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ON THE KOBAYASHI AND CARATHÉODORY DISTANCES OF BOUNDED SYMMETRIC DOMAINS

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1. Let U denote the unit disk in the complex plane $\boldsymbol{C}$ and let ρ be the Poincaré distance in U. ρ is given by

$$\rho(z, w)=\frac{1}{2}\log\frac{1+\left|\dfrac{z-w}{1-\bar{w}z}\right|}{1-\left|\dfrac{z-w}{1-\bar{w}z}\right|} \qquad (z, w\in U).$$

Since any automorphism ϕ of U with $\phi(w)=0$ is given by

$$\phi(z)=e^{i\theta}\frac{z-w}{1-\bar{w}z}$$

with some $\theta\in\boldsymbol{R}$, the distance ρ is also represented in the following:

$$\rho(z, w)=\inf\left\{\frac{1}{2}\log\frac{1+r}{1-r}: 0<r<1 \text{ and } rU\ni\phi(z) \text{ for some } \phi\in\mathrm{Aut}(U) \text{ with } \phi(w)=0\right\},$$

where $\mathrm{Aut}(U)$ denotes the group of automorphisms of U and rU denotes the set $\{z\in\boldsymbol{C}: |z|<r\}$. Furthermore, we have

$$\{z\in U: \rho(0, z)<\alpha\}=rU, \qquad r=\frac{e^{2\alpha}-1}{e^{2\alpha}+1}.$$

In this note we show that the Kobayashi-Carathéodory distance of a bounded symmetric domain has the same property. (It is known that, for a bounded symmetric domain, the Kobayashi distance and the Carathéodory distance coincide [3]). Namely, let D be a bounded symmetric domain given in a canonical realization in the complex N-space $\boldsymbol{C}^N$, and let k_D be the Kobayashi-Carathéodory distance of D. Then we get

$$k_D(z, w)=\inf\left\{\frac{1}{2}\log\frac{1+r}{1-r}: 0<r<1 \text{ and } rD\ni\phi(z) \text{ for some } \phi\in\mathrm{Aut}(D) \text{ with } \phi(w)=0\right\},$$

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and

$$\{z\in D: k_D(0, z)<\alpha\}=rD, \qquad r=\frac{e^{2\alpha}-1}{e^{2\alpha}+1},$$

where $\mathrm{Aut}(D)$ denotes the group of automorphisms of D and rD denotes the set $\{rz: z\in D\}$ in $\boldsymbol{C}^N$.

2. In this section we consider domains D in $\boldsymbol{C}^N$ satisfying the following conditions:

(a) D is homogeneous,

(b) D is bounded,

(c) D is convex and contains the origin 0.

The following facts follow from (b) and (c).

(i) If $0<r<1$, then $\overline{rD}$ is a compact subset of D.

(ii) If $0<r_1<r_2<1$, then $\overline{r_1D}\subset r_2D$.

(iii) If K is a compact subset of D, then $K\subset rD$ for some r with $0<r<1$.

Here $\overline{rD}$ denotes the closure of rD in $\boldsymbol{C}^N$.

(a) and (iii) enable us to introduce the functions $d_w^D: D\to[0, \infty)$ (where $w\in D$) and $d_D: D\times D\to[0, \infty)$;

$$d_w^D(z)=\inf\left\{\frac{1}{2}\log\frac{1+r}{1-r}: 0<r<1 \text{ and } rD\ni\phi(z) \text{ for some } \phi\in\mathrm{Aut}(D) \text{ with } \phi(w)=0\right\},$$

$$d_D(z, w)=\min\{d_w^D(z), d_z^D(w)\}.$$

We note that, for the unit disk U in $\boldsymbol{C}$, we have

$$d_w^U(z)=d_z^U(w)=d_U(z, w)=\rho(z, w) \qquad (z, w\in U).$$

PROPOSITION 1. *The functions d_w^D and d_D have the following properties:*

(1) *$d_w^D(z)\geqq 0$, and $d_w^D(z)=0$ implies $z=w$; $d_D(z, w)\geqq 0$, and $d_D(z, w)=0$ implies $z=w$.*

(2) *$d_D(z, w)=d_D(w, z)$.*

(3) *If $z\to\partial D$, then $d_w^D(z)\to\infty$, $d_z^D(w)\to\infty$ and $d_D(z, w)\to\infty$, where ∂D denotes the boundary of D.*

(4) *If $\phi\in\mathrm{Aut}(D)$, then $d_{\phi(w)}^D(\phi(z))=d_w^D(z)$ and $d_D(\phi(z), \phi(w))=d_D(z, w)$.*

(5) *If $w\in D$ and $\alpha>0$, then*

$$\{z\in D: d_w^D(z)<\alpha\}=\bigcup_{\phi\in\mathrm{Aut}(D),\, \phi(0)=w}\phi(rD), \qquad r=\frac{e^{2\alpha}-1}{e^{2\alpha}+1}.$$

Proof. (2), (4) and (5) are immediate consequences of the definitions of d_w^D and d_D.

To prove (1) let z and w be distinct points in D. Suppose there exists a sequence $\{\phi_n\}$ of elements of $\mathrm{Aut}(D)$ such that $\phi_n(w)=0$ for all n and $\phi_n(z)\to 0$ as $n\to\infty$. Since D is bounded, $\{\phi_n\}$ is a normal family. Hence we may suppose that $\{\phi_n\}$ converges, uniformly on every compact subset of D, to a holomorphic mapping $\psi: D\to \boldsymbol{C}^N$. Since $\psi(w)=\lim_{n\to\infty}\phi_n(w)=0\in D$, $\psi(D)\not\subset\partial D$ and so $\psi\in\mathrm{Aut}(D)$ ([5] p. 78). Thus we are led to a contradiction that $0=\psi(z)\neq\psi(w)=0$. Hence there exists a positive number δ such that $\delta B_N\not\ni\phi(z)$ for all $\phi\in\mathrm{Aut}(D)$ with $\phi(w)=0$. Here B_N denotes the unit ball in $\boldsymbol{C}^N$. Since D is bounded, $\delta B_N\supset r_0 D$ for some r_0 with $0<r_0<1$. Now it follows from (ii) that $d_w^D(z)\geqq\frac{1}{2}\log\frac{1+r_0}{1-r_0}>0$, and (1) follows.

Next we prove (3). Suppose that there exists a positive number α such that, for any compact subset K of D, there is a point $z\notin K$ with $d_z^D(w)<\alpha$. Then we can choose a sequence $\{z^{(n)}\}$ of points in D such that $\{z^{(n)}\}$ tends to a boundary point ζ and such that $d_{z^{(n)}}^D(w)<\alpha$ for all n. By the definition of $d_{z^{(n)}}^D(w)$, we can choose sequences $\{r_n\}$ and $\{\phi_n\}$ such that

$$0<r_n<1\,, \qquad \frac{1}{2}\log\frac{1+r_n}{1-r_n}<\alpha$$

and

$$\phi_n\in\mathrm{Aut}(D)\,,\quad \phi_n(z^{(n)})=0\,,\quad \phi_n(w)\in r_n D\,.$$

Since $\{\phi_n^{-1}\}$ is a normal family and since $\phi_n(w)\in\overline{rD}$ where $r=(e^{2\alpha}-1)/(e^{2\alpha}+1)$, we may assume that $\{\phi_n^{-1}\}$ converges, uniformly on every compact subset of D, to a holomorphic mapping $\psi: D\to\boldsymbol{C}^N$ and that $\phi_n(w)\to w^*\in\overline{rD}$. Hence we have

$$\|\psi(w^*)-w\|\leqq\|\psi(w^*)-\psi(\phi_n(w))\|+\|\psi(\phi_n(w))-\phi_n^{-1}(\phi_n(w))\|\longrightarrow 0 \qquad \text{as } n\to\infty\,,$$

where $\|\ \ \|$ denotes the euclidean norm in $\boldsymbol{C}^N$, then $\psi(w^*)=w$. This implies that $\psi\in\mathrm{Aut}(D)$ ([5] p. 78). But this contradicts that $\psi(0)=\lim_{n\to\infty}\phi_n^{-1}(0)=\lim_{n\to\infty}z^{(n)}=\zeta\in\partial D$. Thus we conclude that $d_z^D(w)\to\infty$ if $z\to\partial D$. Similarly we can prove that $d_w^D(z)\to\infty$ if $z\to\partial D$.

The function d_D may not satisfy the triangle inequality. Following Kobayashi [2], we introduce $\tilde{d}_D$ by setting

$$\tilde{d}_D(z, w)=\inf\sum_{j=0}^{k-1} d_D(z^{(j)}, z^{(j+1)})\,,$$

where the infimum is taken over all finite sequences $\{z^{(0)}, z^{(1)}, \cdots, z^{(k)}\}$ with $z^{(0)}=z$ and $z^{(k)}=w$. Then $\tilde{d}_D$ is a pseudodistance on D.

Next we consider domains D which satisfy conditions (a), (b), (c) and an additional condition

(d) D is circular.

The following lemma tells us that $d_w^D(z)$ decreases under holomorphic mappings:

LEMMA 1. *Let D_1 and D_2 be domains in $\boldsymbol{C}^{N_1}$ and $\boldsymbol{C}^{N_2}$, respectively, which* *satisfy conditions* (a)~(d). *If* $F: D_1 \to D_2$ *is a holomorphic mapping, then*

$$d_{F(w)}^{D_2}(F(z)) \leqq d_w^{D_1}(z) \qquad (z, w \in D_1).$$

Proof. We shall first prove the inequality

$$d_0^{D_2}(G(z)) \leqq d_0^{D_1}(z) \qquad (z \in D_1)$$

for a holomorphic mapping $G: D_1 \to D_2$ with $G(0)=0$. Let $z \in D_1$ and $d_0^{D_1}(z) < \alpha$. Then there exists an r, $0<r<1$, such that $\frac{1}{2}\log\frac{1+r}{1-r} < \alpha$ and $rD_1 \ni \phi(z)$ for some $\phi \in \mathrm{Aut}(D_1)$ with $\phi(0)=0$. Put $H=G \circ \phi^{-1}$. Since $H: D_1 \to D_2$ is a holomorphic mapping with $H(0)=0$ and since D_1 and D_2 satisfy conditions (b)~(d), we have $H(rD_1) \subset rD_2$ ([6] p. 161). Hence $G(z)=H(\phi(z)) \in rD_2$, and therefore $d_0^{D_2}(G(z)) < \alpha$. Thus we have

$$d_0^{D_2}(G(z)) \leqq d_0^{D_1}(z) \qquad (z \in D_1).$$

From this inequality we have, for any $z, w \in D_1$,

$$d_{F(w)}^{D_2}(F(z)) = d_0^{D_2}((\psi_2 \circ F \circ \psi_1^{-1})(\psi_1(z))) \leqq d_0^{D_1}(\psi_1(z)) = d_w^{D_1}(z)$$

by taking $\psi_1 \in \mathrm{Aut}(D_1)$ and $\psi_2 \in \mathrm{Aut}(D_2)$ with $\psi_1(w)=0$, $\psi_2(F(w))=0$.

If D is a domain in $\boldsymbol{C}^N$ which is holomorphically equivalent to a domain $\tilde{D}$ (i.e. there is a biholomorphic mapping of D onto $\tilde{D}$) satisfying conditions (a)~(d), we define d_w^D by

$$d_w^D(z) = d_{\psi(w)}^{\tilde{D}}(\psi(z)),$$

where ψ is a biholomorphic mapping of D onto $\tilde{D}$. Note that this definition does not depend on choices of $\tilde{D}$ and ψ. (It follows from Lemma 1). Hence we can also define d_D and $\tilde{d}_D$ for D. The functions d_w^D and d_D have properties (1)~(4) in Proposition 1. Further the following proposition is an immediate consequence of Lemma 1.

PROPOSITION 2. *Let* D_1 *and* D_2 *be domains in* $\boldsymbol{C}^{N_1}$ *and* $\boldsymbol{C}^{N_2}$, *respectively, which are holomorphically equivalent to domains satisfying conditions* (a)~(d). *If* $F: D_1 \to D_2$ *is a holomorphic mapping, then*

$$d_{F(w)}^{D_2}(F(z)) \leqq d_w^{D_1}(z) \qquad (z, w \in D_1)$$

and

$$d_{D_2}(F(z), F(w)) \leqq d_{D_1}(z, w) \qquad (z, w \in D_1).$$

Let k_D and c_D denote the Kobayashi pseudodistance and the Carathéodory pseudodistance of D, respectively. If D is holomorphically equivalent to a domain satisfying conditions (a)~(d), then it follows from Proposition 2 that

$$c_D \leqq \tilde{d}_D \leqq k_D.$$

3. We shall now turn our attention to bounded symmetric domains. It is known that every bounded symmetric domain is holomorphically equivalent to a domain in $\boldsymbol{C}^N$ satisfying (a)~(d) [4]. (Conversely, every domain in $\boldsymbol{C}^N$ satisfying conditions (a), (c), (d) is symmetric). Hence d_w^D, d_D and $\tilde{d}_D$ are defined on bounded symmetric domains D. In this section we shall show that, for a bounded symmetric domain D, d_w^D, d_D and $\tilde{d}_D$ coincide with k_D and c_D. Our proof will follow Kobayashi's argument ([3] p. 52) which was used to prove that $k_D=c_D$ for bounded symmetric domains D.

LEMMA 2. *Let U^N be the unit polydisk in $\boldsymbol{C}^N$. Then*

$$d_w^{U^N}(z)=d_z^{U^N}(w)=\max\{\rho(z_j, w_j): j=1, \cdots, N\},$$

where $z=(z_1, \cdots, z_N)$ and $w=(w_1, \cdots, w_N)$.

Proof. Let $z^*=(z_1^*, \cdots, z_N^*)$ and $w^*=(w_1^*, \cdots, w_N^*)$ be points in U^N and let $d_{w^*}^{U^N}(z^*)<\alpha$. Then $\frac{1}{2}\log\frac{1+r}{1-r}<\alpha$ and $rU^N\ni\phi(z^*)$ for some r, $0<r<1$, and for some $\phi\in\mathrm{Aut}(U^N)$ with $\phi(w^*)=0$. Now ϕ has a form

$$\phi(z)=(\phi_1(z_{p(1)}), \cdots, \phi_N(z_{p(N)})), \qquad z=(z_1, \cdots, z_N),$$

$$\phi_j(\zeta)=\varepsilon_j\frac{\zeta-w_{p(j)}^*}{1-\overline{w_{p(j)}^*}\zeta} \qquad (j=1, \cdots, N; \zeta\in U),$$

where $\varepsilon_j\in\boldsymbol{C}$, $|\varepsilon_j|=1$ and p is a permutation of the integers from 1 to N ([5] p. 68). Since $\phi_j\in\mathrm{Aut}(U)$, $\phi_j(w_{p(j)}^*)=0$ and $\phi_j(z_{p(j)}^*)\in rU$, we have

$$\rho(z_j^*, w_j^*)=d_{w_j^*}^U(z_j^*)\leqq\frac{1}{2}\log\frac{1+r}{1-r}<\alpha \qquad (j=1, \cdots, N).$$

Since α was arbitrary, we have

$$\max\{\rho(z_j^*, w_j^*): j=1, \cdots, N\}\leqq d_{w^*}^{U^N}(z^*).$$

To prove the inequality in the opposite direction, let $\rho(z_j^*, w_j^*)<\alpha$ for all j. Then we can choose an r, $0<r<1$, and $\phi_j\in\mathrm{Aut}(U)$, $j=1, \cdots, N$, such that $\frac{1}{2}\log\frac{1+r}{1-r}<\alpha$ and $\phi_j(z_j^*)\in rU$ and $\phi_j(w_j^*)=0$. Put

$$\phi(z)=(\phi_1(z_1), \cdots, \phi_N(z_N)), \quad z=(z_1, \cdots, z_N).$$

Then ϕ is an element of $\mathrm{Aut}(U^N)$ satisfying $\phi(w^*)=0$ and $\phi(z^*)\in rU^N$. Hence

$$d_{w^*}^{U^N}(z^*)\leqq\frac{1}{2}\log\frac{1+r}{1-r}<\alpha.$$

Thus we obtain that

$$d_{w^*}^{U^N}(z^*)=\max\{\rho(z_j^*, w_j^*): j=1, \cdots, N\}.$$

Since $\rho(z_j^*, w_j^*)=\rho(w_j^*, z_j^*)$, we have also

$$d_{w^*}^{U^N}(z^*)=d_{z^*}^{U^N}(w^*).$$

THEOREM 1. *If D is a bounded symmetric domain in $\boldsymbol{C}^N$, then*

$$k_D(z, w)=c_D(z, w)=d_w^D(z) \qquad (z, w\in D).$$

Proof. It is known that D is holomorphically equivalent to a domain D^* which has the following properties:

(α) $D^*\cap \boldsymbol{C}^l=U^l$ and $D^*\subset U^N$,

(β) for any $z^*, w^*\in D^*$, there exists a $\phi\in \mathrm{Aut}(D^*)$ such that $\phi(w^*)=0$ and $\phi(z^*)=\zeta=(\zeta_1, \cdots, \zeta_l, 0, \cdots, 0)$ where $|\zeta_j|<1$, $j=1, \cdots, l$.

Here l is the rank of D. Since the injections $U^l\to D^*$ and $D^*\to U^N$ are distance-decreasing (by Proposition 2), we have

$$d_0^{U^N}(\zeta)\leqq d_0^{D^*}(\zeta)\leqq d_0^{U^l}(\zeta')$$

where $\zeta=(\zeta', 0'')$, $\zeta'=(\zeta_1, \cdots, \zeta_l)$, $0''=(0, \cdots, 0)$. From Lemma 2 we have

$$d_0^{U^N}(\zeta)=d_0^{U^l}(\zeta')=\max\{\rho(\zeta_j, 0): j=1, \cdots, l\},$$

and then we obtain

$$d_0^{D^*}(\zeta)=\max\{\rho(\zeta_j, 0): j=1, \cdots, l\}.$$

It is also known that

$$k_{D^*}(\zeta, 0)=c_{D^*}(\zeta, 0)=\max\{\rho(\zeta_j, 0): j=1, \cdots, l\}$$

([3] p. 52). Hence it follows from (β) that, for $z^*, w^*\in D^*$,

$$d_{w^*}^{D^*}(z^*)=k_{D^*}(z^*, w^*)=c_{D^*}(z^*, w^*).$$

This implies that

$$d_w^D(z)=k_D(z, w)=c_D(z, w) \qquad (z, w\in D).$$

COROLLARY 1. *If D is a bounded symmetric domain in $\boldsymbol{C}^N$, then*

$$d_D(z, w)=\tilde{d}_D(z, w)=d_w^D(z)=d_z^D(w) \qquad (z, w\in D).$$

Every bounded symmetric domain in $\boldsymbol{C}^N$ is holomorphically equivalent to a domain D in $\boldsymbol{C}^N$ which satisfies conditions (a)~(d) and

(e) the isotropy group K of 0 in $\mathrm{Aut}(D)$ acts by complex linear transformations.

We shall call such a domain D a domain given in a canonical realization. The following is an immediate consequence of Theorem 1 and (5) in Proposition 1.

THEOREM 2. *Let D be a bounded symmetric domain given in a canonical realization in $\boldsymbol{C}^N$. Then*

$$k_D(z, w)=\inf\left\{\frac{1}{2}\log\frac{1+r}{1-r} : 0<r<1 \text{ and } rD\ni\phi(z) \text{ for some } \phi\in\mathrm{Aut}(D) \text{ with } \phi(w)=0\right\}$$

and

$$\{z\in D : k_D(0, z)<\alpha\}=rD, \qquad r=\frac{e^{2\alpha}-1}{e^{2\alpha}+1}.$$

COROLLARY 2. *Let D be a bounded symmetric domain given in a canonical realization in $\boldsymbol{C}^N$ and let $w\in\partial D$. If $0\leqq t_1<t_2<t_3<1$, then*

$$k_D(t_1w, t_3w)=k_D(t_1w, t_2w)+k_D(t_2w, t_3w).$$

Proof. We shall first show that

$$k_D(sw, tw)=k_D(0, tw)-k_D(0, sw)$$

for $0<s<t<1$. Since D satisfies conditions (c) and (d), $\{\zeta w : \zeta\in\boldsymbol{C}, |\zeta|<1\}\subset D$. Hence

$$f_w(\zeta)=\zeta w$$

is a holomorphic mapping of U into D, and so

$$k_D(sw, tw)\leqq k_U(s, t)=\frac{1}{2}\log\frac{1+\dfrac{t-s}{1-st}}{1-\dfrac{t-s}{1-st}}=\frac{1}{2}\log\frac{1+t}{1-t}-\frac{1}{2}\log\frac{1+s}{1-s}$$

$$=k_D(0, tw)-k_D(0, sw)$$

by Theorem 2. But the triangle inequality yields

$$k_D(sw, tw)\geqq k_D(0, tw)-k_D(0, sw).$$

Thus we obtain the equality

$$k_D(sw, tw)=k_D(0, tw)-k_D(0, sw) \qquad (0<s<t<1).$$

Using this equality we have, for $0\leqq t_1<t_2<t_3<1$,

$$\begin{aligned} k_D(t_1w, t_3w)&=k_D(0, t_3w)-k_D(0, t_1w)\\ &=\{k_D(0, t_3w)-k_D(0, t_2w)\}+\{k_D(0, t_2w)-k_D(0, t_1w)\}\\ &=k_D(t_2w, t_3w)+k_D(t_1w, t_2w). \end{aligned}$$

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