lie image of an isoparametric hypersurface N_0 with four principal curvatures. Assume that at some point $p_1\in N$, the intersection of N with the normal geodesic γ at p_1 makes a regular octagon $p_1p_2\cdots p_8$ as in Fig.* 9.1, *where γ is common at each p_i. Then N is itself isoparametric.*

Proof. By a parallel transformation, if necessary, we may assume that $\lambda\equiv\cot(\pi/8)\equiv-\tau$ and $\mu\equiv\cot(3\pi/8)\equiv-\nu$ on N_0. Let N be the image of N_0 by some $A\in O(n+1,2)$. Let q_ι be the inverse image point of p_ι. Then A maps the regular octagon at q_1 to the regular octagon at p_1. This means, up to isometry, that if we put

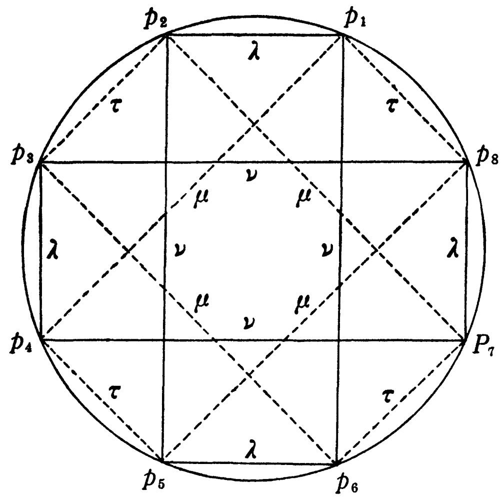

Fig. 9.1.

$$k_1(q_i)=k_1(p_i)={}^t\left(\mathbf{0},\ \cos\frac{(i-1)}{4}\pi,\ \sin\frac{(i-1)}{4}\pi,\ 1,\ 0\right)$$

$$k_2(q_i)=k_2(p_i)={}^t\left(\mathbf{0},\ (-1)^i\sin\frac{(i-1)}{4}\pi,\ (-1)^{i-1}\cos\frac{(i-1)}{4}\pi,\ 0,\ 1\right)$$

for $i=1, 2, \cdots, 8$, and $\mathbf{0}=(\underbrace{0, \cdots, 0}_{n-1})$, then

$$\begin{aligned}&\alpha(q_i)Ak_1(q_i)+\beta(q_i)Ak_2(q_i)=k_1(p_i)\\&\gamma(q_i)Ak_1(q_i)+\delta(q_i)Ak_2(q_i)=k_2(p_i),\end{aligned}\tag{9.1}$$

where

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}$$

and $a, b, c, d\colon N_0\to\boldsymbol{R}$ are given by

$$Ak_1(q_i)={}^t(*, \cdots, *, a(q_i), b(q_i))$$

$$Ak_2(q_i)={}^t(*, \cdots, *, c(q_i), d(q_i)).$$

Let $V^{n+3}=\boldsymbol{R}^{n-1}\times\boldsymbol{R}^4_2$, where $\boldsymbol{R}^4_2=\mathrm{span}({}^t(\mathbf{0}, 1, 0, 0, 0), {}^t(\mathbf{0}, 0, 1, 0, 0), {}^t(\mathbf{0}, 0, 0, 1, 0), {}^t(\mathbf{0}, 0, 0, 0, 1))$. Then since $\mathrm{span}_{1\leq i\leq 8}(k_1(q_i), k_2(q_i))=\boldsymbol{R}^4_2$ and $\mathrm{span}_{1\leq i\leq 8}(Ak_1(q_i), Ak_2(q_i))=\boldsymbol{R}^4_2$, we get

$$A=\begin{pmatrix}A_1 & 0\\ B & A_2\end{pmatrix}.$$

From Lemma 3.1, it follows $B=0$ and $A_1\in O(n-1)$. Now, if we check (9.1) carefully, we get $A_2 = \pm$idendity. q. e. d.

Now, our image hypersurface of M by A_p has the regular octagon at the image point of p as was shown in §8. Thus for M to be a Lie image of an isoparametric hypersurface, this image by A_p itself must be isoparametric. Unfortunately, we can not show this without additional assumption on M.

THEOREM II. *Let M be a compact embedded Dupin hypersurface with four principal curvatures $\lambda > \mu > \nu > \tau$ in $\bar{M}(c)$. If $[\lambda, \tau; \mu, \nu]$ is constant on M, and if for each λ-leaf L^λ (τ-leaf L^τ, resp.), there exists a ν-leaf L^ν_λ (μ-leaf L^μ_τ, resp.) such that $L^\mu_q \cap L^\nu_\lambda \neq \emptyset$ for all $q \in L^\lambda$ ($L^\nu_q \cap L^\mu_\tau \neq \emptyset$ for all $q \in L^\tau$, resp.), then M is Lie equivalent to an isoparametric hypersurface in a sphere.*

Now, we prove this theorem. For simplicity, we denote by $\tilde{M}$ the image hypersurface of M by A_p, on which the relation in Fig. 9.1 is satisfied.

LEMMA 9.2. *For L^λ_1, $L^\nu_\lambda = L^\nu_4$. Similarly, for L^λ_3 (L^λ_5, L^λ_7, resp.), $L^\nu_\lambda = L^\nu_1$ (L^ν_3, L^ν_2, resp.), and for L^τ_2 (L^τ_4, L^τ_6, L^τ_8, resp.), $L^\mu_\tau = L^\mu_5$ (L^μ_2, L^μ_1, L^μ_3, resp.).*

Proof. Suppose that $L^\nu_\lambda \neq L^\nu_4$. Let $q_1 \in L^\mu_1 \cap L^\nu_\lambda$ and $q_2 \in L^\lambda_2 \cap L^\nu_\lambda$. Then we can write

$$q_1 = c^\mu_1 + \|p_1 - c^\mu_1\| \sin\varphi u + \cos\varphi (p_1 - c^\mu_1),$$

$$q_2 = c^\mu_2 + \|p_2 - c^\mu_2\| \sin\phi v + \cos\phi (p_2 - c^\mu_2),$$

for some $u \in T_{p_1} L^\mu_1$ and $v \in T_{p_2} L^\mu_2$, where, denoting by n_i the unit normal vector to $\tilde{M}$ at p_i, we know (see Remark 6.2, and (2.2) of [5]),

$$c^\mu_i = p_i + \frac{H^\mu_i}{\|H^\mu_i\|^2} = p_i + \frac{\mu(p_i) n_i - p_i}{\mu(p_i)^2 + 1}, \qquad i = 1, 2.$$

Obviously $\varphi \neq 0 \neq \phi$, and by $L^\nu_\lambda \neq L^\nu_4$, $\varphi \neq \pi \neq \phi$. Note that the focal point of L^μ_i is given by

$$f^\mu_i = \cos\frac{3\pi}{8} p_i + \sin\frac{3\pi}{8} n_i, \qquad i = 1, 2.$$

Now, let f^ν be the focal point of L^ν_λ. Then since q_i lies on $f^\mu_i f^\nu$, q_1 and q_2 lies on the geodesic two sphere determined by f^μ_1, f^μ_2 and f^ν. Since we have

$$f^\mu_i \in \mathrm{span}(p_1, n_1) = \mathrm{span}(p_2, n_2)$$

$$q_1 \in \mathrm{span}(p_1, n_1, u)$$

$$q_2 \in \mathrm{span}(p_2, n_2, v) = \mathrm{span}(p_1, n_1, v),$$

we conclude that u and v should be parallel along γ. But then apparently, L^μ_1 would intersect L^μ_2, a contradiction. Thus we have $L^\nu_\lambda = L^\nu_4$. q. e. d.

PROPOSITION 9.3. *Among tangent spaces of leaves at p_i's, TL_1^λ, TL_3^ν, TL_4^ν, TL_5^λ are parallel with respect to the connection of S^n. Other parallel families are $\{TL_2^\tau, TL_1^\mu, TL_8^\mu, TL_7^\tau\}$, $\{TL_3^\lambda, TL_2^\nu, TL_1^\nu, TL_7^\lambda\}$ and $\{TL_4^\tau, TL_3^\mu, TL_2^\mu, TL_1^\tau\}$.*

Proof. Put $TL_1^\lambda=X$, $TL_1^\mu=Y$, $TL_1^\nu=Z$, $TL_1^\tau=W$. In the following, "$=$" means "be parallel to" with respect to the connection of S^n. Note that $L_1^\lambda\ni q\rightarrow L_q^\mu\cap L_4^\nu\in L_4^\nu$ is an onto diffeo because of $\dim L_1^\lambda=\dim L_4^\nu=m_1$. Since f_1^λ and f_q^μ lies on the normal geodesic γ_q at q, and since $f_4^\nu=-f_1^\lambda$, $L_q^\mu\cap L_4^\nu$ also lies on γ_q. Thus L_4^ν is contained in the (m_1+1)-dimensional geodesic sphere determined by f_1^λ and L_1^λ, and obviously intersects γ_q orthogonally for any $q\in L_1^\lambda$. Therefore we conclude that $TL_4^\nu=TL_1^\lambda$. Similarly, we get $TL_6^\tau=Y$, $TL_3^\lambda=Z$, $TL_3^\mu=W$. Then from $T_iM:=T_{p_i}M=T_{p_1}M$, it follows that $TL_4^\tau=W$, $TL_6^\lambda=X$ and so $TL_2^\mu=W$, $TL_3^\nu=X$. Thus we get $TL_2^\tau=Y$, $TL_2^\nu=Z$ and so $TL_5^\mu=Y$, $TL_7^\lambda=Z$.

q. e. d.

Remark 9.4. Without the assumption on leaves, there are infinitely many cases in the relation among tangent spaces of leaves.

By Proposition 9.3, we see that for any point $q\in L_1^\lambda$, the normal geodesic γ_q cuts $\tilde{M}$ orthogonally at eight points $q_1=q$, $q_2, \cdots, q_8$, numbered in order along γ_q, where $q_1, q_2\in L_1^\lambda$, $q_3, q_8\in L_3^\nu$, $q_4, q_7\in L_4^\nu$ and $q_5, q_6\in L_6^\lambda$. In the same way, for any point $q\in L_i^*$, $*=\lambda, \mu, \nu, \tau$, and $i=1, 2, \cdots, 8$, it holds that, the intersection of the normal geodesic γ_q with $\tilde{M}$ makes a regular octagon. Note that at such points of intersection, all $\Lambda_{\alpha\alpha}^\beta=0$, for any α, β such that $\alpha\notin[\beta]$ (see Remark 6.2). Thus we have:

Proof of Theorem II. Let x be any point of $\tilde{M}$. Then we can find $x_1\in L_1^\lambda$, $x_2\in L_{x_1}^\mu$ and $x_3\in L_{x_2}^\nu$ such that $x\in L_{x_3}^\tau$. At x_1, the relation of the normal geodesic and $\tilde{M}$ is the same as in Fig. 9.1 where we replace p_1 by x_1 and other points of intersection by suitable symboles. Next at x_2, and then at x_3, we can show the same thing. Finally at x, we conclude that $\Lambda_{\alpha\alpha}^\beta(x)=0$ for all α and β such that $\alpha\notin[\beta]$. Thus the Lie image of the Dupin hypersurface M by $A_p=A_5\cdot A_4\cdot A_3\cdot C\cdot A_2\cdot A_1$ is isoparametric. q. e. d.

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